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V. S. Ilkiv, T. V. Maherovska

EXACT ESTIMATE FOR THE MEASURE OF THE LEVEL SET OF THE MODULUS OF A FUNCTION WITH HIGH–ORDER CONSTANT–SIGN DERIVATIVE

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For functions $f \in \mathbf{C}^n[a,b]$ such that $|f^{(n)}(x)| \geq \delta$ on $[a,b]$, we establish an exact estimate of the Lebesgue measure $\operatorname{meas} G_n(\varepsilon,\delta,f)$ of the set $G_n(\varepsilon,\delta,f)$ of points $x \in [a,b]$, for which $|f(x)| \leq \varepsilon$, namely, we prove a sharp inequality

$$\operatorname{meas} G_n(\varepsilon,\delta,f) \leq \min\left(b-a,\, 4\sqrt[n]{\tfrac{n!}{2}} \cdot \sqrt[n]{\tfrac{\varepsilon}{\delta}}\right).$$

On the base of Chebyshev polynomials T_n, we construct *extremal polynomials*, for which the inequality becomes an equality.

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Для функций $f \in \mathbf{C}^n[a,b]$ таких, что $|f^{(n)}(x)| \geq \delta$ на $[a,b]$, установлена точная оценка меры Лебега $\operatorname{meas} G_n(\varepsilon,\delta,f)$ множества $G_n(\varepsilon,\delta,f)$ тех точек $x \in [a,b]$, для которых $|f(x)| \leq \varepsilon$, а именно, доказано неулучшаемое неравенство

$$\operatorname{meas} G_n(\varepsilon,\delta,f) \leq \min\left(b-a,\, 4\sqrt[n]{\tfrac{n!}{2}} \cdot \sqrt[n]{\tfrac{\varepsilon}{\delta}}\right).$$

На основе многочленов Чебышова T_n построены *экстремальные многочлены*, для которых неравенство превращается в равенство.

Introduction. In problems for partial differential equations, which are connected with the problem of small denominators, there arises the question on establishing the lower estimate $|f_k(x)| > \varepsilon_k > 0$, where the vector k runs through the set of integer vectors $\mathbb{Z}^p$ at $p \geq 1$, for smooth functions f_k satisfying the inequality $\left|f_k^{(n_k)}(x)\right| > \delta_k > 0$ on the segment $[a,b]$, where $n_k \geq 1$.

A metric approach [1, 2] allows to obtain such estimates for almost all numbers $x \in [a,b]$ and for the vectors big by the norm k in the case of convergence of the series

$$\sum\nolimits_{k\in\mathbb{Z}^p} \operatorname{meas}\{x \in [a,b] \colon |f_k(x)| \leq \varepsilon_k\}.$$

To construct convergent majorant series, it is enough to majorize the elements of this series. Such estimates have been obtained in [3] with certain restrictions on the functions f_k and the numbers ε_k. Those restrictions have been eliminated in [4], and, in [5, 6, 7, 8, 9, 11], there have been improved the estimates and transferred them to wider classes of functions.

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The metric theory of Diophantine approximations uses widely the estimates of measures of the sets considered [10, 12], as well as the theory of entire and meromorphic functions [14, 13, 15], measure and integration theory [16] etc.

Problem statement. Pyartli Lemma and its generalizations. Let $[a, b] \subset \mathbb{R}$ be an arbitrary interval of positive length, $\mathbf{C}[a, b]$ and $\mathbf{C}^n[a, b]$, where $n \geq 1$, be spaces of continuous and n times continuously differentiable[1] functions on $[a, b]$,

$$\mathbf{C}^n_+[a, b] = \big\{f \in \mathbf{C}^n[a, b]\colon \min_{x\in[a,b]} f^{(n)}(x) \geq 0\big\}, \quad \mathbf{C}^n_-[a, b] = \big\{f \in \mathbf{C}^n[a, b]\colon -f \in \mathbf{C}^n_+[a, b]\big\}.$$

