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FUZZY HYPERSPACE MONAD

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The hyperspace of a fuzzy metric space is defined by J. Rodríguez-López and S. Romaguera. In this paper, it is shown that the hyperspace construction determines a functor on the category of fuzzy metric spaces and nonexpanding maps.

We also prove that this functor determines a monad on this category and that the G-symmetric power functor can be extended over the Kleisli category of this monad.

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Доказано, что гиперпространство нечеткого метрического пространства, определеное Родрігесом-Лопесом и Ромагуэрой определяет функтор на категории нечетких метрических пространств и их нерастягивающих отображений. Этот функтор дополняется до монады, на категорию Клейсли которой продолжается функтор G-симметрической степени.

1. Introduction. The paper deals with the fuzzy metric spaces in the sense of [4], which is a modification of the notion introduced in [9]. The definition from [4] determines the class of spaces that are tightly connected with the class of metrizable topological spaces.

In [15], it is shown that there exists a natural fuzzy metric, called the Hausdorff fuzzy metric, on the hyperspace (the set of nonempty compact subsets) of a fuzzy metric space. In this paper we show that the construction of the Hausdorff fuzzy metric allows us to define the hyperspace functor in the category of fuzzy metric spaces and nonexpanding maps.

Note that the hyperspace construction determines a monad in different categories: compact Hausdorff spaces, uniform spaces [17], ultrametric spaces and nonexpanding maps etc. One of the main results of this paper (Theorem 1) states that the hyperspace functor determines a monad in the category of fuzzy metric spaces and nonexpanding maps.

Finally, we show that the G-symmetric power functor admits an extension onto the Kleisli category of the hyperspace monad (i.e. the category of fuzzy metric spaces and nonexpanding compact-valued maps).

2. Preliminaries.

2.1. Fuzzy metric spaces. The notion of fuzzy metric space, in one of its forms, is introduced by Kramosil and Michalek [9]. In the present paper we use the version of this concept given in the paper [4] by George and Veeramani.

Definition 1. A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is a *continuous t-norm* if $*$ is satisfying the following conditions:

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(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, and $a, b, c, d \in [0,1]$.

The following are examples of t-norms: $a*b = ab$; $a*b = \min\{a, b\}$, $a*b = \max\{a+b-1, 0\}$ (Łukasiewicz t-norm).

Definition 2. A 3-tuple $(X, M, *)$ is said to be *a fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(i) $M(x, y, t) > 0$,

(ii) $M(x, y, t) = 1$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(v) the function $M(x, y, -)\colon (0, \infty) \to [0,1]$ is continuous.

It is proved in [4] that in a fuzzy metric space X, the function $M(x, y, -)$ is non-decreasing for all $x, y \in X$.

The following notion is introduced in [4] (see Definition 2.6 therein).

Definition 3. Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0,1)$, $t > 0$ and $x \in X$. The set

$$B(x, r, t) = \{y \in X \mid M(x, y, t) > 1 - r\}$$

is called the *open ball* with center x and radius r with respect to t.

The family of all open balls in a fuzzy metric space $(X, M, *)$ forms a base of a topology in X; this topology is denoted by τ_M and is known to be metrizable (see [4]).

If $(X, M, *)$ is a fuzzy metric space and $Y \subset X$, then, clearly,

$$M_Y = M|(Y \times Y \times (0, \infty))\colon Y \times Y \times (0, \infty) \to [0,1]$$

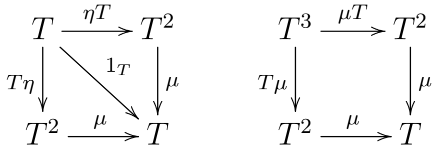
is a fuzzy metric on the set Y. We say that the fuzzy metric M_Y is *induced* on Y by M.

