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# STOCHASTIC OPTIMIZATION PROCEDURE CONVERGENCE WITH MARKOV SWITCHING IN THE AVERAGING SCHEME

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We established sufficient conditions for the convergence of the multi-dimensional continuous stochastic optimization procedure in the case of direct dependence of the regression function on the environment, which described by Markov switching. Additional conditions on Lyapunov function of the averaged pure gradient system have been aquired in the assumption of exponential stability for the averaged evolutionary system according to the Markov process stationary distribution.

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Установлены достаточные условия сходимости многомерной непрерывной процедуры стохастической оптимизации в случае непосредственной зависимости функции регрессии от влияния внешней среды, которая описывается марковскими переключениями. При экспоненциальной устойчивости усредненной по стационарному распределению марковского процесса эволюционной системы получено дополнительные условия на функцию Ляпунова усреднённой чисто градиентной системы.

**Introduction.** The investigations of Robbins and Monroe are dedicated to stochastic approximation procedure for solving the regression equation. Kiefer and Wolfowitz [1] have proposed the stochastic approximation procedure to find the maximum point of the regression function when the explicit form of the function is previously unknown in assuming that extremum point is unique. In [2] have been proved the convergence of procedures in continuous and discrete cases by imposing conditions on the regression function and by using the properties of functions such as Lyapunov functions.

In this article it has been investigated the continuous stochastic optimization procedure of the maximum point $u_0 \in R^d$ search of the regression function $C(u, x)$ which directly depends on the external environment with Markov switching [3]. The convergence of the proposed procedure has been proved by using the coupled Markov process generator and its asymptotic representation of perturbed Lyapunov function.

**1. Problem definition and designation.** Let $C(u, x), u \in R^d$, — a regression function, which reaches a maximum in the point $u_0, u_0 \in R^d$. The second component $x$ of the regression function characteristic describes the influence of external factors that are described by evenly

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ergodic Markov processes $x(t), t \geq 0$, in the dimensional phase states space $(X, \mathbf{X})$. The Markov process generator is defined by the relation:

$$Q\varphi(x) = q(x) \int\limits_X P(x, dy) \left[\varphi(y) - \varphi(x)\right],$$

on the characteristic Banach space $\mathbf{B}(X)$ of real bounded limited continuous functions $\varphi(x), x \in X$, with the norm

$$\|\varphi(x)\| = \sup_{x \in X} |\varphi(x)|,$$

where $P(x, B), x \in X, B \in \mathbf{X}$ — the stochastic kernel [4], $q(x) = g^{-1}(x)$, $g(x) = E\Theta_x$, $\Theta_x$ — the sojourn time of Markov process in state $x$.

The stationary distribution $\pi(B), B \in \mathbf{X}$ of Markov process $x(t), t \geq 0$ is determined by the relations

$$\pi(dx)q(x) = q\rho(dx), \;\; q = \int\limits_X \pi(dx)q(x),$$

where $\rho(dx)$ — stationary distribution of the embedded Markov chain $x(n), n \geq 0$. For the generator $Q$ of the Markov process $x(t), t \geq 0$, the potential is determined by the relation $R_0 = \Pi - (\Pi + Q)^{-1}$, where $\Pi\varphi(x) = \int_X \pi(dx)\varphi(x)$ is the projection on the zeros subspace of the operator $Q$: $N_Q = \{\varphi : Q\varphi = 0\}$ [3].

The continuous stochastic optimization procedure of the regression functions $C(u, x)$ is set in ergodic Markov environment by the stochastic differential equation:

$$du^\varepsilon(t) = a(t)\nabla_b C(u^\varepsilon(t), x(t/\varepsilon))dt, \tag{1}$$

where $\nabla_b C(u, x) = \left\{\dfrac{C(u_i^+, x) - C(u_i^-, x)}{2b(t)}, i = \overline{1, d}\right\}$, $u^\pm = u_i \pm b(t)e_i, i = \overline{1, d}$, $e_i = (0, \ldots, 1, 0, \ldots, 0)$

