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d-MP-MODULES AND THEIR LOCALIZATIONS

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We investigate the properties of d-MP-modules, stated using the operator $(\,)_{\#}$. In particular, behavior of prime differential submodules under localizations is determined. We also study the interrelation between quasi-prime and differentially prime submodules of module over an arbitrary associative differential ring. It is established that a differential module is d-MP-module if every its differentially prime submodule is prime differential.

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Исследуются свойства d-MP-модулей, формулируемые при помощи оператора $(\,)_{\#}$, в частности, установлено поведение первичных дифференциальных подмодулей во при к локализациях. Также исследуется связь между квазипервичными и дифференциально первичными подмодулями модулей над произвольным ассоциативным дифференциальным кольцом. Установлено, что если каждый дифференциально первичный подмодуль является первичным дифференциальным, то сам модуль является d-MP-модулем.

1. Introduction. Notation and conventions. d-MP-rings were introduced by H. Gorman [1], [2] as differential rings for which the radical of every differential ideal is differential. Maximal among differential ideals of d-MP-rings are prime. d-MP-ring properties were further studied by A. Nowicki [3]. W. Keigher [4] introduced so called special differential rings and proved they are equivalent to d-MP-rings. In recent papers in differential algebraic geometry such rings are often called Keigher rings [6]. The concept of d-MP-module arose quite naturally from the concept of d-MP-ring. Many recent studies have focused on prime submodules and their analogues. However, few investigations have been focused on the differential analogues of prime submodules. The paper is devoted to the further study of properties of d-MP-modules and the interrelation between quasi-prime and differentially prime submodules.

Unless otherwise specified, all rings are assumed to be associative with nonzero identity, and all modules are unitary left modules. By an ideal we always mean a two-sided ideal. The notion *differential ring* will refer to a ring R endowed with the set $\Delta = \{\delta_1, \delta_2, \ldots, \delta_n\}$ of n pairwise commutative ring derivations $\delta_i\colon R \to R$. In what follows, M denotes a left *differential R-module*; the differential structure on M is defined by the set $D = \{d_1, d_2, \ldots, d_n\}$ of pairwise commutative module derivations $d_i\colon M \to M$, consistent with the corresponding ring derivations. Assume that at least one derivation of Δ and D is nontrivial.

For any subset X of M, $\mathrm{rad}(X)$ will denote the radical of X, i. e. the intersection of all prime submodules in M containing X.

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For $r \in R$, $m \in M$ introduce the following notations:

$$r^{(i_1,\dots,i_n)} = (\delta_1^{i_1} \circ \ldots \circ \delta_n^{i_n})(r), \qquad m^{(i_1,\dots,i_n)} = (d_1^{i_1} \circ \ldots \circ d_n^{i_n})(m),$$

$$r^{(\infty)} = \{r^{(i_1,\dots,i_n)} | i_1, i_2, \dots, i_n \in \mathbb{N} \cup \{0\}\}, \qquad m^{(\infty)} = \{m^{(i_1,\dots,i_n)} | i_1, i_2, \dots, i_n \in \mathbb{N} \cup \{0\}\}.$$

Let $[r]$ be the least differential ideal containing $r \in R$, and let $[m]$ be the least differential submodule containing $m \in M$. Note that $[r] = (r^{(\infty)})$, $[m] = (m^{(\infty)})$.

If X is an arbitrary subset of the differential R–module M, denote

$$X_{\#} = \{x \in M | \, x^{(i_1,i_2,\dots,i_n)} \in X, \quad \forall i_1, i_2, \dots, i_n \in \mathbb{N} \cup \{0\}\}.$$

The operator $()_{\#}$ has the following properties: $X_{\#}$ is differentially closed for any subset X of M; the union and the intersection of any family of differentially closed subsets is differentially closed; finite products and sums of differentially closed subsets are differentially closed; image and preimage of differentially closed subsets under differential homomorphisms are differentially closed; for any subset X of D-module M, $X_{\#}$ is the largest differentially closed subset of M contained in X.

Remind that a differential R–module M is called *differentially prime* [13] if $\mathrm{Ann}_l(N) = \mathrm{Ann}_l(M)$ for every nonzero differential submodule N of M. A differential submodule N of M is called *differentially prime* [13] if M/N is differentially prime. Obviously, differentially prime ideals, prime differential submodules, and prime differential ideals, maximal differential submodules are differentially prime. Quotient modules of differentially prime modules are differentially prime.

