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ALMOST PERIODIC MEASURES WITH RESPECT TO DIFFERENT METRICS

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We study measures on $\mathbb{R}^k$ that almost periodic in the weak sense and in the sense of Sodin-Tsirelson's metric. Also, we describe a connection between these classes of almost periodic measures.

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Изучаются меры в $\mathbb{R}^k$, которые являются почти периодическими в слабом смысле, а также в смысле метрики Содина-Цирельсона. Описана связь между этими классами почти периодических мер.

Introduction. In our paper we investigate properties of almost periodic measures on $\mathbb{R}^k$. The notion of an almost periodic measure is a natural generalization of the notion of an almost periodic function.

In Section 1 we consider properties of almost periodic Radon measures with respect to the uniform metric. Particularly, we specify the results concerning boundedness and density given in [7], [5], [2] and [3].

In Section 2 we study general properties of the Sodin-Tsirelson distance (see [8]) between measures. Besides, we consider the connection between the convergence with respect to the Sodin-Tsirelson distance and other types of convergence.

In Section 3 we introduce the notion of an almost periodic measure with respect to the Sodin-Tsirelson distance. Also we investigate the connection between measures that are almost periodic in the usual sense and with respect to the Sodin-Tsirelson distance.

Notations. Throughout the work we denote the i-coordinate of a point $x \in \mathbb{R}^k$ by x^i, an open k-dimensional ball of radius R with center $x \in \mathbb{R}^k$ by $B(x,R)$, k-dimensional cube $\{y \in \mathbb{R}^k \mid x^i - L/2 \le y^i < x^i + L/2, i = \overline{1,k}\}$ by $Q(x,L)$. For any set $A \subset \mathbb{R}^k$ and $\rho > 0$ we write $A_\rho = \cup_{x\in A} B(x,\rho)$. We denote usual Euclidean norm by $|x|$. For any measure μ on $\mathbb{R}^k$ and $\tau \in \mathbb{R}^k$ we put $\mu^\tau(E) = \mu(E+\tau)$ for any Borel set $E \subset \mathbb{R}^k$.

1. The properties of almost periodic measures. Let g be a continuous function on $\mathbb{R}^k$. A vector $\tau \in \mathbb{R}^k$ is called an *ε-almost period* of g if

$$(\forall\, x \in \mathbb{R}^k): \quad |g(x) - g(x+\tau)| < \varepsilon.$$

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A set $E \subset \mathbb{R}^k$ is called *relatively dense* if there is $L < \infty$ such that every ball of radius L has a nonempty intersection with E. It is obvious that here we can replace a ball by a cube.

A continuous function on $\mathbb{R}^k$ is called *almost periodic* if for every $\varepsilon > 0$ the set of its ε-almost periods is relatively dense in $\mathbb{R}^k$.

Definition 1 ([1, 7])**.** A Radon (complex-valued) measure μ on $\mathbb{R}^k$ is called *almost periodic* if for each compactly supported continuous function φ on $\mathbb{R}^k$ the convolution

$$\varphi * \mu(z) = \int_{\mathbb{R}^k} \varphi(z-y)d\mu(y)$$

is almost periodic on $\mathbb{R}^k$.

In other words, a Radon measure μ is *almost periodic* if for each compactly supported continuous function φ on $\mathbb{R}^k$ and any $\varepsilon > 0$ there exists a relatively dense in $\mathbb{R}^k$ set $E_{\varepsilon,\varphi}$ *(*the *set of (ε, φ)-almost periods of μ)* with the property

$$|\varphi * \mu(z) - \varphi * \mu(z+\tau)| \le \varepsilon \quad \forall \tau \in E_{\varepsilon,\varphi},\ \forall z \in \mathbb{R}^k.$$

Note that the sum and the difference of any two (ε, φ)-almost periods are $(2\varepsilon, \varphi)$-almost periods.

We will say that measures μ_n converge *uniformly weakly* to some measure μ if

$$\varphi * \mu_n(x) \underset{n\to\infty}{\to} \varphi * \mu(x)$$

uniformly on $\mathbb{R}^k$ for every continuous compactly supported function φ.

Recall some properties of almost periodic measures.

