УДК 512.552

N. S. Khripchenko

FINITARY INCIDENCE ALGEBRAS OF QUASIORDERS

N. S. Khripchenko. *Finitary incidence algebras of quasiorders*, Mat. Stud. **34** (2010), 30–37.

We extend the notion of the finitary incidence algebra $FI(P,R)$ (see [3]) to the case when $P(\preccurlyeq)$ is an arbitrary quasiordered set and R is an associative unital ring. For each pair (P,R) we build the preadditive category $\mathcal{C}(P,R)$ with $\overline{P}=P/_{\sim}$ as the set of objects. The isomorphism theorem for the finitary algebras is proved in the following weakened form: if R and S are the indecomposable rings, then $FI(P,R)\cong FI(Q,S)$ implies $\mathcal{C}(P,Q)\cong\mathcal{C}(Q,S)$. The results by Voss (Illinois J. Math., 1980, **24**, 624–638) and the isomorphism theorem for weak incidence algebras (Int.J.Math. Math. Sci, 2004, **53**, 2835–2845) are obtained as the consequences.

Н. С. Хрипченко. *Финитарные алгебры инцидентности квазипорядков* // Мат. Студії. – 2010. – Т.34, №1. – С.30–37.

Понятие финитарной алгебры инцидентности $FI(P,R)$ (см. [3]) обобщается на случай, когда $P(\preccurlyeq)$ произвольное квазиупорядоченное множество, а R ассоциативное кольцо с единицей. Каждой паре (P,R) ставится в соответствие предаддитивная категория $\mathcal{C}(P,R)$ с множеством объектов $\overline{P}=P/_{\sim}$. Доказывается ослабленный вариант теоремы об изоморфизме для финитарных алгебр: если R и S — неразложимые кольца, то $FI(P,R)\cong FI(Q,S)$ влечет $\mathcal{C}(P,Q)\cong\mathcal{C}(Q,S)$. Как следствия, получены результаты Восса (Illinois J. Math., 1980, **24**, 624–638) и теорема об изоморфизме для слабых алгебр инцидентности (Int.J.Math. Math. Sci, 2004, **53**, 2835–2845).

Introduction. Let P be a partially ordered set (poset), R an associative ring with identity. Recall that an incidence algebra $I(P,R)$ of the locally finite poset P over the ring R is the set of the formal sums of the form

$$\alpha=\sum\nolimits_{x\le y}\alpha(x,y)[x,y], \tag{1}$$

where $\alpha(x,y)\in R$, $[x,y]=\{z\in P\colon x\le z\le y\}$ is a segment of the partial order. If P is not locally finite, then the multiplication (convolution) of any two sums is not defined in general, and $I(P,R)$ is nothing but the R-module. Therefore in [3] the notion of the finitary incidence algebra $FI(P,R)$ was defined. For an arbitrary poset this algebra consists of the so-called finitary series — the sums of the form (1), such that for any $[x,y]$ there exists only a finite number of $[u,v]\subset[x,y]$, $u<v$ with $\alpha(u,v)\ne 0$. The descriptions of the invertible elements [3, Theorem 2] and the idempotents [3, Theorem 3] of $FI(P,R)$ were obtained, as a consequence the positive solution of the isomorphism problem for the finitary incidence algebras over a fixed field F was given.

In this article we extend the notion of the finitary incidence algebra to the case when $P(\preccurlyeq)$ is a quasiordered set. It is easy to see that the definition of the finitary series given

2000 *Mathematics Subject Classification:* 16G20, 16S50, 16S60, 18B35, 18E05.

for the partial order cannot be directly extended. Indeed, there exist the series of the form $\sum_{x',x''\sim x}\alpha(x',x'')[x',x'']$, where $\sim$ is the equivalence relation on P induced by the quasiorder. All the segments $[u,v]$, where $u\prec v$, appear in such series with zero coefficients. Therefore, the condition to be finitary does not impose the restriction on such series, and so their convolution is not defined in the general case.

Let us do as follows: consider $\overline{P}=P/_\sim$ with the induced order $\leq$. Let $\overline{x}$ denote the equivalence class of an element $x\in P$. Bring the formal series

$$\widetilde{\alpha}=\sum_{\overline{x}\leq\overline{y}}\widetilde{\alpha}_{\overline{x}\overline{y}}[\overline{x},\overline{y}] \tag{2}$$