For the average regression function

$$C(u) = \int\limits_X \pi(dx)C(u, x)$$

consider the gradient evolutionary system

$$\frac{du}{dt} = \operatorname{grad} C(u), \operatorname{grad} C(u) = \left\{\frac{\partial C(u)}{\partial u_i}, i = \overline{1, d}\right\}. \tag{2}$$

## 2. The convergence of the stochastic optimization procedure.

**Theorem 1.** *Let the Lyapunov function $V(u), u \in R^d$ for the averaged system (2) satisfy the conditions*

*C1: (the exponential stability of the evolution system (2))* $C'(u)V'(u) \leq -c_0 V(u), c_0 > 0$;

*C2:* $|V'(u)| \leq c_1(1 + V(u)), c_1 > 0$;

*C3:* $|\nabla_b C(u, x)R_0[\nabla_b \widetilde{C}(u, x)V'(u)]'| \leq c_2(1 + V(u)), c_2 > 0$, $\widetilde{C}(u, x) := C(u, x) - C(u)$.

*The regression function $C(u,x)$ according to evolution $u$ satisfies the global Lipschitz condition*

*C4:* $|\nabla_b C(u) - C'(u)| \leq c_3 b(t), c_3 > 0.$

*The functions $a(t), b(t), t \geq 0$ satisfy the convergence conditions of the stochastic optimization procedure:*

*C5:* $\int\limits_0^{+\infty} a(t)dt = \infty,\ \int\limits_0^{+\infty} a(t)b(t)dt < \infty,\ a(t) > 0,\ b(t) > 0.$

*Then for each initial value $u^\varepsilon(0) \in R^d$ the stochastic optimization procedure (1) at arbitrary $\varepsilon \leq \varepsilon_0$, $\varepsilon_0$ – sufficiently small, coincides with probability 1 with the maximum point $u_0$.*

**Theorem 2.** *Let the Lyapunov function $V(u), u \in R^d$ for the averaged system (2) satisfy the conditions*

*C1: (the exponential stability of the evolution system (2))* $C'(u)V'(u) \leq -c_0 V(u), c_0 > 0;$

*C2:* $|V'(u)| \leq c_1(1 + V(u)), c_1 > 0;$

*C3:* $|\nabla_b C(u,x) R_0 [\nabla_b \widetilde{C}(u,x) V'(u)]'| \leq c_2(1 + V(u)), c_2 > 0\ , \widetilde{C}(u,x) := C(u,x) - C(u).$

*The regression function $C(u,x)$ according to evolution $u$ satisfies the global Lipschitz condition*

*C4:* $|\nabla_b C(u) - C'(u)| \leq c_3 b^2(t), c_3 > 0.$

*The functions $a(t), b(t), t \geq 0$ satisfy the conditions of the convergence of the stochastic optimization procedure*

*C5:* $\int\limits_0^{+\infty} a(t)dt = \infty,\ \int\limits_0^{+\infty} a(t)b^2(t)dt < \infty, a(t) > 0,\ b(t) > 0.$

*Then for each initial value $u^\varepsilon(0) \in R^d$ the stochastic optimization procedure (1) at arbitrary $\varepsilon \leq \varepsilon_0$, $\varepsilon_0$ — sufficiently small, coincides with probability 1 with the maximum point $u_0$.*

Firstly, for proving Theorem 1, we construct the stochastic optimization procedure generator and exact its asymptotic representation.

## 3. The properties of the procedure generator.

**Lemma 1.** *The coupled Markov process generator*

$$u^\varepsilon(t), x_t^\varepsilon = x(t/\varepsilon), t \geq 0 \tag{3}$$

*on the Banach space $\boldsymbol{B}(R^d, X)$ real-valued functions $\varphi(u,x) \in C^{2,0}(R^d, X)$ a presented*

$$\boldsymbol{L}_t^\varepsilon \varphi(u,x) = \varepsilon^{-1} Q\varphi(u,x) + a(t)\nabla_b \boldsymbol{C}(u)\varphi(u,x),$$

*where* $\nabla_b \boldsymbol{C}(u)\varphi(u,x) = \nabla_b C(u,x)\varphi'_u(u,x).$

