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TOWARD THE THEORY OF GENERALIZED QUASI-ISOMETRIES

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This note is devoted to the study of the so-called finitely bi-Lipschitz mappings that are far-reaching generalizations of isometries as well as quasi-isometries. It is establish a number of criteria for homeomorphic extension to the boundary of the finitely bi-Lipschitz homeomorphisms f between domains in $\mathbb{R}^n$, $n \geqslant 2$, whose outer dilatations $K_O(x, f)$ satisfy integral constraints of the type $\int \Phi(K_O^{n-1}(x, f))\, dm(x) < \infty$ with a convex increasing function $\Phi\colon [0, \infty] \to [0, \infty]$. Note that integral conditions on the function Φ found by us are not only sufficient but also necessary for a continuous extension of f to the boundary.

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Работа посвящена изучению так называемых конечно билипшицевых отображений, которые является далеко идущими обобщениями изометрий и квазиизометрий. Получен ряд критериев для гомеоморфного продолжения на границу конечно билипшицевых гомеоморфизмов f между областями в $\mathbb{R}^n$, $n \geqslant 2$, внешние дилатации $K_O(x, f)$ которых удовлетворяют интегральным ограничениям $\int \Phi(K_O^{n-1}(x, f))\, dm(x) < \infty$ с возрастающей выпуклой функцией $\Phi\colon [0, \infty] \to [0, \infty]$. Отметим, что интегральные условия на функцию Φ, найденные нами, является не только достаточными, но и необходимыми для непрерывного продолжения f на границу.

1. Introduction. In the theory of mappings quasiconformal in the mean, integral conditions of the type

$$\int_D \Phi(K(x))\, dm(x) < \infty \tag{1}$$

are applied to various characteristics K of these mappings, see e.g. [1], [2], [4], [8]–[11], [14], [15] and [17]. Here $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, $n \geqslant 2$. Investigations of classes with the integral conditions (1) are also actual in the connection with the recent development of the so-called mappings with finite distortion, see related references e.g. in the monographs [5] and [12]. The present paper is a natural continuation of our previous papers [6] and [7], see also Chapters 9 and 10 in the monograph [12], that have been devoted to mappings with integral constraints for dilatations of other types turned out to be useful under the study of mappings with the constraints of the type (1).

Following [6], see also Chapter 10 in [12], we say that a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n \geqslant 2$, is called *finitely bi-Lipschitz* if

$$0 < l(x, f) \leqslant L(x, f) < \infty \qquad \forall\ x \in D \tag{2}$$

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where

$$L(x,f) = \limsup_{y\to x} \frac{|f(x)-f(y)|}{|x-y|}$$

and

$$l(x,f) = \liminf_{y\to x} \frac{|f(x)-f(y)|}{|x-y|} \ .$$

Note that a finitely bi-Lipschitz homeomorphism $f\colon D\to D'$ is differentiable with $J(x,f)\neq 0$ a.e. and has (N)-property.

Recall that the *outer dilatation* of a mapping $f\colon D\to\mathbb{R}^n$, $n\geqslant 2$, at a point $x\in D$ of differentiability for f with $J(x,f)\neq 0$ is the quantity

$$K_O(x,f) = \frac{||f'(x)||^n}{|J(x,f)|} \ .$$

As usual, here $f'(x)$ denotes the Jacobian matrix of f at the point x, $J(x,f)=\det f'(x)$ is its determinant and

$$||f'(x)|| = \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|} \ .$$

2. Preliminaries. Combining Lemma 4.1 and Theorem 5.5 in [6], see also Lemma 10.3 and Theorem 10.11 in [12], and Theorem 10.1 in [7], see also Theorem 9.8 in [12], we obtain the following statement.