For $\delta > 0$ denote $\mathbf{C}^n_{\pm,\delta}[a, b] = \mathbf{C}^n_{+,\delta}[a, b] \cup \mathbf{C}^n_{-,\delta}[a, b]$, where $\mathbf{C}^n_{+,\delta}[a, b]$ and $\mathbf{C}^n_{-,\delta}[a, b]$ are the following sets of functions

$$\mathbf{C}^n_{+,\delta}[a, b] = \Big\{\frac{\delta}{n!}x^n + f(x)\colon f \in \mathbf{C}^n_+[a, b]\Big\}, \quad \mathbf{C}^n_{-,\delta}[a, b] = \big\{f \in \mathbf{C}^n[a, b]\colon -f \in \mathbf{C}^n_{+,\delta}[a, b]\big\}.$$

Hence the following inequalities characterizing the function from the introduced sets hold: $\min_{x\in[a,b]} f^{(n)}(x) \geq \delta$, when $f \in \mathbf{C}^n_{+,\delta}[a, b]$, $\max_{x\in[a,b]} f^{(n)}(x) \leq -\delta$, when $f \in \mathbf{C}^n_{-,\delta}[a, b]$, and $\min_{x\in[a,b]} |f^{(n)}(x)| \geq \delta$, when $f \in \mathbf{C}^n_{\pm,\delta}[a, b]$.

The set $G_n(\varepsilon, \delta, f)$, where $\varepsilon > 0$, for the function f from the set $\mathbf{C}^n_{\pm,\delta}[a, b]$ is given by the formula

$$G_n(\varepsilon, \delta, f) = \{x \in [a, b]\colon |f(x)| \leq \varepsilon\}. \tag{1}$$

This set, in particular, can be empty or a singleton set or a segment $[a, b]$, so (depending on a, b, ε and δ) its measure can attain values of zero and $b - a$, or any interjacent value.

The estimate of the measure $\operatorname{meas} G_n(\varepsilon, \delta, f)$ of the set $G_n(\varepsilon, \delta, f)$ has been obtained in the paper of Pyartli [3]. We shall formulate this result as Lemma [8].

Lemma 1 (Pyartli)**.** *Let $f \in \mathbf{C}^n_{\pm,\delta}[a, b] \cap \mathbf{C}^{n+1}[a, b]$ and at every point $x \in [a, b]$ the following inequality holds $\max_{1\leq i\leq n+1} \big|f^{(i)}(x)\big| \leq M$, with*

$$\varepsilon < \frac{\delta}{2} \cdot \min\Big(1, \frac{\delta^n}{\big(2^n - 2\big)^n M^n}\Big), \tag{2}$$

moreover, the Lebesgue measure of the set $G_n(\varepsilon, \delta, f)$ satisfies the estimate

$$\operatorname{meas} G_n(\varepsilon, \delta, f) \leq C_n \sqrt[n]{\frac{\varepsilon}{\delta}}, \tag{3}$$

where

$$C_n = \sqrt[n]{2}\,(2n + 1)\big(2^n - 2\big). \tag{4}$$

The proof of the stronger version of this Lemma is carried out in [4] by induction. The authors do not impose constraints (2) on the number ε and weaken the smoothness of the function f, but do not determine the constant C_n and its dependence on the parameter n.

Lemma 2. *If $f \in \mathbf{C}^n_{\pm,\delta}[a, b]$, then inequality* (3) *holds with a certain positive constant C_n, which depends*[2] *on the number n only.*

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[1] Derivative at the extreme points a and b are considered as unidirectional derivatives.

[2] There is an obvious dependence of the measure of the set $G_n(\varepsilon, \delta, f)$ on the numbers a and b — the endpoints of the segment $[a, b]$, which we consider to be fixed. If $\min_{x\in[a,b]} |f(x)| < \varepsilon < \max_{x\in[a,b]} |f(x)|$, then the function $\operatorname{meas} G_n(\varepsilon, \delta, f)$ is non–decreasing in variable ε, moreover $0 < \operatorname{meas} G_n(\varepsilon, \delta, f) < b-a$; in other cases: $\operatorname{meas} G_n(\varepsilon, \delta, f) = b - a$, when $\varepsilon \geq \max_{x\in[a,b]} |f(x)|$, and $\operatorname{meas} G_n(\varepsilon, \delta, f) = 0$, when $\varepsilon \leq \min_{x\in[a,b]} |f(x)|$.