Let $(X, M, *)$ and $(X', M', *)$ be fuzzy metric spaces. A map $f\colon X \to X'$ is called *nonexpanding* if $M'(f(x), f(y), t) \ge M(x, y, t)$, for all $x, y \in X$ and $t > 0$. For our purposes, it is sufficient to consider the class of fuzzy metric spaces with the same fixed norm (e.g., $*$). The fuzzy metric spaces (with the norm $*$) and nonexpanding maps form a category, which we denote by $\mathcal{FMS}(*)$.

2.1. Monads and Kleisli categories. Recall some necessary definitions from the category theory; see, e.g., [3] for the proof.

For a category $\mathcal{C}$ we denote by $|\mathcal{C}|$ the class of objects of $\mathcal{C}$. If $X, Y \in |\mathcal{C}|$, then $\mathcal{C}(X, Y)$ denotes the set of morphisms from X to Y in $\mathcal{C}$.

Let $\mathcal{C}$ be a category. If T is an endofunctor in $\mathcal{C}$ and $\eta\colon 1_{\mathcal{C}} \to T$ and $\mu\colon T^2 \equiv TT \to T$ are natural transformations, then $\mathbb{T} = (T, \eta, \mu)$ is called a *monad* if and only if the following diagrams commute:



Then η is called the *unity* and μ the *multiplication* of $\mathbb{T}$. The functor T is often referred as the *functorial part* of $\mathbb{T}$.

A morphism of a monad $\mathbb{T} = (T, \eta, \mu)$ into a monad $\mathbb{T}' = (T', \eta', \mu')$ on $\mathcal{C}$ is a natural transformation $\varphi\colon T \to T'$ such that the diagrams

$$\begin{array}{ccc} 1_{\mathcal{C}} & \xrightarrow{\eta} & T \\ & \searrow^{\eta'} & \downarrow{\scriptstyle\varphi} \\ & & T' \end{array} \qquad \begin{array}{ccc} T^2 & \xrightarrow{\varphi_{T'}T\varphi} & T'^2 \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu'} \\ T & \xrightarrow[\varphi]{} & T' \end{array}$$

are commutative. If all the components of φ are inclusions, then the monad $\mathbb{T}$ is said to ba a submonad of $\mathbb{T}'$.

The *Kleisli category of a monad* $\mathbb{T}$ is the category $\mathcal{C}_{\mathbb{T}}$ defined as follows: $|\mathcal{C}_{\mathbb{T}}| = |\mathcal{C}|$, $\mathcal{C}_{\mathbb{T}}(X, Y) = \mathcal{C}(X, TY)$, and the composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y, Z)$ is given by $g * f = \mu Z \circ Tg \circ f$.

Define the functor $I\colon \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ by $IX = X$, $X \in |\mathcal{C}|$ and $If = \eta Y \circ f$ for $f \in \mathcal{C}(X, Y)$.

A functor $\overline{F}\colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ is called an *extension of the functor* $F\colon \mathcal{C} \to \mathcal{C}$ *on the Kleisli category* $\mathcal{C}_{\mathbb{T}}$ if $IF = \overline{F}I$.

The following is criterion of existence of an extension of a functor onto the Kleisli category.

Proposition 1. *There exists a bijective correspondence between extensions of functor F onto the Kleisli category $\mathcal{C}_{\mathbb{T}}$ of monad $\mathbb{T}$ and natural transformations $\xi\colon FT \to TF$ satisfying*

1. $\xi \circ F\eta = \eta F$;
2. $\mu F \circ T\xi \circ \xi T = \xi \circ F\mu$.

See, e.g., [3] for the proof.

3. Fuzzy hyperspace monad. The fuzzy hyperspace is defined in [15]. We use slightly different style of notations than that in this paper.

Let $(X, M, *)$ be a fuzzy metric space. Denote by $\exp X$ the set of all nonempty compact subsets in X. Let $B \in \exp X$. For any $a \in X$ and $t \in (0, \infty)$, let

$$M(a, B, t) = \sup\{M(a, b, t) \mid b \in B\}$$

(see [18]).

The Hausdorff fuzzy metric M_H on $\exp X$ is defined by the formula:

$$M_H(A, B, t) = \min\left\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\right\}$$

(see [15]).