Let S be a *dm*–system of R. A non-empty subset $S^\star$ of the differential module M over R is called an *Sdm–system* of the module M [13] if for any $s \in S$ and $x \in X$ there exist $r \in R$ and $i_1, i_2, \dots, i_n \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $srx^{(i_1,i_2,\dots,i_n)} \in S^\star$. If all the module derivations are trivial, we obtain the notion of an *Sm*–system of a module over non-commutative ring. For a regular differential module the above concept transforms into *dm*–system, which is introduced in [10].

A differential submodule $\mathcal{N}$ of a differential module M is differentially prime if and only if $M \backslash \mathcal{N}$ is an *Sdm*–system of M for some *dm*–system S of the ring R. A differential submodule $\mathcal{P}$ of the differential module M is differentially prime if and only if $IN \subseteq \mathcal{P}$ follows $N \subseteq \mathcal{P}$ or $I \subseteq \mathrm{Ann}_l(M/\mathcal{P})$ for every differential ideal I and every differential submodule N of M. The latter condition is equivalent to $[r][m] \subseteq \mathcal{P}$ follows $m \in \mathcal{P}$ or $r \in \mathrm{Ann}_l(M/\mathcal{P})$ for any $r \in R$ and $m \in M$. (see [13])

A differential submodule $\mathcal{Q}$ of the differential R–module M is called *quasi-prime* if there exists an *Sm*–system $S^\star$ of M such that $\mathcal{Q}$ is a maximal among the differential submodules of M not meeting $S^\star$. Prime differential submodules, as well as maximal differential submodules, are quasi-prime. Nontrivial examples of quasi-prime submodules are obtained as $\mathcal{P}_{\#}$, where $\mathcal{P}$ is a prime differential submodule of M. Differentially prime and quasi-prime submodules coincide in differentially noetherian modules.

Any unexplained notation and terminology will be standard (see [7], [8]).

2. ***d-MP*-modules.** A differential R–module M is called *d-MP-module* if for any prime differential submodule N of M the submodule $N_{\#}$ is a prime differential submodules of M.

In [13] it is proved that for a *d-MP*-module M the following conditions are equivalent:

1. any quasi-prime submodule N of M is prime;
2. any quasi-prime submodule N of M is radical, i. e. $\mathrm{rad}(N) = N$;

3. any prime submodule, minimal over some differential submodule, is differential;
4. radical of each differential submodule is a differential submodule.

It is therefore easy to see that in a d-MP-module maximal among differential submodules are prime, which explains the notation d–MP.

Denote by $\mathfrak{p} = \mathrm{Ann}_l(M/\mathcal{N})$ the associated prime submodule of N.

Lemma 1. *Let M be a d-MP-module. If $\mathcal{N}$ is a $\mathfrak{p}$–prime differential submodule of M then $\mathcal{N}_\#$ is a $\mathfrak{p}_\#$–prime differential submodule of M. Moreover,*

$$\mathfrak{p}_\# = \mathrm{Ann}_l(M/\mathcal{N})_\# = \mathrm{Ann}_l(M/\mathcal{N}_\#).$$

Proof. Without loss of generality consider ordinary differential rings and modules. Let $r \in \mathrm{Ann}_l(M/\mathcal{N}_\#)$. Then $rM \subseteq \mathcal{N}_\#$. Thus, $(rm)^{(n)} \in N$ for all $m \in M$, $n \in \mathbb{N} \cup \{0\}$, i. e. $rm \in N$, $(rm)' = r'm + rm' \in \mathcal{N}$, $(rm)'' = r''m + 2r'm' + rm'' \in \mathcal{N}$, $\ldots$, $(rm)^{(n)} = \sum_{k=0}^{n} C_n^k r^{(n-k)} m^{(k)} \in \mathcal{N}$. Since M is a differential module, $rm' \in \mathcal{N}_\#$, so $rm' \in \mathcal{N}$. By induction $r^{(i)}m^{(j)} \in \mathcal{N}$ for all $i, j \in \mathbb{N} \cup \{0\}$. Then $r^{(i)}m \in \mathcal{N}$. It follows that $r^{(i)}M \subseteq \mathcal{N}$. Then $r^{(i)} \in \mathrm{Ann}_l(M/\mathcal{N})$, i. e. $r \in \mathrm{Ann}_l(M/\mathcal{N})_\#$.