in correspondence with the sum $\alpha=\sum_{x\preccurlyeq y}\alpha(x,y)[x,y]$. Here $\widetilde{\alpha}_{\overline{x}\overline{y}}$ is the matrix over R, the rows and columns of which are indexed by the elements of the sets $\overline{x}$ and $\overline{y}$, respectively, and $\widetilde{\alpha}_{\overline{x}\overline{y}}(x',y')=\alpha(x',y')$ for the arbitrary $x'\sim x, y'\sim y$. This correspondence is one-to-one. If P is locally finite, then all matrices, which appear in the sum, are finite. It is not hard to see that $\widetilde{\alpha}+\widetilde{\beta}$ corresponds to $\alpha+\beta$, and the series $\sum_{\overline{x}\leq\overline{y}}\left(\sum_{\overline{x}\leq\overline{z}\leq\overline{y}}\widetilde{\alpha}_{\overline{x}\overline{z}}\widetilde{\beta}_{\overline{z}\overline{y}}\right)[\overline{x},\overline{y}]$ corresponds to the product $\alpha\beta$. The indicated correspondence says about the possibility to extend the notion of the finitary incidence algebra. For this purpose we introduce the concept of *a partially ordered category (pocategory)* and define the finitary incidence ring of such a category. Then we formulate the definition of the finitary incidence algebra in the language of categories. To this end we associate with each pair (P,R), where P is a quasiordered set and R is a ring with identity, a pocategory $\mathcal{C}(P,R)$. The finitary incidence ring $FI(\mathcal{C}(P,R))$ of this category is called *the finitary incidence algebra of the quasiordered set P over the ring R* and denoted by $FI(P,R)$. We prove the weakened variant of the isomorphism theorem: if R and S are the indecomposable rings, then the isomorphism $FI(P,R)\cong FI(Q,S)$ implies the isomorphism of the categories $\mathcal{C}(P,R)$ and $\mathcal{C}(Q,S)$. As the consequences, we obtain the generalizations of the results by Voss [5] proved for the locally finite sets. After that we discuss some conditions, under which the isomorphism of the categories implies the isomorphism of P and Q. As another consequence, we obtain the generalization of the isomorphism theorem for weak incidence algebras [4].

1. Pocategories and their finitary incidence rings. Recall that a category $\mathcal{C}$ is preadditive if for any $x,y\in Ob\,\mathcal{C}$ the set $Mor(x,y)$ has a structure of abelian group under addition and this structure agrees with the composition of the morphisms. Define a binary relation $\leq$ on the set of objects of the preadditive category $\mathcal{C}$ as follows:

$$x\leq y \Longleftrightarrow Mor(x,y)\neq 0_{xy}$$

and call $\mathcal{C}$ *partially ordered (pocategory)* if $\leq$ is a partial order. Consider the set of formal sums of the form

$$\alpha=\sum_{x\leq y}\alpha_{xy}[x,y], \tag{3}$$

where $\alpha_{xy}\in Mor(x,y)$, $[x,y]$ is a segment of the partial order. The formal sum (3) is called a finitary series if for any $x,y\in Ob\,\mathcal{C}$, $x<y$ there exists only a finite number of $[u,v]\subset[x,y]$, such that $u<v$ and $\alpha_{uv}\neq 0_{uv}$. The set of the finitary series is denoted by $FI(\mathcal{C})$.

The addition of the finitary series is inherited from the addition of the morphisms. Moreover, $0_{xy}[x,y]=0$ for all $x,y\in P$. The multiplication is given by the convolution:

$$\alpha\beta=\sum_{x\leq y}\left(\sum_{x\leq z\leq y}\alpha_{xz}\beta_{zy}\right)[x,y],$$

where $\alpha_{xz}\beta_{zy}\in Mor(x,y)$ is the composition of the morphisms.

The proofs of the following three theorems slightly differ from the proofs of the Theorem 1–3 from [3], but we produce them here for the account to be full.

Theorem 1. *$FI(\mathcal{C})$ is an associative ring with identity $\delta = \sum_{x\in Ob\,\mathcal{C}} \mathrm{id}_x[x,x]$ (id_x is the identity morphism).*

Proof. Obviously, it is sufficient to prove that if $\alpha,\beta \in FI(\mathcal{C})$, then $\alpha\beta \in FI(\mathcal{C})$.

Suppose that $\gamma = \alpha\beta \notin FI(\mathcal{C})$. This means that we can find such $x, y \in Ob\,\mathcal{C}$, $x \le y$ that there exists an infinite number of subsegments $[u_i, v_i] \subset [x, y]$ $(i = 1, 2, \ldots)$, for which $u_i < v_i$ and $\gamma_{u_iv_i} \ne 0_{u_iv_i}$. At least one of these sets $\{u_i\}$, $\{v_i\}$ must be infinite; for example, let $|\{u_i\}| = \infty$.

It follows from $\gamma_{u_iv_i} \ne 0_{u_iv_i}$ that for each i there is $w_i \in [u_i, v_i]$, such that $\alpha_{u_iw_i} \ne 0_{u_iw_i}$, $\beta_{w_iv_i} \ne 0_{w_iv_i}$. Since $\alpha \in FI(\mathcal{C})$ and $[u_i, w_i] \subset [x, y]$, we have $u_i = w_i$ for an infinite number of indexes. But then $\beta_{u_iv_i} \ne 0_{u_iv_i}$ for this set of indexes, which is impossible because $u_i < v_i$. □

Call $FI(\mathcal{C})$ *a finitary incidence ring of the pocategory $\mathcal{C}$*. This notion is a generalization of the notion of a finitary incidence algebra of a poset. Indeed, for a given ring R and a poset P consider the pocategory $\mathcal{C}$, such that $Ob\,\mathcal{C} = P$, and $Mor(x, y)$ is the additive group of the ring R for the arbitrary $x \le y$ (the composition of the morphisms is the product in R). Then $FI(\mathcal{C})$, obviously, coincides with the finitary algebra $FI(P, R)$.