Proposition 1. *Let D and D' be bounded domains in $\mathbb{R}^n$, $n\geq 2$, which either are convex or have smooth boundaries. If $f\colon D\to D'$ is a finitely bi-Lipschitz homeomorphism with*

$$\int_0^{\delta(x_0)} \frac{dr}{||\,K_O||_{\,n-1}(x_0,r)} = \infty \qquad \forall\ x_0\in\partial D \tag{3}$$

for some $\delta(x_0)\in(0,d(x_0))$ where $d(x_0)=\sup_{x\in D}\,|x-x_0|$ and

$$||\,K_O||_{\,n-1}(x_0,r) = \Big(\int\limits_{D\cap S(x_0,r)} K_O^{n-1}(x,f)\,d\mathcal{A}\Big)^{\frac{1}{n-1}},$$

then f has a homeomorphic extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ onto $\overline{D'}$.

Here as usual $S(x_0,r)$ denotes the sphere $|x-x_0|=r$.

Remark 1. Furthermore, it follows from our results in [6] and [7], see also Chapters 9 and 10 in [12], Proposition 1 is valid for mappings with finite area distortion (and the so-called lower Q-homeomorphisms with the change of $K_O(x,f)$ by $Q(x)$ in (3)) and for much more general domains with the so–called weakly flat boundaries. The latter class of domains include, in particular, the so–called QED-domains (quasiextremal domains by Gehring–Martio) and the so–called uniform domains by Martio–Sarvas, see [3] and [13].

3. The main lemma. The following statement is a generalization of Lemma 3.1 from [16].

Lemma 1. *Let $K\colon \mathbb{B}^n \to [0,\infty]$, $n \geq 2$, be a measurable function and let $\Phi\colon [0,\infty] \to [0,\infty]$ be a convex increasing function. Then*

$$\int_0^1 \frac{dr}{rk^{\frac{1}{p}}(r)} \geqslant \frac{1}{n}\int_{eM}^{\infty} \frac{d\tau}{\tau\Big[\Phi^{-1}(\tau)\Big]^{\frac{1}{p}}} \qquad \forall\, p\in(0,\infty) \tag{4}$$

where $k(r)$ is the average of the function $K(x)$ over the sphere $|x| = r$,

$$M{:=}\,⨍_{\mathbb{B}^n} \Phi(K(x))\, dm(x) \tag{5}$$

is the average of the function $\Phi\circ K$ over the unit ball $\mathbb{B}^n$.

Remark 2. Note that (4) under every $p\in(0,\infty)$ is equivalent to

$$\int_0^1 \frac{dr}{rk^{\frac{1}{p}}(r)} \geqslant \frac{1}{n}\int_{eM}^{\infty} \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} \qquad \text{where} \qquad \Phi_p(t){:=}\Phi(t^p)\ . \tag{6}$$

Proof of Lemma 1. The result is obvious if $M=\infty$ because then the integral in the right hand side of (4) is zero. Hence, we assume further that $M<\infty$. Moreover, we may also assume that $\Phi(0)>0$ and hence, that $M>0$ (the case $\Phi(0)=0$ is reduced to it by approximation of $\Phi(t)$ through cutting off its graph lower the line $\tau=\delta>0$). Denote

$$t_* = \sup\nolimits_{\Phi_p(t)=\tau_0} t, \quad \tau_0 = \Phi(0) > 0. \tag{7}$$

Setting

$$H_p(t){:=}\log\,\Phi_p(t), \tag{8}$$

we observe that

$$H_p^{-1}(\eta) = \Phi_p^{-1}(e^\eta), \quad \Phi_p^{-1}(\tau) = H_p^{-1}(\log\,\tau). \tag{9}$$

Thus, we obtain that

$$k^{\frac{1}{p}}(r) = H_p^{-1}\Big(\log\frac{h(r)}{r^n}\Big) = H_p^{-1}\Big(n\log\frac{1}{r} + \log\, h(r)\Big) \quad \forall\, r\in R_* \tag{10}$$

where $h(r){:=}r^n\Phi(k(r)) = r^n\Phi_p\Big(k^{\frac{1}{p}}(r)\Big)$ and $R_* = \{r\in(0,1)\colon k^{\frac{1}{p}}(r) > t_*\}$. Then also

$$k^{\frac{1}{p}}(e^{-s}) = H_p^{-1}\Big(ns + \log\, h(e^{-s})\Big) \quad \forall\, s\in S_* \tag{11}$$

where $S_* = \{s\in(0,\infty)\colon k^{\frac{1}{p}}(e^{-s}) > t_*\}$.