Conversely, suppose $r \in \mathrm{Ann}_l(M/\mathcal{N})_\#$. Then for all $n \in \mathbb{N} \cup \{0\}$, $r^{(n)} \in \mathrm{Ann}_l(M/\mathcal{N})$, i. e. $r^{(n)}M \subseteq \mathcal{N}$. Then for all $m \in M$, $r^{(n)}m \in \mathcal{N}$, i. e. $rm \in \mathcal{N}$, $r'm \in \mathcal{N}$, $r''m \in \mathcal{N}$, and so on. Then $(rm)' = r'm + rm' \in \mathcal{N}$. By induction it is easy to prove that $(rm)^{(n)} \in \mathcal{N}$ for all $n \in \mathbb{N} \cup \{0\}$. Hence $r \in \mathrm{Ann}_l(M/\mathcal{N}_\#)$.

Since $\mathcal{N}$ is a prime submodule of M, then $\mathfrak{p} = \mathrm{Ann}_l(M/\mathcal{N})$ is a prime ideal of R, so $\mathfrak{p}_\# = \mathrm{Ann}_l(M/\mathcal{N})_\#$ is also prime. □

Remind that *Ritt algebra* is a commutative differential ring containing the field of rationals (as a subfield of the ring of constants).

Proposition 1. *A differential module over Ritt algebra is a d-MP-module.*

Proof. Let $\mathcal{N}$ be a prime differential submodule of the left differential module M over the Ritt algebra R. Suppose that $m \notin \mathcal{N}_\#$ i $r \notin \mathrm{Ann}_l(M/\mathcal{N}_\#)$. Then there exist such $n, k \in \mathbb{N} \cup \{0\}$, that $m^{(n)} \notin \mathcal{N}$, $r^{(k)} \notin \mathrm{Ann}_l(M/\mathcal{N})$ and for all $\alpha < n$, $\beta < k$, $m^{(\alpha)} \in \mathcal{N}$ and $r^{(\beta)} \in \mathrm{Ann}_l(M/\mathcal{N})$. Since $\mathcal{N}$ is a prime submodule, then $r^{(k)}m^{(n)} \notin \mathcal{N}$. Then by Leibnitz rule,

$$(rm)^{(k+n)} = \sum_{i=0}^{k+n} C_{k+n}^{i} r^{(k+n-i)} m^{(i)} = \sum_{\alpha+\beta=k+n, \alpha\neq k, \beta\neq n} \mathbf{C} r^{(\alpha)} m^{(\beta)} + C_{k+n}^{n} r^{(k)} m^{(n)},$$

where $\mathbf{C} \in \mathbb{Q}$.

The first summand lies in $\mathcal{N}$ by assumption. The second summand $C_{k+n}^{n} r^{(k)} m^{(n)} \notin \mathcal{N}$, because $m^{(n)} \notin \mathcal{N}$ and $C_{k+n}^{n} r^{(k)} \notin \mathrm{Ann}_l(M/\mathcal{N})$. Then $(rm)^{(k+n)} \notin \mathcal{N}$, i. e. $rm \notin \mathcal{N}_\#$. Therefore $\mathcal{N}_\#$ is a prime submodule of M. □

Any Ritt algebra is a d-MP-module over the field of rational numbers. Every differentially simple module is a d-MP-module. Differential vector space over an arbitrary field is a d-MP-module. Any differential module with trivial derivations is a d-MP-module, since $N_\# = N$ holds for each submodule.