Theorem 2. *A series $\alpha \in FI(\mathcal{C})$ is invertible iff α_{xx} is an invertible element of the ring $Mor(x, x)$ for all $x \in Ob\,\mathcal{C}$.*

Proof. Necessity. Let $\alpha\beta = \beta\alpha = \delta$. Then $\alpha_{xx}\beta_{xx} = \beta_{xx}\alpha_{xx} = \delta_{xx} = \mathrm{id}_x$, hence α_{xx} is invertible.

Sufficiency. Suppose that α_{xx} is invertible for all $x \in Ob\,\mathcal{C}$. We prove that α is right invertible, the left invertibility is proved similarly. Let $[x, y]$ be a segment of the partial order $\le$. A series β, which is right inverse to α, exists if $\sum_{x\le z\le y} \alpha_{xz}\beta_{zy} = \delta_{xy}$. In the case when $x = y$ this means that $\beta_{xx} = \alpha_{xx}^{-1}$. And if $x < y$, then

$$\beta_{xy} = -\alpha_{xx}^{-1} \sum_{x<z\le y} \alpha_{xz}\beta_{zy} \tag{4}$$

(note that the sum in the right side is defined since α_{xz} is non-zero only for a finite number of the elements $z \in [x, y]$). Prove that a solution of the equation (4) can be computed recursively in a finite number of steps and belongs to $FI(P, R)$.

Denote by $C_\alpha(x, y)$ a number of the subsegments $[u, v] \subseteq [x, y]$, such that $u < v$ and $\alpha_{uv} \ne 0_{uv}$. By the definition of the finitary series $C_\alpha(x, y)$ is finite. Build β by the induction on $C_\alpha(x, y)$.

If $C_\alpha(x, y) = 0$, then $\beta_{xy} = 0_{xy}$. If $C_\alpha(x, y) = 1$ and $x < z \le y$, $\alpha_{xz} \ne 0_{xz}$ then

$$\beta_{xy} = -\alpha_{xx}^{-1}\alpha_{xz}\beta_{zy} = \begin{cases} -\alpha_{xx}^{-1}\alpha_{xy}\beta_{yy}, & \text{if } z = y, \\ 0, & \text{if } z < y \text{ (since } C_\alpha(z, y) = 0). \end{cases}$$

And if $\alpha_{xz} = 0_{xz}$ for all $x < z \le y$, then according to (4) $\beta_{xy} = 0_{xy}$.

Now suppose that β_{uv} is defined for all u, v, such that $C_\alpha(u, v) < n$. Let $C_\alpha(x, y) = n$ and $x < z \le y$, $\alpha_{xz} \ne 0_{xz}$ (if such z doesn't exist, then $\beta_{xy} = 0_{xy}$). Then $C_\alpha(z, y) \le n - 1$, and by the induction hypothesis β_{zy} is defined. Thus every summand in the right side of (4) is defined, and since the sum is finite, β_{xy} is also defined.

Prove that β is a finitary series. We use again the induction on $C_\alpha(x,y)$. If $C_\alpha(x,y)=0$, then $\beta_{uv}=0_{uv}$ for all $[u,v]\subset[x,y]$, $u<v$ since $C_\alpha(u,v)=0$. If $C_\alpha(x,y)=1$ and $x\le u<v\le y$, $\alpha_{uv}\ne 0_{uv}$, then using the recurrence relation for β we can show that $[u,v]$ is the only non-trivial subsegment of $[x,y]$, at which β may be non-zero. Now suppose that each segment $[x,y]$ with $C_\alpha(x,y)<n$ contains a finite number of $[u,v]$, such that $u<v$ and $\beta(u,v)\ne 0_{uv}$. Consider $[x,y]$ with $C_\alpha(x,y)=n$. Suppose that there is an infinite number of the subsegments $[u_i,v_i]\subset[x,y]$, $u_i<v_i$, such that $\beta_{u_iv_i}\ne 0_{u_iv_i}$. It follows from

$$\beta_{u_iv_i}=-\alpha_{u_iu_i}^{-1}\textstyle\sum_{u_i<w\le v_i}\alpha_{u_iw}\beta_{wv_i}$$

that for each i there is w_i, $u_i<w_i\le v_i$, such that $\alpha_{u_iw_i}\ne 0_{u_iw_i}$, $\beta_{w_iv_i}\ne 0_{w_iv_i}$. Since α is a finitary series, the sets $\{u_i\}$, $\{w_i\}$ are finite. But by the supposition $[u_i,v_i]$ is infinite, therefore $\{v_i\}$ is infinite. Hence there are $z=w_{i_0}$ and an infinite number of v_i, such that $\beta_{zv_i}\ne 0_{zv_i}$. But $C_\alpha(z,y)<n$ (since $\alpha_{u_iz}\ne 0_{u_iz}$). This contradicts the induction hypothesis. □

A series $\alpha_D=\sum_{x\in Ob\,\mathcal{C}}\alpha_{xx}[x,x]$ is called *a diagonal* of an element $\alpha\in FI(\mathcal{C})$. Accordingly α is called *diagonal* if $\alpha_D=\alpha$. It is easy to show that $(\alpha\beta)_D=\alpha_D\beta_D$; in particular, $\alpha_D^2=\alpha_D$ if $\alpha^2=\alpha$.