Theorem 1. *If almost periodic Radon measures converge uniformly weakly to some measure μ, then μ is almost periodic as well.*

The proof follows from properties of almost periodic functions and the definition of an almost periodic measure.

The property of boundedness of almost periodic measures is given by the following theorem.

Theorem 2 (See [7], Theorem 2.1 with $S = \{0\},\ p = 0$)**.** *Let μ be an almost periodic Radon measure. Then there exists $M < \infty$ such that the condition*

$$|\mu|(B(c,1)) \le M \quad \text{for all } c \in \mathbb{R}^k \tag{1}$$

is fulfilled. Here $|\mu|$ is the variation of the measure μ.

The next theorem is an analogue of the Bochner criterion for almost periodic Radon measures.

Theorem 3 (See [7], Theorem 2.2 with $S = \{0\},\ p = 0$)**.** *A Radon measure μ is almost periodic if and only if for every sequence $(h_n) \subset \mathbb{R}^k$ there is a subsequence $(h_{n'})$ such that the measures $\mu^{h_{n'}}$ converge uniformly weakly to some measure μ'.*

Almost periodic Radon measures also possess the following property.

Theorem 4. *Let μ be an almost periodic Radon measure. Then for any $\varepsilon > 0$ there exists $r_0 < \infty$ such that for any $r > r_0$ and $c, t \in \mathbb{R}^k$ the inequality*

$$|\mu(Q(c,r)) - \mu(Q(c+t,r))| < \varepsilon r^k$$

is fulfilled.

Proof. Take an arbitrary $\varepsilon > 0$ and $R > 2^{k+6}Mk/\varepsilon$, where M is the constant from (1). Next, take a continuous function $\varphi : \mathbb{R}^k \to [0,1]$ with a support in $Q(\mathbf{0}, R+1)$ such that $\varphi(x) \equiv 1$ on $Q(\mathbf{0}, R)$. There exists some $L = L(R) < \infty$ such that every cube with an edge of length L contains a $(1,\varphi)$-almost period of the measure μ. Take an arbitrary vector $\alpha \in \mathbb{R}^k$. If τ is a $(1,\varphi)$-almost period of the measure μ, then

$$\begin{gathered}
|\mu(Q(\alpha,R)) - \mu(Q(\alpha+\tau,R))| = \\
= \left| \int_{Q(\alpha,R)} \varphi(x-\alpha)d\mu(x) - \int_{Q(\alpha+\tau,R)} \varphi(x-\alpha-\tau)d\mu(x) \right| \le \\
\le \left| \int_{Q(\alpha,R+1)\setminus Q(\alpha,R)} \varphi(x-\alpha)d\mu(x) \right| + \left| \int_{Q(\alpha+\tau,R+1)\setminus Q(\alpha+\tau,R)} \varphi(x-\tau-\alpha)d\mu(x) \right| + \\
+ \left| \int_{Q(\alpha,R+1)} \varphi(x-\alpha)d\mu(x) - \int_{Q(\alpha+\tau,R+1)} \varphi(x-\tau-\alpha)d\mu(x) \right| \le \\
\le |\mu|(Q(\alpha,R+1) \setminus Q(\alpha,R)) + |\mu|(Q(\alpha+\tau,R+1) \setminus Q(\alpha+\tau,R)) + 1.
\end{gathered}$$

The quantity of cubes with edge of length 1 covering the difference $Q(\alpha, R+1) \setminus Q(\alpha, R)$ is less than $4k(R+1)^{k-1}$. Therefore, (1) yields

$$|\mu(Q(\alpha,R)) - \mu(Q(\alpha+\tau,R))| \le 8Mk(R+1)^{k-1} + 1 < 2^{k+3}MkR^{k-1}.$$

For $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^k$ there exist $\alpha_j \in \mathbb{R}^k$ ($j \in \{1, 2, \ldots, n^k\}$), such that

$$\begin{gathered}
\left| \frac{1}{(nR)^k}\mu(Q(\alpha,nR)) - \frac{1}{(nR)^k}\mu(Q(\alpha+\tau,nR)) \right| = \\
= \left| \frac{1}{(nR)^k}\sum_{j=1}^{n^k}\mu(Q(\alpha_j,R)) - \frac{1}{(nR)^k}\sum_{j=1}^{n^k}\mu(Q(\alpha_j+\tau,R)) \right| \le \qquad (2) \\
\le \frac{1}{(nR)^k} n^k \max\{|\mu(Q(\alpha_j,R)) - \mu(Q(\alpha_j+\tau,R))| : 1 \le j \le n^k\} \le \frac{2^{k+3}Mk}{R}.
\end{gathered}$$