Proposition 2. *If M is a d-MP-module and N is a differential submodule of M then M/N is a d-MP-module.*

Proof. The statement follows easily from the structure of prime submodules of the factor module M/N and the definition of d-MP-module. □

Proposition 3. *If N is a radical submodule of d-MP-module M then $N_{\#}$ is a radical submodule of M.*

Proof. Since the radical submodule coincides with the intersection of all prime submodules which contain it, and the operator $()_{\#}$ preserves intersections, $()_{\#}$ also preserves radical submodules. □

Proposition 4. *If M is a d-MP-module and $f\colon M \to \widetilde{M}$ is a differential module epimorphism then $\widetilde{M}$ is a d-MP-module.*

Proof. Denote $L = \operatorname{Ker}(f)$. The epimorphism $f\colon M \to \widetilde{M}$ induces a bijective correspondence

$$\overline{f}\colon \{\mathcal{K} \in \operatorname{Spec}(M)\,|\operatorname{Ker} f = L \subseteq \mathcal{K}\} \to \left\{\mathcal{N} \in \operatorname{Spec}(\widetilde{M})\right\}$$

between the prime submodules $\mathcal{K}$ of the module M containing the kernel of the homomorphism $L = \operatorname{Ker}(f)$ and prime submodules $\mathcal{N}$ of $\widetilde{M}$. Then $\overline{f} = f\,\big|_{\{\mathcal{K}\in\operatorname{Spec}(M)|L\subseteq\mathcal{K}\}}$ is a bijective correspondence. Since $(0) \subseteq \mathcal{N}$, the preimage of zero must be contained in $\mathcal{K} = \overline{f}^{-1}(\mathcal{N})$, i. e. $L \subseteq \mathcal{K}$. Then $\mathcal{N} \in \operatorname{Spec}(\widetilde{M})$, and it follows from the properties of $()_{\#}$ that

$$\mathcal{N}_{\#} = \overline{f}\left(\overline{f}^{-1}(\mathcal{N}_{\#})\right) = \overline{f}\left(\left(\overline{f}^{-1}(\mathcal{N})\right)_{\#}\right).$$

The preimage $\overline{f}^{-1}(\mathcal{N})$ of prime differential submodule is a prime differential submodule of $\widetilde{M}$, and so $\left(\overline{f}^{-1}(\mathcal{N})\right)_{\#}$ is a prime differential submodule of M. Then $\mathcal{K}_{\#}$ is also a prime differential submodule of $\widetilde{M}$. □

Proposition 5. *Let $M_1, \ldots, M_n$ be an arbitrary finite family of differential modules and let $M = M_1 \times \cdots \times M_n$ be their product. Then M is a d-MP-module if and only if every M_i is a d-MP-module.*

Proof. Let M be a d-MP-module over R. Consider canonical projections $\pi_i\colon M \to M_i$. By Proposition 4, all M_i are d-MP-modules.

Conversely, suppose that $\mathcal{N}$ is a prime submodule of M, all M_i are d-MP-modules, let also $\pi_i\colon M \to M_i$ be canonical projections for all $i = 1, 2, \ldots, n$. Then $\pi_k(\mathcal{N}) = \mathcal{N}_k$ is a prime submodule of M_k for some k, $1 \leqslant k \leqslant n$ i $\pi_j(\mathcal{N}) = M_j$ for $j \neq k$. It follows that $\mathcal{N} = \pi_k^{-1}(\mathcal{N}_k)$, therefore

$$\mathcal{N}_{\#} = (\pi_k^{-1}(\mathcal{N}_k))_{\#} = \pi_k^{-1}((\mathcal{N}_k)_{\#})$$

is a prime submodule as a preimage of the prime submodule. Therefore, each M_k is a d-MP-module. □

3. Localizations of d-MP-modules. It is known that the construction of the ring of fractions of a commutative ring by a multiplicatively closed subset can be extended to the differential case simply by defining derivations on the ring of fractions $S^{-1}R$ in terms of derivations on R. The same is true for modules.

Let R be a differential ring with the derivation $\delta\colon R \to R$, M is a left differential R-module with the derivation $d\colon M \to M$, S is a multiplicatively closed subset of R, $s \in S$, $r \in R$,