Theorem 3. *Each idempotent $\alpha\in FI(\mathcal{C})$ is conjugate to its diagonal α_D.*

Proof. Let $\rho=\alpha-\alpha_D$. It follows from $\alpha^2=\alpha$ that

$$\alpha_D\rho+\rho\alpha_D=\rho-\rho^2. \tag{5}$$

Multiplying this equality by α_D on the left we obtain

$$\alpha_D\rho\alpha_D+\alpha_D\rho^2=0. \tag{6}$$

Set $\beta=\delta+(2\alpha_D-\delta)\rho$. Since $\beta_D=\delta_D$, β is invertible by the Theorem 2. By (6) we have $\beta\alpha=(\delta+2\alpha_D\rho-\rho)(\alpha_D+\rho)=\alpha_D-\rho\alpha_D+\rho-\rho^2$, $\alpha_D\beta=\alpha_D(\delta+2\alpha_D\rho-\rho)=\alpha_D+\alpha_D\rho$. From (5) we obtain $\beta\alpha=\alpha_D\beta$. Hence $\alpha=\beta^{-1}\alpha_D\beta$. □

2. The isomorphism problem. In this section we define the notion of a finitary incidence algebra for quasiorders in the language of categories and consider the isomorphism problem for such algebras.

Let R be a fixed ring with identity, $P(\preccurlyeq)$ an arbitrary quasiordered set. Denote by $\sim$ the natural equivalence relation on P, such that $x\sim y$ whenever $x\preccurlyeq y$ and $y\preccurlyeq x$. Let $M_{X\times Y}(R)$ denote an abelian group of infinite matrices over R, the rows and columns of which are indexed by the elements of the sets X and Y, respectively, and each row of which has a finite number of non-zero elements. Such matrices are called row-finite; it is easy to see that the product of any two row-finite matrices is again a row-finite matrix. Bring the pocategory $\mathcal{C}(P,R)$ in correspondence with the pair (P,R) in the following way

1. $Ob\,\mathcal{C}(P,R)=\overline{P}=P/_\sim$ with the induced order $\le$.
2. For any two equivalence classes $\overline{x},\overline{y}\in\overline{P},\overline{x}\le\overline{y}$ define $Mor(\overline{x},\overline{y})=M_{\overline{x}\times\overline{y}}(R)$ (and $Mor(\overline{x},\overline{y})=0_{\overline{xy}}$ whenever $\overline{x}\nleq\overline{y}$). The composition $\alpha_{\overline{xz}}\beta_{\overline{zy}}$ is the matrix product.

Denote the finitary incidence ring of the obtained category by $FI(P,R)$. Obviously, $FI(P,R)$ is an algebra over R called a *finitary incidence algebra of a quasiordered set P over a ring R.*

As in the classic case, the elements of $FI(P,R)$ can be represented as the functions on the set of the segments of P, namely, $\alpha(x,y)$ means an element of the matrix $\alpha_{\overline{xy}}$, which is situated on the intersection of the row x and the column y. Moreover, the convolution of the functions corresponds to the product of the series. If P is locally finite, then the indicated correspondence defines an isomorphism $FI(P,R) \cong I(P,R)$.

For short we write $\alpha_{\overline{x}}$, instead of $\alpha_{\overline{x}\overline{x}}[\overline{x},\overline{x}]$, and call this series *a projection of* α onto the equivalence class $\overline{x}$. Obviously, $\alpha_{\overline{x}} = \delta_{\overline{x}}\alpha\delta_{\overline{x}}$, $(\alpha\beta)_{\overline{x}} = \alpha_{\overline{x}}\beta_{\overline{x}}$.

Recall that an idempotent $\alpha \neq 0$ is *primitive* if it cannot be represented as a sum of any two non-zero orthogonal idempotents.

We need the idempotents of the special form δ_x defined for an arbitrary element $x \in P$ as follows:

$$\delta_x(u,v) = \begin{cases} 1, & \text{if } u = v = x, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

If the ring R is indecomposable, then δ_x is primitive. Note that the idempotents δ_x and δ_y are conjugates iff $x \sim y$.

The following theorem is the main result of our article.

Theorem 4. *Let P and Q be arbitrary quasiordered sets, R and S indecomposable rings with identity. Suppose that $FI(P,R) \cong FI(Q,S)$ as rings. Then $\mathcal{C}(P,R) \cong \mathcal{C}(Q,S)$.*