Now, by the Jensen inequality and convexity of Φ we have that

$$\int_0^\infty h(e^{-s})\, ds = \int_0^1 h(r)\,\frac{dr}{r} = \int_0^1 \Phi(k(r))\, r^{n-1}\, dr \leqslant \tag{12}$$
$$\leqslant \int_0^1 \Big(⨍_{S(r)} \Phi(K(x))\, d\mathcal{A}\Big)\, r^{n-1} dr \leqslant \frac{\Omega_n}{\omega_{n-1}}\cdot M = \frac{M}{n}$$

where we use the mean value of the function $\Phi_p\circ K$ over the sphere $S(r) = \{x\in\mathbb{B}^n\colon |x| = r\}$ with respect to the area measure. As usual, here Ω_n and ω_{n-1} is the volume of the unit ball and the area of the unit sphere in $\mathbb{R}^n$, respectively. Then arguing by contradiction, it is easy to see that

$$|T| = \int_T ds \leqslant \frac{1}{n} \tag{13}$$

where $T = \{s \in (0,\infty)\colon h(e^{-s}) > M\}$. Next, let us show that

$$k^{\frac{1}{p}}\Big(e^{-s}\Big) \leqslant H_p^{-1}\Big(ns + \log\, M\Big) \quad \forall\ s \in (0,\infty) \setminus T_* \tag{14}$$

where $T_* = T \cap S_*$. Note that $(0,\infty)\setminus T_* = [(0,\infty)\setminus S_*] \cup [(0,\infty)\setminus T] = [(0,\infty)\setminus S_*] \cup [S_* \setminus T]$. Inequality (14) holds for $s \in S_* \setminus T$ by (11) because H_p^{-1} is a non-decreasing function. Note also that by (7)

$$e^{ns}M > \Phi(0) = \tau_0 \quad \forall\ s \in (0,\infty) \tag{15}$$

and then by (9)

$$t_* < \Phi_p^{-1}(e^{ns}M) = H_p^{-1}\Big(ns + \log\, M\Big) \quad \forall\ s \in (0,\infty). \tag{16}$$

Consequently, (14) holds for $s \in (0,\infty) \setminus S_*$, too. Thus, (14) is true.

Since H_p^{-1} is non-decreasing, we have by (13) and (14) that

$$\int_0^1 \frac{dr}{rk^{\frac{1}{p}}(r)} = \int_0^\infty \frac{ds}{k^{\frac{1}{p}}(e^{-s})} \geqslant \int_{(0,\infty)\setminus T_*} \frac{ds}{H_p^{-1}(ns+\Delta)} \geqslant \tag{17}$$

$$\geqslant \int_{|T_*|}^\infty \frac{ds}{H_p^{-1}(ns+\Delta)} \geqslant \int_{\frac{1}{n}}^\infty \frac{ds}{H_p^{-1}(ns+\Delta)} = \frac{1}{n}\int_{1+\Delta}^\infty \frac{d\eta}{H_p^{-1}(\eta)}$$

where $\Delta = \log\, M$. Note that $1+\Delta = \log\, eM$. Thus,

$$\int_0^1 \frac{dr}{rk^{\frac{1}{p}}(r)} \geqslant \frac{1}{n}\int_{\log\, eM}^\infty \frac{d\eta}{H_p^{-1}(\eta)} \tag{18}$$

and, after the change of variables $\eta = \log\, \tau$, we obtain (6), see (9), and hence (4). $\Box$