We denote by $\exp^2 = \exp\exp$, $\exp^3$,... the iterations of the hyperspace functor. Note that the union map $u = u_X\colon \exp^2 X \to \exp X$, $u(\mathcal{A}) = \cup\mathcal{A}$, is continuous (see, e.g., [11]).

Given a fuzzy metric space $(X, M, *)$, we denote by M_{HH}, M_{HHH}, ... the Hausdorff fuzzy metrics on the sets $\exp^2 X$, $\exp^3 X$, ...

Proposition 2. *The union map* $u\colon (\exp^2 X, M_{HH}, *) \to (\exp X, M_H, *)$ *is nonexpanding.*

Proof. Suppose that $M_{HH}(\mathcal{A},\mathcal{B},t) \geq r$, for some $\mathcal{A},\mathcal{B} \in \exp^2 X$ and $t \in (0,\infty)$. We are going to show that $M_H(\cup\mathcal{A},\cup\mathcal{B},t) \geq r$. Let $a \in \cup\mathcal{A}$. Then there exists $A \in \mathcal{A}$ such that $a \in A$. By the definition of the Hausdorff metric, we see that $M_H(A,\mathcal{B},t) \geq r$. Therefore, there is $B \in \mathcal{B}$ such that $M_H(A,B,t) \geq r$. In particular, this implies that there exists $b \in B \subset \cup\mathcal{B}$ such that $M(a,b,t) \geq r$.

One can similarly show that, for every $b \in \cup\mathcal{B}$ there exists $a \in \cup\mathcal{A}$ such that $M(a,b,t) \geq r$. Summing up, we see that $M_H(\cup\mathcal{A},\cup\mathcal{B},t) \geq r$. □

We denote by $s\colon X \to \exp X$ the singleton map, $s(x) = \{x\}$, for every $x \in X$.

Proposition 3. *The singleton map* $s\colon (X,M,*) \to (\exp X, M_H, *)$ *is nonexpanding.*

Proof. Straightforward. □

Theorem 1. *The triple* $\mathbb{H} = (\exp, s, u)$ *is a monad on the category* $\mathcal{FMS}(*)$.

Proof. That the algebraic identities from the definition of monad hold is a well-known fact (see, e.g., [20]). The previous results guarantee that the maps under consideration are nonexpanding. □

4. Extension of G-symmetric power functor to the Kleisli category. Recall the definition of the G-symmetric power. Let G be a subgroup of the symmetric group S_n. Denote by $\sim=\sim_G$ the following equivalence relation on the n-th power X^n of a set X

$$(x_1,\dots,x_n) \sim (y_1,\dots,y_n) \Leftrightarrow x_i = y_{\sigma(i)} \text{ for some } \sigma \in G.$$

We denote by $[x_1,\dots,x_n]$ the equivalence class containing $(x_1,\dots,x_n)$. The quotient set of the equivalence relation $\sim$ is denoted by $SP^n_G X$.

Now, let $(X,M,*)$ be a fuzzy metric space. Define a function $\tilde{M}\colon SP^n_G X \times SP^n_G X \times (0,\infty) \to [0,1]$ as follows

$$\tilde{M}([x_1,\dots,x_n],[y_1,\dots,y_n],t) = \max_{\sigma\in G} \min_{i=1,\dots,n} M(x_i, y_{\sigma(i)}, t).$$

Proposition 4. *The function* $\tilde{M}$ *is a fuzzy metric on the set* $SP^n_G X$.