*Proof.* Let exact the conditional expectation of test functions $\varphi(u,x)\in C^{2,0}(R^d,X)$

$$E[\varphi(u^\varepsilon(t+\Delta),x^\varepsilon_{t+\Delta})\,|u^\varepsilon(t)=u,x^\varepsilon_t=x]=E_{u,x}\varphi(u^\varepsilon(t+\Delta),x^\varepsilon_{t+\Delta})=$$
$$=E_{u,x}\varphi(u+\int_t^{t+\Delta}a(s)\nabla_b C(u^\varepsilon(s),x(s/\varepsilon))ds,x)I(\Theta>\varepsilon^{-1}\Delta)+$$
$$+E_{u,x}\varphi(u+\int_t^{t+\Delta}a(s)\nabla_b C(u^\varepsilon(s),x(s/\varepsilon))ds,x_{t+\Delta})I(\Theta\le\varepsilon^{-1}\Delta)+o(\Delta).$$

The distribution function of the sojourn time $\Theta_x$ in the state $x$ has the exponential distribution, i.e. there are representations

$$I(\Theta_t>\varepsilon^{-1}\Delta)=e^{-\varepsilon^{-1}q(x)\Delta}=1-\varepsilon^{-1}q(x)\Delta+o(\Delta),$$

and $I(\Theta_t\le\varepsilon^{-1}\Delta)=1-e^{-\varepsilon^{-1}q(x)\Delta}=\varepsilon^{-1}q(x)\Delta+o(\Delta)$, where $q(x)$ — intensity [4].

According to the Taylor formula, for the test function $\varphi(u,x)\in C^{2,0}(R^d,X)$ we get

$$\varphi(u+\int_t^{t+\Delta}a(s)\nabla_b C(u^\varepsilon(s),x(s/\varepsilon))ds,x)=\varphi(u,x)+$$
$$+\varphi'_u(u,x)\int_t^{t+\Delta}a(s)\nabla_b C(u^\varepsilon(s),x(s/\varepsilon))ds=\varphi(u,x)+\nabla_b C(u,x)\varphi'_u(u,x)a(t)\Delta+o(\Delta).$$

Thus,

$$E_{u,x}[\varphi(u,x)+a(t)\nabla_b C(u,x)\varphi'_u(u,x)\Delta][1-\varepsilon^{-1}q(x)\Delta+o(\Delta)]+$$
$$+E_{u,x}[\varphi(u,x^\varepsilon_{t+\Delta})+a(t)\nabla_b C(u,x)\varphi'_u(u,x^\varepsilon_{t+\Delta})\Delta][\varepsilon^{-1}q(x)\Delta+o(\Delta)]=$$
$$=\varphi(u,x)+a(t)\nabla_b C(u,x)\varphi'_u(u,x)\Delta-\varepsilon^{-1}q(x)E_{u,x}\varphi(u,x)\Delta-$$
$$-\varepsilon^{-1}q(x)(a(t)E_{u,x}\nabla_b C(u,x)\varphi'_u(u,x)\Delta^2)+o(\Delta)+\varepsilon^{-1}q(x)E_{u,x}\varphi(u,x^\varepsilon_{t+\Delta})\Delta+o(\Delta)=$$
$$=\varphi(u,x)+a(t)\nabla_b C(u,x)\varphi'_u(u,x)\Delta+\varepsilon^{-1}q(x)\Delta\int_X P(x,dy)[\varphi(u,y)-\varphi(u,x)]+o(\Delta). \quad (4)$$

From (4) we obtain the Markov process (3)

$$\boldsymbol{L}^\varepsilon_t\varphi(u,x)=\lim_{\Delta\to 0}\frac{1}{\Delta}E_{u,x}(\varphi(u^\varepsilon(t+\Delta),x^\varepsilon_{t+\Delta})-\varphi(u,x))=$$
$$=\varepsilon^{-1}Q\varphi(u,x)+a(t)\nabla_b C(u,x)\varphi'_u(u,x).$$

□

Let the perturbed Lyapunov function have the form

$$V^\varepsilon(u,x)=V(u)+\varepsilon a(t)V_1(u,x), \quad (5)$$

where $V(u)\in C^3(R^d)$ — the Lyapunov function of the averaged system (2).