Determination of the sequence C_n have been held in the papers [5, 11, 6, 8, 9, 7]. The results are as follows: $C_n = 2(b-a)n\sqrt[n]{(n+1)!/\max\limits_{x\in[a,b]}|f(x)|}$ in [11], $C_n = n2^{(n+1)/2}$ in [6], $C_n = 2n\sqrt[n]{n!}$ in [8], $C_n = \dfrac{1}{2}(n^2+n+2)3^{(n+1)/2}$ in [9], $C_n = 2n$ in [7]. The latter value is the smallest of the specified ones, but, as noted in [7], is not optimal for the class of functions $\mathbf{C}^n_{\pm,\delta}[a,b]$.

Main results. In this paper, we establish that the sought optimal value is the sequence $C_n = 4\sqrt[n]{n!/2}$, as well as find the functions for which the inequality (3) becomes an equality. We prove two theorems.

Theorem 1. *If ε and δ are arbitrary positive numbers, then*

$$\sup_{f\in\mathbf{C}^n_{\pm,\delta}[a,b]} \operatorname{meas} G_n(\varepsilon,\delta,f) = \min\left(b-a,\ 4\sqrt[n]{\frac{n!}{2}}\cdot\sqrt[n]{\frac{\varepsilon}{\delta}}\right). \tag{5}$$

In the case of a complex variable, for *polynomials*, one can use [13, c. 275–276], [14], [19, c. 78] the analogue of Lemma 2, namely the Cartan lemma [17], [18, c. 31–33].

Lemma 3 (Cartan)**.** *Let a function f be a unital polynomial of degree n, i.e. $f^{(n)}(z) = n! = \delta$ for all $z\in\mathbb{C}$. Then the inequality $|f(z)| < \varepsilon$ on the complex plane holds on a set that is covered by a system of circles, with the sum of their diameters less than $4n\sqrt[n]{\dfrac{\varepsilon}{\delta}}$.*

The question of whether the constant $4n$ is sharp, i.e. in the equality similar to (5), in the complex case remains open [19, c. 78].

In contrast to Theorem 1 that deals with estimating the measure of the set defined by the function f, in the following main Theorem 2, we estimate the measure of an arbitrary (measurable) set.

Theorem 2. *Let E be a measurable subset of the interval $[a,b]$, function f belong to the space $\mathbf{C}^n[a,b]$ and its derivative $f^{(n)}$ does not vanish on $[a,b]$, then*[3]

$$\operatorname{meas} E \le \min\left(4\sqrt[n]{\frac{n!}{2}}\cdot\sqrt[n]{\frac{\sup\limits_{x\in E}|f(x)|}{\min\limits_{x\in[a,b]}|f^{(n)}(x)|}},\ b-a\right). \tag{6}$$

Proof of the main results. Proofs of Theorem 1 and Theorem 2 are based on two lemmas (Lemma 4 and Lemma 5).

Lemma 4. *Let $f\in\mathbf{C}^n_{\pm,\delta}[a,b]$, and function F be defined on the interval $[a_1,b_1]$ on the base of the function f, where $a_1 = \left(\dfrac{2\delta}{\varepsilon n!}\right)^{1/n}\cdot a$, $b_1 = \left(\dfrac{2\delta}{\varepsilon n!}\right)^{1/n}\cdot b$, by the formula*

$$F(y) = \frac{2}{\varepsilon}\cdot f\left(\left(\frac{\varepsilon n!}{2\delta}\right)^{1/n}\cdot y\right), \tag{7}$$

[1]The minimum value on the set $[a,b]$ in formula (6) can be replaced with the minimum on the set $[\inf E, \sup E]$.

then $F \in \mathbf{C}^n_{\pm,n!}[a_1,b_1]$ *and the following equality holds:*

$$\operatorname{meas} G_n(\varepsilon,\delta,f) = \Big(\frac{\varepsilon n!}{2\delta}\Big)^{1/n} \cdot \operatorname{meas} G_n(2,n!,F), \tag{8}$$

moreover the point x *belongs to the set* $G_n(\varepsilon,\delta,f)$ *if and only if the point* $y = \Big(\frac{2\delta}{\varepsilon n!}\Big)^{1/n} \cdot x$ *belongs to the set* $G_n(2,n!,F) \equiv \{y \in [a_1,b_1] \colon |F(y)| \le 2\}$.