Proof. First define a bijection between $\overline{P} = Ob\,\mathcal{C}(P,R)$ and $\overline{Q} = Ob\,\mathcal{C}(Q,S)$. Let $\Phi\colon FI(P, R) \to FI(Q,S)$ be an isomorphism. Consider the equivalence class $\overline{x} \in \overline{P}$ and choose an arbitrary element $x' \in \overline{x}$. The image $\Phi(\delta^P_{x'})$ is a primitive idempotent, and by Theorem 3 it is conjugate to the idempotent $\Phi(\delta^P_{x'})_D$ (to avoid the confusion we write the upper index, corresponding to the set, which finitary incidence algebra our series belongs to). Since the last one is also primitive, there is $\overline{y} \in \overline{Q}$, such that $\Phi(\delta^P_{x'})_D$ coincides with $\Phi(\delta^P_{x'})_{\overline{y}}$. According to the remark before the theorem $\overline{y}$ does not depend on the choice of the representative $x' \in \overline{x}$. Hence Φ induces the mapping $\varphi\colon \overline{P} \to \overline{Q}$, such that $\Phi(\delta^P_{x'})$ is conjugate to $\Phi(\delta^P_{x'})_{\varphi(\overline{x})}$. Similarly we can define $\psi\colon \overline{Q} \to \overline{P}$. Show that they are mutually inverse. Suppose $\varphi(\overline{x}) \neq \overline{v}$. Then for each $x' \in \overline{x}$: $\delta^Q_{\overline{v}}\Phi(\delta^P_{x'})_{\varphi(\overline{x})} = 0$, i. e. $\delta^Q_{\overline{v}}\beta\Phi(\delta^P_{x'})\beta^{-1} = 0$ for some invertible element β. Hence $\Phi^{-1}(\delta^Q_{\overline{v}})\Phi^{-1}(\beta)\delta^P_{x'} = 0$. This means that $(\Phi^{-1}(\delta^Q_{\overline{v}})\Phi^{-1}(\beta))_{\overline{x}} = 0$, and thus $\psi(\overline{v}) \neq \overline{x}$. The fact that $\psi(\overline{v}) \neq \overline{x}$ implies $\varphi(\overline{x}) \neq \overline{v}$ is proved similarly. So $\psi = \varphi^{-1}$.

Consider the idempotent $\delta^P_{\overline{x}} \in FI(P,R)$ (the projection of the identity δ onto the class $\overline{x}$) and prove that

$$\Phi(\delta^P_{\overline{x}})_D = \delta^Q_{\varphi(\overline{x})}. \tag{8}$$

First suppose that there are $v', v'' \in \overline{v} \neq \varphi(\overline{x})$, such that $\delta^Q_{v'}\Phi(\delta^P_{\overline{x}})\delta^Q_{v''} \neq 0$. Then we have $\Phi^{-1}(\delta^Q_{v'})\delta^P_{\overline{x}}\Phi^{-1}(\delta^Q_{v''}) \neq 0$. But $\Phi^{-1}(\delta^Q_{v'})$ and $\Phi^{-1}(\delta^Q_{v''})$ are conjugates to $\Phi^{-1}(\delta^Q_{v'})_{\varphi^{-1}(\overline{v})}$ and $\Phi^{-1}(\delta^Q_{v''})_{\varphi^{-1}(\overline{v})}$, respectively. This means that there are the invertible elements β and γ, such that $\Phi^{-1}(\delta^Q_{v'})_{\varphi^{-1}(\overline{v})}\beta^{-1}\delta^P_{\overline{x}}\gamma\Phi^{-1}(\delta^Q_{v''})_{\varphi^{-1}(\overline{v})}$ differs from 0. The last statement contradicts the fact that $\varphi^{-1}(\overline{v}) \neq \overline{x}$. Thus $\Phi(\delta^P_{\overline{x}})_D$ coincides with its projection onto $\varphi(\overline{x})$. Similarly, $\Phi^{-1}(\delta^Q_{\varphi(\overline{x})})_D = \Phi^{-1}(\delta^Q_{\varphi(\overline{x})})_{\overline{x}}$. Therefore $\Phi^{-1}(\delta^Q_{\varphi(\overline{x})})$ is conjugate to $\delta^P_{\overline{x}}\Phi^{-1}(\delta^Q_{\varphi(\overline{x})})\delta^P_{\overline{x}}$. Consequently $\delta^Q_{\varphi(\overline{x})}$ is conjugate to $\Phi(\delta^P_{\overline{x}})\delta^Q_{\varphi(\overline{x})}\Phi(\delta^P_{\overline{x}})$. Since $\Phi(\delta^P_{\overline{x}})$ is an idempotent and the diagonal of $\delta^P_{\overline{x}}$ is stable under conjugation, we obtain that $\Phi(\delta^P_{\overline{x}})_D = \delta^Q_{\varphi(\overline{x})}$.

Now define $\varphi_{\overline{x},\overline{y}}\colon Mor(\overline{x},\overline{y}) \to Mor(\varphi(\overline{x}),\varphi(\overline{y}))$. For each class $\overline{x} \in \overline{P}$ fix an invertible element $\beta \in FI(Q,S)$, such that $\Phi(\delta^P_{\overline{x}}) = \beta\delta^Q_{\varphi(\overline{x})}\beta^{-1}$. Identify the morphism $\alpha_{\overline{x}\overline{y}}$ with the series $\alpha = \alpha_{\overline{x}\overline{y}}[\overline{x},\overline{y}]$. Then $\alpha = \delta^P_{\overline{x}}\alpha\delta^P_{\overline{y}}$. Therefore $\Phi(\alpha) = \Phi(\delta^P_{\overline{x}})\Phi(\alpha)\Phi(\delta^P_{\overline{y}})$. According to (8) and Theorem 3 we have the equality $\beta_1^{-1}\Phi(\alpha)\beta_2 = \delta^Q_{\varphi(\overline{x})}\beta_1^{-1}\Phi(\alpha)\beta_2\delta^Q_{\varphi(\overline{y})}$, where β_1, β_2 are the invertible elements, corresponding to $\overline{x}$ and $\overline{y}$. Thus $\varphi_{\overline{x},\overline{y}}(\alpha) = \beta_1^{-1}\Phi(\alpha)\beta_2$ defines the required mapping (here the morphism is also identified with the series). Since β_1 and β_2 are invertible and the multiplication is distributive, it is an isomorphism of abelian groups. Obviously, $\varphi_{\overline{x},\overline{x}}(\delta^P_{\overline{x}}) = \delta^Q_{\varphi(\overline{x})}$. Prove that $\varphi_{\overline{x},\overline{y}}$ preserves the composition. Take $\alpha' = \alpha'_{\overline{yz}}[\overline{y},\overline{z}]$. Then $\varphi_{\overline{y},\overline{z}}(\alpha') = \beta_2^{-1}\Phi(\alpha')\beta_3$ for fixed β_3. Thus