Let $t \in \mathbb{R}^k$ be arbitrary. Take $(1,\varphi)$-almost period τ such that $t \in Q(\tau, L)$. Since $Q(\alpha+t, nR)$ and $Q(\alpha+\tau, nR)$ are contained in $Q(\alpha+t, nR+L)$, we have

$$\begin{gathered}
|\mu(Q(\alpha,nR)) - \mu(Q(\alpha+t,nR))| \le |\mu(Q(\alpha,nR)) - \mu(Q(\alpha+\tau,nR))| + \\
+ |\mu|(Q(\alpha+t,nR+L) \setminus Q(\alpha+\tau,nR)) + |\mu|(Q(\alpha+t,nR+L) \setminus Q(\alpha+t,nR)).
\end{gathered}$$

The quantity of cubes with edge of length 1 covering the difference $Q(\alpha, nR+L) \setminus Q(\alpha, nR)$ is less than $4kL(nR+L)^{k-1}$. Therefore, for all $\alpha \in \mathbb{R}^k$ and $n > L/R$ we get

$$|\mu|(Q(\alpha+t,nR+L) \setminus Q(\alpha+t,nR)) \le 4MLk(nR+L)^{k-1} < 2^{k+1}MLk(nR)^{k-1}.$$

Analogously, $|\mu|(Q(\alpha+t,nR+L)\setminus Q(\alpha+\tau,nR))<2^{k+1}MLk(nR)^{k-1}$. Using (2), we obtain

$$\left|\frac{\mu(Q(\alpha,nR))}{(nR)^k}-\frac{\mu(Q(\alpha+t,nR))}{(nR)^k}\right|<\frac{2^{k+3}Mk}{R}+\frac{2^{k+2}MLk}{nR}.$$

Taking into account our choice of R, for any $n>n_0=\max\left\{\frac{2^{k+4}MLk}{R\varepsilon},L/R\right\}$ we have

$$\left|\frac{\mu(Q(\alpha,nR))}{(nR)^k}-\frac{\mu(Q(\alpha+t,nR))}{(nR)^k}\right|<\varepsilon/2\ \text{ uniformly in } \alpha\in\mathbb{R}^k.$$

Now consider an arbitrary cube $Q(c,r)$, where $r>n_0R$, and take $n>n_0$ such that $nR\le r\le(n+1)R$. Since the quantity of cubes with edge of length 1 covering the difference $Q(c,r)\setminus Q(c,nR)$ is less than $4kr^{k-1}R$, we get

$$\frac{1}{r^k}\left|\mu(Q(c,r))-\mu(Q(c+t,r))\right|\le$$

$$\le\frac{1}{r^k}|\mu|(Q(c,r)\setminus Q(c,nR))+\frac{1}{r^k}|\mu|(Q(c+t,r)\setminus Q(c+t,nR))+$$

$$+\frac{1}{r^k}\left|\mu(Q(c,nR))-\mu(Q(c+t,nR))\right|<\frac{8Mkr^{k-1}R}{r^k}+$$

$$+\frac{1}{(nR)^k}\left|\mu(Q(c,nR))-\mu(Q(c+t,nR))\right|<\frac{8MkR}{r}+\varepsilon/2.$$

Thus, the claim of the theorem holds for $r>\max\left\{\frac{16MkR}{\varepsilon},n_0R\right\}$. □

Remark. For the special class of measures a stronger assertion was proved in [5]. Namely, for any discrete positive integer-valued almost periodic measure μ there exists a constant C such that for any convex bounded set $A\subset\mathbb{R}^k$ and for any $t\in\mathbb{R}^k$

$$|\mu(A)-\mu(A+t)|<C\,((\operatorname{diam}A)^{k-1}+1). \tag{3}$$

However, the following example shows that (3) does not hold for arbitrary almost periodic measures.