Proof. Let $\widehat{x} \in G_n(\varepsilon,\delta,f)$ and $f \in \mathbf{C}^n_{\pm,\delta}[a,b]$, i.e. the following inequality hold $|f(\widehat{x})| \le \varepsilon$ and $|f^{(n)}(x)| \ge \delta$, where $x \in [a,b]$, then, by formula (7),

$$|F^{(n)}(y)| = \frac{2}{\varepsilon} \cdot \bigg(\Big(\frac{\varepsilon n!}{2\delta}\Big)^{1/n}\bigg)^n \cdot \bigg| f^{(n)}\bigg(\Big(\frac{\varepsilon n!}{2\delta}\Big)^{1/n} \cdot y\bigg)\bigg| = \frac{n!}{\delta} \cdot |f^{(n)}(x)| \ge n!,$$

when $\Big(\frac{2\delta}{\varepsilon n!}\Big)^{1/n} \cdot x = y \in [a_1,b_1]$, and

$$|F(\widehat{y})| = \bigg| F\bigg(\Big(\frac{2\delta}{\varepsilon n!}\Big)^{1/n} \cdot \widehat{x}\bigg)\bigg| = \frac{2}{\varepsilon} \cdot |f(\widehat{x})| \le \frac{2}{\varepsilon} \cdot \varepsilon = 2,$$

when $\widehat{y} = \Big(\frac{2\delta}{\varepsilon n!}\Big)^{1/n} \cdot \widehat{x}$.

The first formula implies the sought inclusion $F \in \mathbf{C}^n_{\pm,n!}[a_1,b_1]$, and the second one implies that $\widehat{y}$ belongs to the set $G_n(2,n!,F)$.

If $\widehat{x} \notin G_n(\varepsilon,\delta,f)$, i.e. $|f(\widehat{x})| > \varepsilon$, then, from the second formula we obtain $|F(\widehat{y})| > 2$, i.e. $\widehat{y} \notin G_n(2,n!,F)$.

The mapping $\widehat{y} \mapsto \Big(\frac{\varepsilon n!}{2\delta}\Big)^{1/n} \cdot \widehat{y} = \widehat{x}$ is a bijection of the set $G_n(2,n!,F)$ onto the set $G_n(\varepsilon,\delta,f)$, so equality (8) holds. □

Lemma 5. *If* $F \in \mathbf{C}^n_{\pm,n!}[a_1,b_1]$, *where* $a_1 \le b_1$, *then*

$$\operatorname{meas} G_n(2,n!,F) \le \min(4, b_1 - a_1). \tag{9}$$

Proof. Let $F[\xi_0,\xi_1,\dots,\xi_n]$ be the n–order divided difference of the function $F \in \mathbf{C}^n_{\pm,n!}[a_1,b_1]$ at points $\xi_0,\xi_1,\dots,\xi_n$, then in the case of $a_1 \le \xi_0 < \xi_1 < \dots < \xi_n \le b_1$ the following *two* equalities hold [20, c. 39–48], [21]

$$F[\xi_0,\xi_1,\dots,\xi_n] = \sum_{j=0}^{n} (-1)^{n-j} F(\xi_j) \prod_{\alpha=0,\,\alpha\ne j}^{n} |\xi_j - \xi_\alpha|^{-1}, \tag{10}$$

$$F[\xi_0,\xi_1,\dots,\xi_n] = \int_{\Delta_n} F^{(n)}\big(g(t_1,\dots,t_n)\big)\, dt_1 \cdots dt_n, \tag{11}$$

where $\Delta_n = \big\{(t_1,\dots,t_n) \in [0,1]^n \colon t_1 \le \dots \le t_n\big\}$ is the n-dimensional simplex whose measure $\operatorname{meas} \Delta_n = \int_{\Delta_n} dt_1 \cdots dt_n$ is $1/n!$,

$$g(t_1,\dots,t_n) = \xi_0 + \sum_{j=1}^{n} (\xi_j - \xi_{j-1}) t_j \equiv (1 - t_1)\xi_0 + \sum_{j=1}^{n-1} (t_j - t_{j+1})\xi_j + t_n \xi_n.$$

The function g takes values from the interval $[a_1, b_1]$ for elements of the simplex Δ_n.