$m \in M$. Then $S^{-1}R$–module $S^{-1}M$ is differential [2] with the derivations $\overline{\delta}\colon S^{-1}R \to S^{-1}R$, $\overline{d}\colon S^{-1}M \to S^{-1}M$, defined by the rules:

$$\overline{\delta}(r/s) = (\delta(r)s - r\delta(s))/s^2,$$

$$\overline{d}(m/s) = (d(m)s - m\delta(s))/s^2.$$

Lemma 2. *Let R be a differential ring, S be a multiplicatively closed subset of R, M be a left differential R–module, $S^\star$ be a S–multiplicatively closed subset of M, and $\mathcal{N}$ be a prime submodule of M, maximal in $M \backslash S^\star$. Then in a differential module of fractions, the following equality for prime submodules holds*

$$\left(S^{-1}\mathcal{N}\right)_\# = S^{-1}\mathcal{N}_\#.$$

Proof. Without loss of generality prove the equality for ordinary differential rings. Let $m/s \in S^{-1}\mathcal{N}_\#$. Prove that $(m/s)^{(n)} \in S^{-1}\mathcal{N}$ by induction. For $n = 0$ we have $m/s \in S^{-1}\mathcal{N}$. Suppose that $(m/s)^{(k)} \in S^{-1}\mathcal{N}$ for $k < n$, where $n \geqslant 1$. It is easy to check that $(m/1)^{(n)} = m^{(n)}/1$ for all $n \in \mathbb{N} \cup \{0\}$ and $m \in M$. Since $(m/s) \cdot (s/1) = m/1$ in $S^{-1}M$, we have $(m/1)^{(n)} = \sum_{k=0}^{n-1} C_n^k (m/s)^{(k)} \cdot (s/1)^{(n-k)} + (m/s)^{(n)} \cdot s/1$. Hence

$$(m/s)^{(n)} = \Big(m^{(n)}/1 - \sum_{k=0}^{n-1} C_n^k (m/s)^{(k)} \cdot (s^{(n-k)}/1)\Big) \cdot (1/s).$$

Then $(m/s)^{(n)} \in S^{-1}\mathcal{N}$. Therefore $S^{-1}\mathcal{N}_\# \subseteq (S^{-1}\mathcal{N})_\#$.

On the other hand, let $m/s \in (S^{-1}\mathcal{N})_\#$. It follows from the equality

$$m^{(n)}/1 = \sum_{k=0}^{n-1} C_n^k (m/s)^{(k)} \cdot (s/1)^{(n-k)} + (m/s)^{(n)} \cdot s/1,$$

that $m^{(n)}/1 \in S^{-1}\mathcal{N}$. Since $\mathcal{N}$ is a prime submodule of M, maximal among submodules not meeting $S^\star$, $S^{-1}\mathcal{N}$ is a prime submodule of the $S^{-1}R$-module $S^{-1}M$ [11]. Therefore $m/s \in S^{-1}\mathcal{N}_\#$. □

Theorem 1. *Let M be a d-MP-module over R, S is a multiplicatively closed subset of R, $S^\star$ is a S–multiplicatively closed subset of the module M. The differential module of fractions $S^{-1}M$ is a d-MP-module.*

Proof. Let $\mathcal{N}$ be a prime differential submodule of the differential module M, not meeting $S^\star$. Then $\mathcal{N}_\#$ is also a prime submodule of M and $\mathcal{N}_\#$ does not meet $S^\star$. Then $\mathcal{N} \cap S^\star = \varnothing$ and $\mathcal{N}_\# \cap S^\star = \varnothing$ implies that $\mathrm{Ann}_l(M/\mathcal{N}) \cap S = \varnothing$ and $\mathrm{Ann}_l(M/\mathcal{N}_\#) \cap S = \varnothing$ [11], and $S^{-1}\mathcal{N}$ is a $S^{-1}\mathfrak{p}$–prime submodule of the module of fractions $S^{-1}M$, $S^{-1}\mathcal{N}_\#$ is a $S^{-1}\mathfrak{p}_\#$–prime submodule of the module of fractions $S^{-1}M$, where $\mathfrak{p} = \mathrm{Ann}_l(M/\mathcal{N})$ [12]. By Lemma 2, $(S^{-1}\mathcal{N})_\# = S^{-1}\mathcal{N}_\#$, so $(S^{-1}\mathcal{N})_\#$ is a prime submodule of $S^{-1}M$, i. e. $S^{-1}M$ is a d-MP-module. □

Corollary 1. *A differential R–module M is a d-MP-module if and only of its localization $M_\mathfrak{p}$ is a d-MP-module for every prime differential ideal $\mathfrak{p}$ of the ring R.*

Proof. If M is a d-MP-module, then by Proposition 1, each localization $R_{\mathfrak{p}}$ is a d-MP-module.