Example 1. Consider a positive almost periodic function

$$f(x)=\frac{1}{2}\sum\nolimits_{k=0}^{\infty}\frac{1}{(2k+1)^2}\sin\frac{x}{(2k+1)^2}+c,$$

where $c>-\inf\left\{\frac{1}{2}\sum\nolimits_{k=0}^{\infty}\frac{1}{(2k+1)^2}\sin\frac{x}{(2k+1)^2}:x\in\mathbb{R}\right\}$. Define a positive measure μ on $\mathbb{R}$ by the equality $\mu(A)=\int_A f(x)dx$ for any bounded Borel set $A\subset\mathbb{R}$.

The function

$$g(x)=\int_0^x(f(t)-c)dt=\frac{1}{2}\sum_{k=0}^{\infty}\left(1-\cos\frac{x}{(2k+1)^2}\right)=\sum\nolimits_{k=0}^{\infty}\sin^2\frac{x}{2(2k+1)^2}$$

is unbounded (to check it out consider the values of the function on the sequence $x_N=((2N+1)!!)^2\pi$). Therefore, the value

$$\mu((-T,0))-\mu((0,T))=\int_{-T}^0 f(x)dx-\int_0^T f(x)dx=2g(T)$$

is unbounded from above for $T\to\infty$, which contradicts to (3) for $k=1$.

Definition 2. For any Radon measure μ the value

$$\Delta = \lim_{T\to\infty} \frac{\mu(Q(\mathbf{0},T))}{T^k}$$

is called the *density* of μ.

Theorem 5. *Any almost periodic Radon measure μ possesses a finite shift invariant density, i.e.,*

$$\Delta = \lim_{T\to\infty} \frac{\mu(Q(\alpha,T))}{T^k}$$

uniformly on $\alpha \in \mathbb{R}^k$.

For the proof see [7], Theorem 2.7 with $S = \{0\}$.

Remark. Another proof follows from Theorem 4 of the present paper. Indeed, we can repeat the proof of Theorem 5 from [5], using Theorem 4 instead of inequality (3) and bound (1) instead of property of discreetness.

In the case of positive measure the following specification of Theorem 5 holds.

Theorem 6. *Every almost periodic positive measure μ has a positive density.*

Proof. Assume that $\mu(Q(\mathbf{0}, 1/2)) = \delta > 0$. Take a positive continuous function $\varphi(x)$ with the support in $Q(\mathbf{0}, 1)$ such that $\varphi(x) = \max \varphi = 1$ for all $x \in Q(\mathbf{0}, 1/2)$. By the assumption, there exists $L > 1$ such that every cube with edge of length L contains a vector τ with the property

$$\left| \int_{\mathbb{R}^k} \varphi(x) d\mu(x) - \int_{\mathbb{R}^k} \varphi(x-\tau) d\mu(x) \right| \le \delta/2.$$

Consequently,

$$\mu(Q(\tau,1)) \ge \int_{Q(\tau,1)} \varphi(x-\tau)d\mu(x) = \int_{\mathbb{R}^k} \varphi(x-\tau)d\mu(x) \ge \int_{\mathbb{R}^k} \varphi(x)d\mu(x) - \delta/2 \ge$$
$$\ge \int_{Q(\mathbf{0},1/2)} \varphi(x)d\mu(x) - \delta/2 = \mu(Q(\mathbf{0},1/2)) - \delta/2 = \delta/2.$$

Therefore, every cube with edge of length $L+1$ contains a vector τ such that $\mu(Q(\tau, 1)) \ge \delta/2$, i.e. $\mu(Q(c, L + 1)) \ge \delta/2$ for all $c \in \mathbb{R}^k$. A cube with edge of length R contains $\left[\frac{R}{L+1}\right]^k$ disjoint cubes with edge of length $L + 1$. Hence,

$$\mu(Q(\mathbf{0}, R)) \ge \left(\frac{R^k}{(L+1)^k} - 1 \right) \frac{\delta}{2}.$$

Therefore,

$$\Delta = \lim_{R\to\infty} \frac{\mu(Q(\mathbf{0},R))}{R^k} \ge \frac{\delta}{2(L+1)^k} > 0.$$