If $F \in \mathbf{C}^n_{\pm,n!}[a_1, b_1]$, then, from formulas (10) and (11) and from the inequality $|F^{(n)}(x)| \geq n!$ for $x \in [a_1, b_1]$, we obtain the bilateral estimate

$$1 \leq \min_{x\in[\xi_0,\xi_n]} |F^{(n)}(x)| \cdot \int_{\Delta_n} dt_1 \cdots dt_n \leq \big|F[\xi_0, \xi_1, \dots, \xi_n]\big| \leq \varkappa \cdot \max_{j=0,1,\dots,n} |F(\xi_j)|, \tag{12}$$

where

$$\varkappa \equiv \varkappa[\xi_0, \xi_1, \dots, \xi_n] = \sum_{j=0}^{n} \prod_{\alpha=0,\,\alpha\neq j}^{n} |\xi_j - \xi_\alpha|^{-1}. \tag{13}$$

If $\xi_j \in G_n(2, n!, F)$ for $j = 0, 1, \dots, n$, then $|F(\xi_j)| \leq 2$ and the inequalities (12) imply that

$$\varkappa[\xi_0, \xi_1, \dots, \xi_n] \geq 1/2. \tag{14}$$

The set $G_n(2, n!, F)$ is a disjoint union of (at most, n) sets which are either singletons or segments $[x_j, y_j]$ of positive length $y_j - x_j$, where $j = 1, \dots, J$, $J \leq n$. Therefore, the measure θ of this set is the sum of the lengths of segments $[x_j, y_j]$, i.e.

$$\operatorname{meas} G_n(2, n!, F) \equiv \theta = \sum_{j=1}^{J} (y_j - x_j).$$

We now introduce a bijective strictly monotone function $\widetilde{\varphi}$, which maps the interval $[0, 1]$ onto set $[x_1, y_1] \cup (x_2, y_2] \cup \cdots \cup (x_J, y_J]$, namely $\widetilde{\varphi}(x) = x_1 + x\theta$, when $x \in \Big[0, \dfrac{y_1 - x_1}{\theta}\Big]$ and $\widetilde{\varphi}(x) = x_j + x\theta$, when $x \in \Big(\sum\limits_{s=1}^{j-1} \dfrac{y_s - x_s}{\theta}, \sum\limits_{s=1}^{j} \dfrac{y_s - x_s}{\theta}\Big]$.

So, the function $\widetilde{\varphi}$ takes values from the set $G_n(2, n!, F)$ and the divided difference of this function, at points η_1, η_2 from $[0, 1]$, is at least, θ, i.e. $\dfrac{\widetilde{\varphi}(\eta_2) - \widetilde{\varphi}(\eta_1)}{\eta_2 - \eta_1} \geq \theta$.

This yields the inequality $\widetilde{\varphi}(\eta_2) - \widetilde{\varphi}(\eta_1) \geq (\eta_2 - \eta_1) \cdot \theta$, if $0 \leq \eta_1 < \eta_2 \leq 1$.

We now select points (in ascending order) $\widetilde{\xi}_0, \widetilde{\xi}_1, \dots, \widetilde{\xi}_n$ from the set $G_n(2, n!, F)$ in order to apply the inequalities (13) and (14).

Since $0 = \sin^2 \dfrac{0 \cdot \pi}{2n} < \sin^2 \dfrac{1 \cdot \pi}{2n} < \sin^2 \dfrac{2\pi}{2n} < \cdots < \sin^2 \dfrac{n\pi}{2n} = 1$, the points $\widetilde{\xi}_j$ can be chosen according to the formula

$$\widetilde{\xi}_j = \widetilde{\varphi}\Big(\sin^2 \frac{j\pi}{2n}\Big), \qquad j = 0, 1, \dots, n.$$

Then, from the estimate $|\widetilde{\xi}_j - \widetilde{\xi}_\alpha| = \Big|\widetilde{\varphi}\Big(\sin^2 \dfrac{j\pi}{2n}\Big) - \widetilde{\varphi}\Big(\sin^2 \dfrac{\alpha\pi}{2n}\Big)\Big| \geq \theta \cdot \Big|\sin^2 \dfrac{j\pi}{2n} - \sin^2 \dfrac{\alpha\pi}{2n}\Big|$ and formula (13), we obtain

$$\varkappa \leq \theta^{-n} \sum_{j=0}^{n} \prod_{\alpha=0,\,\alpha\neq j}^{n} \Big|\sin^2 \frac{j\pi}{2n} - \sin^2 \frac{\alpha\pi}{2n}\Big|^{-1}.$$

From the last assessment and formula (14), excluding the variable $\varkappa$, we obtain the following inequality for the sought value θ:

$$\theta^n \leq 2 \sum_{j=0}^{n} \prod_{\alpha=0,\,\alpha\neq j}^{n} \Big|\sin^2 \frac{j\pi}{2n} - \sin^2 \frac{\alpha\pi}{2n}\Big|^{-1}. \tag{15}$$

To calculate the right–hand side of inequality (15), we shall write down a formula similar to (12) for the case $F = T_n$, where T_n is an n–th order Chebyshev polynomial

$$T_n(x) = \frac{1}{2^{n-1}} \cos(n \arccos x) = x^n + \cdots$$

on the interval $[-1, 1]$.