Since the mapping $t \mapsto t^p$ for every positive p is a sense-preserving homeomorphism $[0,\infty]$ onto $[0,\infty]$ we may rewrite Theorem 2.1 from [16] in the following form which is more convenient for further applications. Here, in (20) and (21), we complete the definition of integrals by ∞ if $\Phi_p(t) = \infty$, correspondingly, $H_p(t) = \infty$, for all $t \geqslant T \in [0,\infty)$. The integral in (21) is understood as the Lebesgue–Stieltjes integral and the integrals in (20) and (22)–(25) as the ordinary Lebesgue integrals.

Proposition 2. *Let $\Phi\colon [0,\infty] \to [0,\infty]$ be an increasing function. Set*

$$H_p(t) = \log \Phi_p(t), \quad \Phi_p(t) = \Phi(t^p), \quad p \in (0,\infty). \tag{19}$$

Then the equality

$$\int_\delta^\infty H_p'(t)\,\frac{dt}{t} = \infty \tag{20}$$

implies the equality

$$\int_\delta^\infty \frac{dH_p(t)}{t} = \infty \tag{21}$$

and (21) is equivalent to

$$\int_{\delta}^{\infty} H_p(t)\,\frac{dt}{t^2} = \infty \tag{22}$$

for some $\delta > 0$*, and (22) is equivalent to every of the equalities:*

$$\int_{0}^{\delta} H_p\Big(\frac{1}{t}\Big)\,dt = \infty \tag{23}$$

for some $\delta > 0$,

$$\int_{\delta_*}^{\infty} \frac{d\eta}{H_p^{-1}(\eta)} = \infty \tag{24}$$

for some $\delta_* > H(+0)$,

$$\int_{\delta_*}^{\infty} \frac{d\tau}{\tau\Phi_p^{-1}(\tau)} = \infty \tag{25}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (20) is equivalent to (21) and hence (20)–(25) are equivalent if Φ *is, in addition, absolutely continuous. In particular, all the conditions (20)–(25) are equivalent if* Φ *is convex and increasing.*

It is easy to see that conditions (20)–(25) become weaker if p increases, see e.g. (22). It is necessary to give one more explanation. From the right hand sides of conditions (20)–(25) we have in mind $+\infty$. If $\Phi_p(t) = 0$ for $t \in [0, t_*]$, then $H_p(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H_p'(t) = 0$ for $t \in [0, t_*]$. Note that conditions (21) and (22) exclude that t_* belongs to the interval of integrability because of the left hand sides of (21) and (22) are either equal to $-\infty$ or indeterminate. Hence, we may assume in (20)–(23) that $\delta > t_0$, correspondingly, $\delta < 1/t_0$ where $t_0\colon = \sup_{\Phi_p(t)=0} t$, $t_0 = 0$ if $\Phi_p(0) > 0$.

4. The main result. Combining Proposition 1 and Lemma 1, we come to the following statement.

Theorem 1. *Let* D *and* D' *be bounded domains in* $\mathbb{R}^n$*,* $n \geq 2$*, which either are convex or have smooth boundaries. Suppose that* $f\colon D \to D'$ *is a finitely bi-Lipschitz homeomorphism with*

$$\int_D \Phi(K_O^{n-1}(x,f))\,dm(x) < \infty \tag{26}$$

for a convex increasing function $\Phi\colon [0,\infty] \to [0,\infty]$*. If*

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau\Big[\Phi^{-1}(\tau)\Big]^{\frac{1}{n-1}}} = \infty \tag{27}$$

for some $\delta_0 > \tau_0\colon = \Phi(0)$*, then* f *has a homeomorphic extension* $\overline{f}$ *to* $\overline{D}$ *that maps* $\overline{D}$ *onto* $\overline{D'}$*.*