**Lemma 2.** *Let the generator $\boldsymbol{L}^\varepsilon_t$ is on the perturbed Lyapunov function $V^\varepsilon(u,x)$, then*

$$\boldsymbol{L}^\varepsilon_t V^\varepsilon(u,x)=a(t)\boldsymbol{L}V(u)+\varepsilon a^2(t)\Theta_t(x)V(u),$$

*where the marginal generator is $\boldsymbol{L}V(u)=\nabla_b C(u)V'(u)$, and the remainder*

$$\Theta_t(x)V(u)=\nabla_b C(u,x)R_0[\nabla_b\widetilde{C}(u,x)V'(u)]'.$$

*Proof.* The generator $\boldsymbol{L}_t^\varepsilon$ on the perturbed Lyapunov function (5) is represented as

$$\boldsymbol{L}_t^\varepsilon V^\varepsilon(u,x) = \big[\varepsilon^{-1}Q + a(t)\nabla_b \boldsymbol{C}(x)\big]\big[V(u) + \varepsilon a(t)V_1(u,x)\big] = \tag{6}$$

$$= \varepsilon^{-1}QV(u) + a(t)QV_1(u,x) + a(t)\nabla_b \boldsymbol{C}(x)V(u) + \varepsilon a^2(t)\nabla_b \boldsymbol{C}(x)V_1(u,x).$$

Considering the solvability conditions $QV(u) = 0$ of the singular perturbation problem (6), we get

$$a(t)QV_1(u,x) + a(t)\nabla_b \boldsymbol{C}(x)V(u) = a(t)\Big[QV_1(u,x) + \nabla_b \boldsymbol{C}(x)V(u)\Big] \equiv a(t)\boldsymbol{L}V(u),$$

where the boundary operator $\boldsymbol{L}$ is represented as $\boldsymbol{L}V(u) = \nabla_b C(u)V'(u)$.

Thus, for the perturbed Lyapunov function $V_1$ we get [4] $V_1 = R_0\nabla_b\widetilde{C}(u,x)V'(u)$. Consequently,

$$\varepsilon a^2(t)\nabla_b \mathbf{C}(x)V_1(u,x) = \varepsilon a^2(t)\nabla_b \mathbf{C}(x)\big[R_0\nabla_b\widetilde{C}(x)V(u)\big] =$$

$$= \varepsilon a^2(t)\nabla_b C(u,x)R_0\big[\nabla_b\widetilde{C}(u,x)V'(u)\big]'$$

So, the remainder has the form $\Theta_t(x)V(u) = \nabla_b C(u,x)R_0\big[\nabla_b\widetilde{C}(u,x)V'(u)\big]'$.

Thus, $\boldsymbol{L}_t^\varepsilon V^\varepsilon(u,x) = a(t)\boldsymbol{L}V(u) + \varepsilon a^2(t)\Theta_t(x)V(u)$. □

**4. Proof of the theorems.** From the theorem condition C1 we get the estimation

$$\nabla_b C(u)V'(u) \le -c_0 V(u) - [C'(u) - \nabla_b C(u)]V'(u),$$

and from the conditions C2-C5 we get the estimates of the remainder $\Theta_t(x)V(u)$ as

$$|\Theta_t(x)V(u)| \le c(1 + V(u)).$$

Then

$$\boldsymbol{L}_t^\varepsilon V^\varepsilon(u,x) \le -a(t)c_0V(u) + (\varepsilon a^2(t)c_4 - c_5 a(t)b(t))(1 + V(u)). \tag{7}$$

From (7) and the theorem about the convergence of Nevelson-Khasminsky stochastic approximation procedure ([2], c.100) we receive the statement of Theorem 1.

The proof of Theorem 2 is realized according to the proof scheme of Theorem 1 taking into account conditions $C4$ and $C5$ of Theorem 2.

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