Since T_n is an n–th order *unital* polynomial, the n–th order divided difference equals to the one for an arbitrary set of points of the interval $[-1, 1]$.

On the other hand, this polynomial attains extreme values $(-1)^{n-j}2^{1-n}$ for $n+1$ times at the points

$$\widetilde{\eta}_j = -\cos\frac{j\pi}{n} = 2\sin^2\frac{j\pi}{2n} - 1, \qquad j = 0, 1, \ldots, n,$$

hence, the equalities $T_n(\widetilde{\eta}_j) = (-1)^{n-j}2^{1-n}$ and $|\widetilde{\eta}_j - \widetilde{\eta}_\alpha| = 2\left|\sin^2\frac{j\pi}{2n} - \sin^2\frac{\alpha\pi}{2n}\right|$ and formula (10) imply the equality:

$$1 = T_n[\widetilde{\eta}_0, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_n] = 2^{1-2n} \sum_{j=0}^{n} \prod_{\alpha=0,\, \alpha\neq j}^{n} \left|\sin^2\frac{j\pi}{2n} - \sin^2\frac{\alpha\pi}{2n}\right|^{-1},$$

from which we obtain a simple convenient formula as follows:

$$2\sum_{j=0}^{n} \prod_{\alpha=0,\, \alpha\neq j}^{n} \left|\sin^2\frac{j\pi}{2n} - \sin^2\frac{\alpha\pi}{2n}\right|^{-1} = 4^n.$$

Now equality (15) yields the estimate $\theta = \operatorname{meas} G_n(2, n!, F) \leq 4$. Since the function F is considered on the interval $[a_1, b_1]$, obviously, $\theta \leq b_1 - a_1$.

The lemma has been proved. □

Proof of Theorem 1. Consider an arbitrary function $f \in \mathbf{C}^n_{\pm,\delta}[a, b]$, then, by Lemma 4, the function F from the formula (7) belongs to the set $\mathbf{C}^n_{\pm,n!}[a_1, b_1]$ and equality (8) holds, where $a_1 = \left(\frac{2\delta}{\varepsilon n!}\right)^{1/n} \cdot a$, $b_1 = \left(\frac{2\delta}{\varepsilon n!}\right)^{1/n} \cdot b$.

From Lemma 5, we use inequality (9), which, together with equality (8), yields the inequality

$$\operatorname{meas} G_n(\varepsilon, \delta, f) \leq \sqrt[n]{\frac{\varepsilon n!}{2\delta}} \cdot \min(4, b_1 - a_1) = \min\left(b - a,\ 4\sqrt[n]{\frac{n!}{2}} \cdot \sqrt[n]{\frac{\varepsilon}{\delta}}\right).$$

From the latter formula, we obtain the sought inequality

$$\sup\{\operatorname{meas} G_n(\varepsilon, \delta, f)\colon f \in \mathbf{C}^n_{\pm,\delta}[a, b]\} \leq \min\left(b - a,\ 4\sqrt[n]{\frac{n!}{2}} \cdot \sqrt[n]{\frac{\varepsilon}{\delta}}\right).$$

To construct the functions f satisfying the equation

$$\operatorname{meas} G_n(\varepsilon, \delta, f) = \min\left(b - a,\ 4\sqrt[n]{\frac{n!}{2}} \cdot \sqrt[n]{\frac{\varepsilon}{\delta}}\right),$$

we select a function F from the set $\mathbf{C}^n_{\pm,n!}[a_1,b_1]$, using the Chebyshev polynomials T_n, namely: $F(x)=2^n\cdot T_n\Big(\dfrac{x}{2}\Big)$.