Proof. Let us verify property (iv) from the definition of fuzzy metric. Let

$$[x_1,\dots,x_n],[y_1,\dots,y_n],[z_1,\dots,z_n] \in SP^n_G X,$$

$t,s \in (0,\infty)$, then there exist $\sigma,\tau \in G$ such that

$$\tilde{M}([x_1,\dots,x_n],[y_1,\dots,y_n],t) = \min_{i=1,\dots,n} M(x_i,y_{\sigma(i)},t),$$
$$\tilde{M}([y_1,\dots,y_n],[z_1,\dots,z_n],s) = \min_{i=1,\dots,n} M(y_i,y_{\tau(i)},s).$$

We have

$$\begin{aligned}
&\tilde{M}([x_1,\dots,x_n],[y_1,\dots,y_n],t) * \tilde{M}([y_1,\dots,y_n],[z_1,\dots,z_n],s)\\
&= \left(\min_{i=1,\dots,n} M(x_i,y_{\sigma(i)},t)\right) * \left(\min_{i=1,\dots,n} M(y_i,y_{\tau(i)},s)\right)\\
&\leq \min_{i=1,\dots,n} M(x_i,y_{\sigma(i)},t) * M(y_{\sigma(i)},z_{\tau(\sigma(i))},s) \leq \min_{i=1,\dots,n} M(x_i,z_{\tau(\sigma(i))},t+s)\\
&\leq \max_{\varrho\in G} \min_{i=1,\dots,n} M(x_i,z_{\varrho(i)},t+s) = \tilde{M}([x_1,\dots,x_n],[z_1,\dots,z_n],t+s).
\end{aligned}$$

The function $t \mapsto \tilde{M}([x_1, \dots, x_n], [y_1, \dots, y_n], t)$ is obtained from the continuous functions $t \mapsto M(x_i, y_j, t)$, $i, j = 1, \dots, n$, by applying the operations max and min and therefore are continuous as well. This proves (v).

The other properties from the definition of fuzzy metric are easy to verify. □

Remark 1. If G is a trivial group, then $SP^n_G(X) = X^n$ and we obtain the following fuzzy metric on X^n:

$$\tilde{M}((x_1, \dots, x_n), (x_1, \dots, x_n)) = \min_{i=1,\dots,n} M(x_i, y_i, t).$$

Given a map $f\colon X \to Y$, define the map $SP^n_G(f)\colon SP^n_G(X) \to SP^n_G(Y)$ by the formula:

$$SP^n_G(f)([x_1, \dots, x_n]) = [f(x_1), \dots, f(x_n)].$$

Proposition 5. *Let $(X, M, *)$ and $(X', M', *)$ be fuzzy metric spaces. If a map $f\colon (X, M, *) \to (Y, N, *)$ is nonexpanding, then so is $SP^n_G(f)$.*

Proof. The straightforward verification is left to the reader. □

We therefore obtain the G-symmetric power functor SP^n_G acting in the category $\mathcal{FMS}(*)$.

Theorem 2. *The functor SP^n_G admits an extension to the Kleisli category $\mathcal{FMS}(*)_{\mathbb{H}}$.*

Proof. Let $(X, M, *)$ be a fuzzy metric space. Define the map

$$d_X\colon SP^n_G \exp X \to \exp SP^n_G X$$

by the formula

$$d_X([A_1, \dots, A_n]) = \{[a_1, \dots, a_n] \mid a_i \in A_i,\ i = 1, \dots, n\},$$

where $A_i \in \exp X$, $i = 1, \dots, n$.

We are going to show that the map d_X is nonexpanding.

Let $[A_1, \dots, A_n], [B_1, \dots, B_n] \in SP^n_G \exp X$. Suppose that

$$\tilde{(M_H)}([A_1, \dots, A_n], [B_1, \dots, B_n], t) \geq r,$$

for some $r \in (0, 1)$. Then there exists $\sigma \in G$ such that

$$\min_{i=1,\dots,n} M_H(A_i, B_{\sigma(i)}, t) \geq r$$

and therefore, for every $i = 1, \dots, n$, we see that $M_H(A_i, B_{\sigma(i)}, t) \geq r$. Let $[a_1, \dots, a_n] \in d_X([A_1, \dots, A_n])$. Then, for every $i = 1, \dots, n$ there exists $b_{\sigma(i)} \in B_{\sigma(i)}$ such that $M(a_i, b_{\sigma(i)}, t) \geq r$. Then $[b_1, \dots, b_n] \in d_X([B_1, \dots, B_n])$ and

$$\tilde{M}([a_1, \dots, a_n], [b_1, \dots, b_n], t) \geq \min_{i=1,\dots,n} M(a_i, b_{\sigma(i)}, t) \geq r.$$

