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BOUNDED ELEMENTARY DIVISOR DOMAINS OF STABLE RANGE 1

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It is proved that any restricted Bezout domain of stable rank 1 in which the Dubrovin condition is satisfied and where a maximal nonprincipal left ideal is an elementary divisor domain.

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Доказано, что ограниченная область Безу стабильного ранга 1, в которой выполняется условие Дубровина и произвольный максимально неглавний левый идеал является идеалом есть кольцом элементарных делителей.

A diagonal reduction of matrices is a unique property of principal domain which does not hold in finitely generated principal ideal domains. In the present paper we consider a bounded Bezout domain of stable range 1 in which Dubrovin's condition holds and where a maximal nonprincipal left ideal is an ideal. It is shown that this domain is an elementary divisor domain.

1. Preliminaries. Let R be an associative ring with $1 \neq 0$. Call some definitions and facts: a Bezout left(right) domain is a ring without zero divisors in which every finitely generated left(right) ideal is principal. It should be remarked that in general, the left(right) Bezout rings are different from the principal left(right) ideal rings with the presence of nonprincipal left(right) ideals. Throughout, R will always denote a Bezout domain, which is not a principal left and right ideals domain. The set of nonprincipal left(right) ideals is not empty and this set is induced with respect to the order of inclusion of ideals. Then according to Zorn's lemma we can consider the maximal elements among nonprincipal left(right) ideals. That is, a left ideal M is a maximal nonprincipal left ideal, if it is a nonprincipal left ideal and $M \subset J$, $M \neq J$ for a left ideal J, always implies that J is a principal left ideal (see [1]).

Definition 1. A nonzero element a from a domain R is *left (right) bounded*, if $Ra(aR)$ contains a nonzero two-side ideal. If this ideal is a principal left ideal Ra' (principal right ideal $a'R$), then the element a' is called a *left (right) limit of an element a*. An left or right limited element is bounded in a Bezout domain (see [2]).

Definition 2. A domain is said to be a *bounded left (right) domain* if each its element is its left (right) limit.

Thus we discuss a bounded domain.

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Theorem 1. *Let R be a bounded Bezout domain in which any maximal nonprincipal left ideal is an ideal. Then an arbitrary maximal nonprincipal left ideal is a maximal nonprincipal right ideal in R* (see [9]).

Definition 3. A ring R is a *ring of stable range* 1, if the condition $aR + bR = R$ for every $a, b \in R$ implies that there exists $t \in R$ such that $(a + bt)R = R$ (see [3]).

The following Dubrovin's condition is widely used in this article: for an arbitrary element $a \in R$ exists an element $a^* \in R$ such that $RaR = a^*R = Ra^*$ (see[4]).

Definition 4. An element of a ring R is called an *atom* if it is not invertible and can not be represented as a product of two non invertible elements.

Definition 5. An element $a \in R$ is a *duo-element* if $aR = Ra$.

Definition 6. Nonzero element a of a ring R without zero divisors is called *factorial* if it is invertible and can be represented as

$$a = b_1 b_2 \dots b_m,$$

where b_i are atoms, $i = 1 \dots n$ (see [10]).

Definition 7. A ring R is called an *elementary divisor ring* if any matrix A over R admits a canonical diagonal reduction, that is for any matrix A of order $n \times m$ over R there exist matrices $P \in GL_n(R)$ and $Q \in GL_m(R)$ such that

$$PAQ = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix},$$

where $GL_n(R)$ is a general linear group, moreover $Rd_{i+1}R \subseteq Rd_i \cap d_iR$, $i = 1, 2, \dots n-1$ (see [8]).

Two matrices A and B are equivalent $(A \sim B)$ as long as there are invertible matrices P and Q over R of the correspond size with $A = PBQ$.

Proposition 1. *Let R be a Bezout domain in which Dubrovin's condition holds. Thus R is an elementary divisor ring if and only if any matrix A,*

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

where $RaR + RbR + RcR = R$, admits a canonical diagonal reduction (see [5]).

2. Main result.

Proposition 2. *Let R be a bounded Bezout domain in which Dubrovin's condition holds and also the condition $RaR = R$ implies that a is a factorial element. Then any nonzero element $a \in R$ can be represented as*

$$a = a^* f = \varphi a^*,$$

where a^* is a duo-element, f and φ are factorial elements.

Proof. Since $RaR = a^*R = Ra^*$, we see $a = a^*f = \varphi a^*$, where $RfR = R\varphi R = R$ (because R is bounded domain). Taking into account restrictions of the ring R, we obtain that f and φ are factorial elements of the ring R. The statement is proved. $\square$

Theorem 2. *Let R be a restricted Bezout domain of stable range 1 in which Dubrovin's condition holds and also the condition $RaR = R$ implies that a is a factorial element. Then R is an elementary divisor ring.*

Proof. Taking into consideration Proposition 2, it is enough to show that the matrix

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

admits a canonical diagonal reduction, where $RaR + RbR + RcR = R$. Since R is a Bezout domain of stable range 1, there are some elements $a, b \in R$ and there exist elements $x, d \in R$ such that $xa + b = d$, where $Ra + Rb = Rd$. Then

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} xa + b & c \\ a & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ a & 0 \end{pmatrix},$$

where $a = a_0 d$ for some element of $a_0 \in R$.

Let $cR + dR = zR$ and $cy + d = z$ for some $y \in R$ (such elements exist according to [5]). Then

$$\begin{pmatrix} d & c \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} d + cy & c \\ a & 0 \end{pmatrix} = \begin{pmatrix} z & c \\ a & 0 \end{pmatrix},$$

where $d = zt, c = zc_0$ for some elements $t, c_0 \in R$. Thus, we proved that the matrix $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ is equivalent to the matrix $\begin{pmatrix} z & \beta \\ \alpha & 0 \end{pmatrix}$, where $\alpha = \alpha_0 zt$, $\beta = z\beta_0$. Since $RaR + RbR + RcR = R$, we obtain

$$RzR + R\alpha R + R\beta R = RaR + RbR + RcR = R$$

Moreover, since $\alpha = \alpha_0 zt$ and $\beta = z\beta_0$, we obtain $R\alpha R + RzR + R\beta R = RzR$. Therefore $RzR = R$. According to the conditions imposed the on ring R we obtain the fact that z is a factorial element. By [1] the matrix $\begin{pmatrix} z & \beta \\ \alpha & 0 \end{pmatrix}$ and consequently $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ admits a canonical diagonal reduction. The theorem is proved. $\square$

As a result of this theorem and taking into account the previous result, we obtain the following proposition.

Proposition 3. *Let R be a bounded Bezout domain in which any maximal nonprincipal left ideal is an ideal. Then in R the condition $RaR = R$ provides a as a factorial element of ring R.*

Proof. Since $RaR = R$ and ring R is bounded the element a is not contained in any maximal nonprincipal left ideal. Because R is a bounded Bezout domain in which any maximal nonprincipal left ideal is two-side, according to [9] the element a is not contained in any maximal nonprincipal right ideal. So, a is an invertible element of the ring R which is not contained in any maximal nonprincipal left and right ideal. Hence a is a factorial element in the ring R. The theorem is proved. $\square$

Because of this conclusion we obtain the following proposition:

Proposition 4. *Let R be a bounded Bezout domain of stable range 1 in which the Dubrovin condition holds and any maximal nonprincipal left ideal is an ideal. Then R is an elementary divisor ring.*

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