$$\varphi_{\overline{x},\overline{y}}(\alpha)\varphi_{\overline{y},\overline{z}}(\alpha') = \beta_1^{-1}\Phi(\alpha)\Phi(\alpha')\beta_3 = \beta_1^{-1}\Phi(\alpha\alpha')\beta_3 = \varphi_{\overline{x},\overline{z}}(\alpha\alpha').$$

□

The isomorphism of the categories $\mathcal{C}(P,R)$ and $\mathcal{C}(Q,S)$ implies, in particular, that the posets $\overline{P}$ and $\overline{Q}$ are isomorphic. As for the equivalence classes, we can assert that $M_{\overline{x}\times\overline{x}}(R) \cong M_{\varphi(\overline{x})\times\varphi(\overline{x})}(S)$ as rings. In the case when the initial sets P and Q are partially ordered, these conditions guarantee the positive solution of the isomorphism problem (i. e. $P \cong Q$ and $R \cong S$). This is not the case for quasiorders, and the problem is reduced to the question when the isomorphism of the above matrix rings implies $|\overline{x}| = |\varphi(\overline{x})|$. We formulate these notes in the following two corollaries, which are the generalizations of the results by Voss [5].

Corollary 1. *Let R and S be indecomposable rings with identity, P and Q arbitrary posets. Then an isomorphism $FI(P,R) \cong FI(Q,S)$ implies $P \cong Q$ and $R \cong S$.*

Corollary 2. *Let R and S be indecomposable rings with identity, such that for any sets X and Y an isomorphism $M_{X\times X}(R) \cong M_{Y\times Y}(S)$ implies $|X| = |Y|$; P and Q the quasiordered sets. Then $FI(P,R) \cong FI(Q,S) \Longrightarrow P \cong Q$.*

In particular, the isomorphism problem is solved positively for the quasiorders with finite equivalence classes (the so-called class finite quasiorders) and indecomposable commutative rings (see [2, Corollary 5.13]). If we reject the commutativity, then one can give an example of the indecomposable ring R, such that $M_{2\times 2}(R) \cong M_{3\times 3}(R)$ (see [1]). But the condition of indecomposability can be significantly weakened if we assume that $R = S$ and $FI(P,R) \cong FI(Q,R)$ as algebras. Namely, it is sufficient to require the existence of at least one primitive idempotent e in R, such that eRe is commutative. Indeed, in this case $FI(P,eRe) = eFI(P,R)e \cong eFI(Q,R)e = FI(Q,eRe)$. Note that the ring eRe is indecomposable since e is its identity. Therefore $\mathcal{C}(P,eRe) \cong \mathcal{C}(Q,eRe)$, which in the case of the finite classes implies $P \cong Q$.

3. The isomorphism problem: appendix. In [4] for a class finite quasiordered set P was introduced the notion of a weak incidence algebra $I^*(P,R)$. It consists of the functions $\alpha \in I(P,R)$, for which the set of pairs $x \prec y$ with $\alpha(x,y) \neq 0$ is finite. Obviously, $I^*(P,R)$ coincides with $I(P,R)$ only in the case when P has finite number of nontrivial segments. In the general case $I^*(P,R)$ is a subalgebra of $FI(P,R)$. Indeed, an element $\alpha \in I^*(P,R)$ can be identified with the finitary series (2), which has a finite number of non-zero coefficients $\widetilde{\alpha}_{\overline{xy}}$ corresponding to $\overline{x} < \overline{y}$. In [4] a positive solution of the isomorphism problem for weak incidence algebras of class finite quasiorders over indecomposable commutative rings with identity was obtained. In this section we introduce the notion of a weak incidence ring of a pocategory. In particular, we obtain a generalization of $I^*(P,R)$ to the case of an arbitrary quasiordered set P. We also prove that in the case of indecomposable rings the isomorphism problem for weak incidence algebras can be reduced to the isomorphism problem for finitary incidence algebras.

Let $\mathcal{C}$ be a pocategory. Denote by $I^*(\mathcal{C})$ the set of formal sums of the form (3), such that the set of $x<y$ with $\alpha_{xy}\neq 0_{xy}$ is finite. $I^*(\mathcal{C})$ is a subring of $FI(\mathcal{C})$, which is called a *weak incidence ring of a pocategory* $\mathcal{C}$. Let us obtain some properties of $I^*(\mathcal{C})$.