One can similarly show that, for every $[b_1, \dots, b_n] \in d_X([B_1, \dots, B_n])$, there exists $[a_1, \dots, a_n] \in d_X([A_1, \dots, A_n])$ such that $\tilde{M}([a_1, \dots, a_n], [b_1, \dots, b_n], t) \geq r$. We conclude that

$$(\tilde{M})_H(d_X([A_1, \dots, A_n]), d_X([B_1, \dots, B_n])) \geq r.$$

Therefore, the map d_X is nonexpanding.

Similarly as in [21] we verify that the natural transformation $d = (d_X)$ satisfies the conditions of Proposition 1. Thus, the functor SP^n_G admits an extension to the Kleisli category $\mathcal{FMS}(*)_{\mathbb{H}}$. □

5. Remarks and open problems. Let $\exp_c X$ denote the subspace of $\exp X$ consisting of continua. Clearly, $\exp_c$ is a subfunctor of exp. This subfunctor determines a submonad $\mathbb{H}_c$ (the continuum hyperspace monad) of the monad $\mathbb{H}$. The above results remain correct if we replace $\mathbb{H}$ by $\mathbb{H}_c$.

For an arbitrary monad $\mathbb{T} = (T, \eta, \mu)$ in $\mathcal{C}$ a pair (X, f), where $f\colon TX \to X$ is a morphism in $\mathcal{C}$, is called a $\mathbb{T}$-*algebra* if the following commute

$$\begin{array}{ccc} X & \xrightarrow{\eta X} & TX \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle f} \\ & & X \end{array} \qquad \begin{array}{ccc} T^2X & \xrightarrow{\mu X} & TX \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} \\ TX & \xrightarrow{f} & X. \end{array}$$

The morphism $f\colon TX \to X$ is then referred as the *structure morphism* of the $\mathbb{T}$-algebra (X, f).

Evidently, the couple $(TX, \mu X)$ is a $\mathbb{T}$-algebra for every X. This algebra is said to be a *free* $\mathbb{T}$-*algebra*, determined by the object X. An arrow $\varphi\colon X \to Y$ is called *a morphism of algebras* $(X, f) \to (Y, g)$ if and only if the diagram

$$\begin{array}{ccc} TX & \xrightarrow{T\phi} & TY \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\phi} & Y \end{array}$$

commutes. $\mathbb{T}$-algebras and maps of algebras form the *Eilenberg-Moore category* $\mathcal{C}^{\mathbb{T}}$. It is well known that the category of $\mathbb{H}$-algebras for the hyperspace monad $\mathbb{H}$ in the category of compact Hausdorff spaces and continuous maps is isomorphic to the category of Lawson semilattices. A similar characterization is obtained for the hyperspace monad in the category of uniform spaces. We leave as an open problem that of characterization of the category of $\mathbb{H}$-algebras for the hyperspace monad $\mathbb{H}$ in the category $\mathcal{FMS}(*)$. Since the characterization in the mentioned categories is based on the corresponding semi-lattice theory, we expect that the characterization of the category of $\mathbb{H}$-algebras for the hyperspace monad $\mathbb{H}$ in the category $\mathcal{FMS}(*)$ opens the door to the development of a fuzzy semi-lattice theory.

One can also consider the hyperspace functor in the category of fuzzy ultrametric spaces and nonexpanding maps. Recall that a fuzzy metric $(X, M, *)$ is called a *fuzzy ultrametric* if the following inequality holds:

$$M(x, y, t) * M(y, z, t) \le M(x, z, t),\ x, y, z \in X,\ t \in (0, \infty)$$

(see, e.g., [16]). The results of the present paper have their counterparts for this category.

One can consider a wider category of fuzzy metric spaces and nonexpanding maps (without fixing the t-norms). It looks plausible that the results of this note have their counterparts also in this wider category.

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