From the properties of Chebyshev polynomials we obtain that $|F(x)|\le 2$ and $F^{(n)}(x)=T_n^{(n)}\Big(\dfrac{x}{2}\Big)=n!$ on the interval $[-2,2]$, hence, extreme functions on the class $\mathbf{C}^n_{\pm,n!}[a_1,b_1]$ will be the following polynomials parameterized on the interval $\big[0,|b_1-a_1-4|\big]$ by parameter t:

$$F_t(x)=\begin{cases}2^n\cdot T_n\Big(\dfrac{x-a_1-2+t}{2}\Big), & \text{при}\ \ b_1-a_1\le 4,\\ 2^n\cdot T_n\Big(\dfrac{x-a_1-2-t}{2}\Big), & \text{при}\ \ b_1-a_1\ge 4.\end{cases}$$

Now, by Lemma 4, we obtain that upon $b-a\lesseqgtr 4\sqrt[n]{\dfrac{\varepsilon n!}{2\delta}}$, the functions

$$f_t(x)=\varepsilon 2^{n-1}\cdot T_n\Big(\frac{1}{2}\Big(\sqrt[n]{\frac{2\delta}{\varepsilon n!}}(x-a)\pm t\Big)-1\Big)$$

respectively satisfy the equality (3) for all $t\in\Big[0,\Big|\sqrt[n]{\dfrac{2\delta}{\varepsilon n!}}(b-a)-4\Big|\Big]$.

The theorem has been proved. □

Proof of Theorem 2. Let E_J be a disjoint union of J intervals $[x_1,y_1],(x_2,y_2],\dots,(x_J,y_J]$ and $F[\xi_0,\xi_1,\dots,\xi_n]$ be the divided difference of the function $f\in\mathbf{C}^n[a,b]$ at points $\xi_0,\xi_1,\dots,\xi_n$, moreover $\xi_0<\xi_1<\dots<\xi_n$ and $\xi_j\in E_J$, where $J\in\mathbb{N}$.

As in Lemma 5, we construct a bijective monotonous function $\varphi_J\colon[0,1]\to E_J$ and choose points $\widetilde{\xi}_j=\varphi_J\Big(\sin^2\dfrac{j\pi}{2n}\Big)$, $j=0,1,\dots,n$, for the divided differences. Since $\operatorname{meas}E_J=\sum\limits_{j=1}^{J}(y_j-x_j)$, the equalities (10), (11) imply the inequalities

$$\frac{1}{n!}\min_{x\in[x_1,y_J]}\big|f^{(n)}(x)\big|\le\big|F(\widetilde{\xi}_0,\widetilde{\xi}_1,\dots,\widetilde{\xi}_n)\big|\le\varkappa\cdot\sup_{x\in E_J}\big|f(x)\big|\le\frac{1}{2}\Big(\frac{4}{\operatorname{meas}E_J}\Big)^n,$$

where $\varkappa=\sum\limits_{j=0}^{n}\prod\limits_{\alpha=0,\alpha\ne j}^{n}\big|\widetilde{\xi}_j-\widetilde{\xi}_\alpha\big|^{-1}$. Hence we obtain the inequality

$$\frac{1}{n!}\min_{x\in[x_1,y_J]}\big|f^{(n)}(x)\big|\le\frac{1}{2}\Big(\frac{4}{\operatorname{meas}E_J}\Big)^n\sup_{x\in E_J}\big|f(x)\big|.$$

From the continuity of f and the possibility to approximate the measurable set E arbitrarily exactly with sets E_J by passage to the limit at $J\to\infty$ in the latter inequality, we obtain the sought inequality (6).

The theorem has been proved. □

Generalization of the statement of Theorem 2 is the inequality of *Heisenberg type* ([22])

$$\frac{\|f_j\|_{\mathbf{C}(E)}\cdot\|\widetilde{f}_n\|_{\mathbf{C}[a,b]}}{2}\ge \mathrm{C}_n^j\cdot\Big(\frac{\operatorname{meas}E}{4}\Big)^{n-j}\qquad (j\in\{0,1,\ \dots,n-1\}),$$

where $\|f\|_{\mathbf{C}(E)}=\sup\limits_{x\in E}|f(x)|$, C_n^j is a binomial number, $f_j=f^{(j)}$, $j\in\{0,1,\dots,n\}$, and $\widetilde{f}_n=1/f_n$.

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Lviv Polytechnic National University,
Lviv National University of Internal Affairs
ilkivv@i.ua

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