Lemma 1. *Let $\alpha\in I^*(\mathcal{C})$, and there exist $\alpha^{-1}\in FI(\mathcal{C})$. Then $\alpha^{-1}\in I^*(\mathcal{C})$.*

Proof. Suppose that there is an infinite number of $[x_i,y_i]$, such that $x_i<y_i$ and $(\alpha^{-1})_{x_iy_i}\neq 0_{x_iy_i}$. From the equalities $\alpha^{-1}\alpha=\alpha\alpha^{-1}=\delta$ we have

$$(\alpha^{-1})_{x_iy_i}=-\alpha^{-1}_{x_ix_i}\sum\nolimits_{x_i<z\le y_i}\alpha_{x_iz}(\alpha^{-1})_{zy_i}, \tag{9}$$

$$(\alpha^{-1})_{x_iy_i}=-\alpha^{-1}_{y_iy_i}\sum\nolimits_{x_i\le z<y_i}(\alpha^{-1})_{x_iz}\alpha_{zy_i}. \tag{10}$$

It follows from (9) that for each i there is such z_i that $\alpha_{x_iz_i}\neq 0_{x_iz_i}$. But the number of such segments is finite, therefore $|\{x_i\}|<\infty$. Similarly, $|\{y_i\}|<\infty$ by (10). A contradiction; hence $\alpha^{-1}\in I^*(\mathcal{C})$. □

Lemma 2. *$\alpha\in I^*(\mathcal{C})$ is idempotent if it is conjugate to α_D in $I^*(\mathcal{C})$.*

Proof. According to the proof of Theorem 3 α is conjugate to α_D by $\beta=\delta+(2\alpha_D-\delta)\rho$, where $\rho=\alpha-\alpha_D$. Since $\alpha_D\in I^*(\mathcal{C})$, $\beta\in I^*(\mathcal{C})$. By the previous lemma $\beta^{-1}\in I^*(\mathcal{C})$. □

Let P be an arbitrary quasiordered set, R an associative ring with the identity. Then $I^*(\mathcal{C}(P,R))$ is a subalgebra of $FI(P,R)$. Denote it by $I^*(P,R)$ and call a *weak incidence algebra of a quasiordered set P over a ring R*. If all classes of P are finite, then this algebra is isomorphic to the weak algebra considered in [4]. In the remainder of this section we study the isomorphism problem for such algebras.

Lemma 3. *Let Φ be an automorphism of $FI(P,R)$, such that $\Phi|_{I^*(P,R)}=\mathrm{id}_{I^*(P,R)}$. Then $\Phi=\mathrm{id}_{FI(P,R)}$.*

Proof. Take an arbitrary $\alpha\in FI(P,R)$. Write the obvious equality:

$$\delta_x\alpha\delta_y=\alpha(x,y)\delta_{xy}, \tag{11}$$

where δ_x is defined by (7), $\delta_{xy}(u,v)=\begin{cases}1, & \text{if } u=x, v=y,\\ 0, & \text{otherwise.}\end{cases}$
Apply Φ to the both sides of (11). As $\delta_x,\delta_y,\delta_{xy}\in I^*(P,R)$, we obtain $\delta_x\Phi(\alpha)\delta_y=\alpha(x,y)\delta_{xy}$, which is equivalent to $\Phi(\alpha)(x,y)=\alpha(x,y)$. Since x and y are the arbitrary elements, $\Phi(\alpha)=\alpha$. □

Theorem 5. *Let P and Q be the arbitrary quasiordered sets, R and S indecomposable rings with identity. Then*

$$I^*(P,R)\cong I^*(Q,S)\Longrightarrow FI(P,R)\cong FI(Q,S).$$

Proof. Let $\Phi\colon I^*(P,R)\to I^*(Q,S)$ be an isomorphism, $\alpha\in I^*(P,R)$. For the arbitrary $u,v\in Q, u\preccurlyeq v$ consider $\delta_u\Phi(\alpha)\delta_v$. Let $\varepsilon'=\Phi^{-1}(\delta_u),\varepsilon''=\Phi^{-1}(\delta_v)$. According to Lemma 2 ε' and ε'' are conjugates to ε'_D and ε''_D, respectively. Since δ_u and δ_v are primitive, there are $x,y\in P$, such that $\varepsilon'_D=\varepsilon'_{\overline{x}}$, $\varepsilon''_D=\varepsilon''_{\overline{y}}$ (as above we write for short $\varepsilon_{\overline{x}}$, instead of $\varepsilon_{\overline{x}\overline{x}}[\overline{x},\overline{x}]$). Thus $\delta_u\Phi(\alpha)\delta_v=\Phi(\beta\varepsilon'_{\overline{x}}\beta^{-1}\alpha\gamma\varepsilon''_{\overline{y}}\gamma^{-1})=\Phi(\beta\varepsilon'_{\overline{x}}\beta^{-1}\alpha_{[x,y]}\gamma\varepsilon''_{\overline{y}}\gamma^{-1})=\delta_u\Phi(\alpha_{[x,y]})\delta_v$, which is equivalent to $\Phi(\alpha)(u,v) = \Phi(\alpha_{[x,y]})(u,v)$. Here and below $\alpha_{[x,y]}$ is the part of the series α, which corresponds to the subsegments of $[x,y]$ (it is easy to see that $\varepsilon'_{\overline{x}}\beta^{-1}\alpha\gamma\varepsilon''_{\overline{y}}$ depends only on the values of α at such subsegments). Note that $[x,y] \neq \varnothing$ (i. e. $x \preccurlyeq y$) since otherwise we can take $\alpha = \Phi^{-1}(\delta_{uv})$ and obtain $\delta_{uv} = 0$.