Remark 3. Note that (27) can be rewritten in the form

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau\Phi_{n-1}^{-1}(\tau)} = \infty \qquad \text{where} \quad \Phi_{n-1}(t){:=}\Phi(t^{n-1})\,. \tag{28}$$

Note also that by Proposition 2 (28) can be replaced with every of the conditions (20)–(24) under $p = n-1$ and, in particular, (22) can be rewritten in the form

$$\int_{\delta}^{\infty} \log\,\Phi(t)\,\frac{dt}{t^{n'}} = +\infty \tag{29}$$

for some $\delta > 0$ where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e. $n' = 2$ for $n = 2$, n' is decreasing in n and $n' = n/(n-1) \to 1$ as $n \to \infty$.

Proof of Theorem 1. Indeed, let us extend the function $K_O(x, f)$ by zero outside of D and set, for fixed $x_0 \in \partial D$,

$$K(x) = K_O^{n-1}(x_0 + xd_0, f)\ , \qquad x \in \mathbb{B}^n$$

with some positive $d_0 < \sup_{z\in D} |z - x_0|$. Then by Lemma 1 with the given $K(x)$ and $p = n-1$ we have that

$$\int_0^1 \frac{dr}{rk^{\frac{1}{n-1}}(r)} \geqslant \frac{1}{n}\int_{eM}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} \tag{30}$$

where $k(r)$ is the average of $K(x)$ over the sphere $|x| = r$ and

$$M = \fint_{\mathbb{B}^n} \Phi(K(x))\,dm(x). \tag{31}$$

Now, after the change of variables $y_0 = x_0 + xd_0$ in (31), we have by condition (26) that

$$M \leqslant N\colon\ = \Phi(0) + \frac{1}{\Omega_n d_0^n}\int_D \Phi(K_O^{n-1}(y, f))\,dm(y) < \infty$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$ and after the substitution $\rho = rd_0$ in the left hand side integral in (30) we obtain that

$$\int_0^{d_0} \frac{d\rho}{||K_O||_{n-1}(x_0, \rho)} \geqslant \frac{1}{n\omega_{n-1}^{\frac{1}{n-1}}}\ \int_{eN}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}},$$

where ω_{n-1} is the area of unit sphere in $\mathbb{R}^n$ and

$$||K_O||_{n-1}(x_0, \rho) = \Big(\int_{|z-x_0|=\rho} K_O^{n-1}(z, f)\,d\mathcal{A}\Big)^{\frac{1}{n-1}}.$$

Note that $N > \Phi(0)$. Thus, we conclude from condition (27) that

$$\int_0^{\delta_0} \frac{d\rho}{||K_O||_{n-1}(x_0, \rho)} = \infty. \tag{32}$$

This is obvious if $\delta\colon\ = eN \leqslant \delta_0$. If $\delta > \delta_0$, then

$$\int_{\delta_0}^{\infty} \frac{\delta\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \int_{\delta}^{\infty} \frac{\delta\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} + \int_{\delta_0}^{\delta} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}},$$

where

$$0 < \int_{\delta_0}^{\delta} \frac{\delta\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} \leqslant \frac{\log\frac{\delta}{\delta_0}}{[\Phi^{-1}(\delta_0)]^{\frac{1}{n-1}}} < \infty$$

because $\Phi^{-1}(\delta_0) > 0$.

Finally, by Proposition 1 and (32) we obtain the statement of Theorem 1. □

Remark 4. Furthemore, by our results in [6] and [7], see also Chapters 9 and 10 in [12], Theorem 1 is valid for mappings with finite area distortion (and the so-called lower Q-homeomorphisms with the change of $K_O(x, f)$ by $Q(x)$ in (3)) and for much more general domains with the so–called weakly flat boundaries, see Remark 1. The latter class of domains include, in particular, the so–called $QED-$domains (quasiextremal domains by Gehring–Martio) and the so–called uniform domains by Martio–Sarvas, see [3] and [13].

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