Conversely, let $\mathfrak{p}$ be a prime ideal of the differential ring R, $S = R\backslash\mathfrak{p}$, $\mathcal{N}$ be a prime submodule of the differential module M such that $\mathcal{N}_{\mathfrak{p}} \neq M_{\mathfrak{p}}$, and let $f\colon M \to M_{\mathfrak{p}}$ be a canonical differential module homomorphism, $f\colon m \mapsto m/1$. It is known that if $\mathcal{N}$ is a prime submodule of R–module M, $\mathcal{N}_{\mathfrak{p}} \neq M_{\mathfrak{p}}$, then $f^{-1}(\mathcal{N}_{\mathfrak{p}}) = \mathcal{N}$, besides, under the condition that $\mathcal{N}_{\mathfrak{p}} \neq M_{\mathfrak{p}}$, $\mathcal{N}_{\mathfrak{p}}$ is a prime submodule. Then

$$\mathcal{N}_{\#} = \left(f^{-1}\left(\mathcal{N}_{\mathfrak{p}}\right)\right)_{\#} = f^{-1}\left(\left(\mathcal{N}_{\mathfrak{p}}\right)_{\#}\right).$$

Since $M_{\mathfrak{p}}$ is a d-MP-module over $R_{\mathfrak{p}}$, the ideal $\left(\mathcal{N}_{\mathfrak{p}}\right)_{\#}$ is prime in $M_{\mathfrak{p}}$, because its preimage $f^{-1}\left(\left(\mathcal{N}_{\mathfrak{p}}\right)_{\#}\right)$ is a prime submodule of M. The equality implies that $\mathcal{N}_{\#}$ is a prime submodule of M, therefore M is a d-MP-module over $R_{\mathfrak{p}}$. □

4. Differentially prime and quasi-prime submodules. The following theorem gives the interrelation between differentially prime and quasi-prime submodules.

Theorem 2. *If $\mathcal{P}$ is a quasi-prime differential submodule of the differential R–module M, then $\mathcal{P}$ is a differentially prime submodule of M.*

Proof. Let $\mathcal{P}$ be a quasi-prime submodule of the differential module M. Suppose that it is not differentially prime. Than there exist $r \in R$, $m \in M$ such that $[r] \cdot [m] \subseteq \mathcal{P}$ and $r \notin \mathrm{Ann}_l(M/\mathcal{P}) = \mathfrak{p}$, $m \notin \mathcal{P}$, i.e. $r \in R\backslash\mathfrak{p}$, $m \in M\backslash\mathcal{P}$ and $[r] \cdot [m] \subseteq \mathcal{P}$. It is clear that $\mathcal{P} \subset \mathcal{P} + [m]$ and $\mathfrak{p} \subset \mathfrak{p} + [r]$ and $(\mathcal{P}\colon M) + [r]$ is a differential ideal of R, $\mathcal{P} + [m]$ is a differential submodule of M. Since $\mathcal{P}$ is maximal among the differential submodule not meeting some Sm-system $S^{\star}$, $(\mathfrak{p} + [r]) \cap S \neq \varnothing$ and $(\mathcal{P} + [m]) \cap S^{\star} \neq \varnothing$. Therefore there exist $s \in S$, $x \in S^{\star}$ such that $s \in \mathfrak{p} + [r]$ and $x \in \mathcal{P} + [m]$. On the other hand, $S^{\star}$ is a Sm-system and $s \in S$, $x \in S^{\star}$ follows the existence of the element $a \in R$ such that $sax \in S^{\star}$. Then

$$sax \in (\mathfrak{p} + [r]) \cdot (\mathcal{P} + [m]) = \mathcal{P} + [r] \cdot [m] \subseteq \mathcal{P}.$$

So $sax \in S^{\star} \cap \mathcal{P} \neq \varnothing$, but it contradicts the assumption that $S^{\star} \cap \mathcal{P} = \varnothing$. Hence $\mathcal{P}$ is a differentially prime submodule. □

Corollary 2. *If every differentially prime submodule of the differential module M is prime, then the module M is a d-MP-module.*

Proof. Since every differentially prime submodule is prime differential submodule, every quasi-prime submodule is prime. It is equivalent to the fact that the submodule is d–MP. □

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