Now let $\alpha \in FI(P,R)$, $[u,v] \subset Q$. Define

$$\overline{\Phi}(\alpha)(u,v) = \Phi(\alpha_{[x,y]})(u,v),$$

where $[x,y] \subset P$ corresponds to $[u,v]$ as indicated above. In the right part of the equality we can write $\Phi(\alpha_{[x,y]})$ since $\alpha_{[x,y]} \in I^*(P,R)$ for each $[x,y]$. Show that $\overline{\Phi}(\alpha)$ is a finitary series. Indeed, for each $[u',v'] \subset [u,v]$: $\overline{\Phi}(\alpha)(u',v') = \Phi(\alpha_{[x',y']})(u',v')$ for the corresponding $[x',y'] \subset P$. But $[x',y'] \subset [x,y]$ since, as shown above, $u \preccurlyeq v$ implies $x \preccurlyeq y$. Therefore, $\alpha_{[x',y']} = (\alpha_{[x,y]})_{[x',y']}$. Hence $\Phi(\alpha_{[x',y']})(u',v') = \Phi(\alpha_{[x,y]})(u',v')$. So

$$\overline{\Phi}(\alpha)(u',v') = \Phi(\alpha_{[x,y]})(u',v'). \tag{12}$$

Thus $\overline{\Phi}(\alpha)_{[u,v]} = \Phi(\alpha_{[x,y]})_{[u,v]}$. Since $\Phi(\alpha_{[x,y]}) \in I^*(Q,S)$, it is in particular finitary. So the values of $\overline{\Phi}(\alpha)$ inside an arbitrary segment coincide with the values of some finitary series inside this segment. Therefore, $\overline{\Phi}(\alpha)$ is finitary.

Prove that $\overline{\Phi}$ is a homomorphism. Indeed, it is not hard to see that $(\alpha\beta)_{[x,y]} = \alpha_{[x,y]}\beta_{[x,y]}$. Hence $\overline{\Phi}(\alpha\beta)(u,v) = (\Phi(\alpha_{[x,y]})\Phi(\beta_{[x,y]}))(u,v)$. By (12) and the definition of the convolution we obtain $(\Phi(\alpha_{[x,y]})\Phi(\beta_{[x,y]}))(u,v) = (\overline{\Phi}(\alpha)\overline{\Phi}(\beta))(u,v)$.

Thus we succeeded in building $\overline{\Phi}$, the extension of Φ. $\overline{\Phi^{-1}}$ is built similarly. Obviously, $(\overline{\Phi^{-1}} \circ \overline{\Phi})|_{I^*(P,R)} = \mathrm{id}_{I^*(P,R)}$, $(\overline{\Phi} \circ \overline{\Phi^{-1}})|_{I^*(Q,S)} = \mathrm{id}_{I*(Q,S)}$. By Lemma 3 $\overline{\Phi^{-1}} \circ \overline{\Phi} = \mathrm{id}_{FI(P,R)}$. Similarly, $\overline{\Phi} \circ \overline{\Phi^{-1}} = \mathrm{id}_{FI(Q,S)}$. Hence $\overline{\Phi}$ is an isomorphism. □

Corollary 3. *Let P and Q be the arbitrary quasiordered sets, R and S indecomposable rings with identity. Suppose that $I^*(P,R) \cong I^*(Q,S)$. Then $\mathcal{C}(P,R) \cong \mathcal{C}(Q,S)$.*

This result is a generalization of [4, Theorem 3.10].

REFERENCES

1. Abrams G., Haefner J. and Á. del Río, *Corrections and addenda to 'The isomorphism problem for incidence rings'*// Pacific J. Math. – 2002. – V.207, No.2 – P. 497–506.
2. Brown W.C. *Matrices over Commutative Rings*, Monographs and Textbooks in Pure and Applied Mathematics, V. 169. Marcel Dekker. New York, 1993.
3. Khripchenko N.S., Novikov B.V. *Finitary incidence algebras*// Communications in Algebra – 2009. – V.37, No.5. – P. 1670–1676.
4. Singh S., Al-Thukair F. *Weak incidence algebra and maximal ring of quotients*// Int.J.Math. Math. Sci. – 2004. – No.53. – P. 2835–2845.
5. Voss E.R. *On the isomorphism problem for incidence rings*// Illinois J. Math. – 1980. – No.24. – P. 624–638.

Department of Mechanics and Mathematics,
Kharkov V. N. Karazin National University,
4 Svobody sq, 61077, Kharkov, Ukraine
NSKhripchenko@mail.ru

Received 29.10.2009
Revised 17.06.2010