In the case of arbitrary positive measure we can consider the measure μ_T such that $\mu_T(E) = \mu(TE)$ for suitable $T < \infty$ and note that the density Δ_T of μ_T is equal to $T^k\Delta$. □

2. The Sodin-Tsirelson metric and its properties. Let μ_1 and μ_2 be two positive measures. Define the *Sodin-Tsirelson distance* (or, otherwise, the *transportation distance* — see [8], [4]) between them by the formula

$$\mathrm{Tra}(\mu_1,\mu_2)=\inf_{\gamma}\ \sup\{|x-y|:(x,y)\in\mathrm{supp}\ \gamma\},$$

where infimum is taken over all positive measures γ on $\mathbb{R}^k\times\mathbb{R}^k$ such that

$$\int\int_{\mathbb{R}^k\times\mathbb{R}^k}\varphi(x)d\gamma(x,y)=\int_{\mathbb{R}^k}\varphi(x)d\mu_1(x), \tag{4}$$

$$\int\int_{\mathbb{R}^k\times\mathbb{R}^k}\varphi(y)d\gamma(x,y)=\int_{\mathbb{R}^k}\varphi(y)d\mu_2(y) \tag{5}$$

for any continuous compactly supported function φ on $\mathbb{R}^k$.

For two positive measures μ_1,μ_2 denote by $Di(\mu_1,\mu_2)$ the infimum of all $r\in(0,\infty)$ such that for any bounded Borel set A the inequalities

$$\mu_1(A)\le\mu_2(A_r),\quad \mu_2(A)\le\mu_1(A_r)$$

are fulfilled (see [8]).

Theorem 7 ([8], Theorem 1.2)**.** $\mathrm{Tra}(\mu_1,\mu_2)=\mathrm{Di}(\mu_1,\mu_2)$.

Throughout this paper we will call Di the Sodin-Tsirelson distance as well.

Evidently, the Sodin-Tsirelson distance satisfies the following axioms of metric:

1. $\mathrm{Di}(\mu_1,\mu_2)\ge 0$;
2. $\mathrm{Di}(\mu_1,\mu_2)=0\Leftrightarrow\mu_1\equiv\mu_2$;
3. $\mathrm{Di}(\mu_1,\mu_2)=\mathrm{Di}(\mu_2,\mu_1)$;
4. $\mathrm{Di}(\mu_1,\mu_2)\le\mathrm{Di}(\mu_1,\mu_3)+\mathrm{Di}(\mu_3,\mu_2)$.

However, the Sodin-Tsirelson distance may take infinite value (for example, the Sodin-Tsirelson distance between Lebesgue measure and the zero measure is equal to $+\infty$).

It is easy to check that for all $t\in\mathbb{R}^k$

$$\mathrm{Di}(\mu_1^t,\mu_2^t)=\mathrm{Di}(\mu_1,\mu_2), \tag{6}$$

$$\mathrm{Di}(\mu,\mu^t)\le|t|. \tag{7}$$

Consider the connection between the convergence in the Sodin-Tsirelson metric and other types of convergence. First, we prove the following theorem.

Theorem 8. *Let μ, ν be positive measures on $\mathbb{R}^k$. Let φ be a continuous compactly supported positive function on $\mathbb{R}^k$. Then*

$$\int_{\mathbb{R}^k}\varphi d\mu\le\int_{\mathbb{R}^k}\varphi d\nu+\mu(\mathrm{supp}\ \varphi)\cdot\omega_\varphi(\mathrm{Di}(\mu,\nu)),$$

where $\omega_\varphi(\delta)=\sup\{|\varphi(x)-\varphi(x')|:\{x,x'\}\subset\mathrm{supp}\ \varphi,|\mathrm{x}-\mathrm{x}'|<\delta\}$ *is the modulus of continuity of* φ.

Proof. Put $G_t = \{ x \in \mathbb{R}^k : \varphi(x) > t\}$ and $\delta_0 = \mathrm{Di}(\mu,\nu)$. By assumption, for arbitrary small $\varepsilon > 0$, $(\forall\, t \in \mathbb{R}^k)\ : \mu(G_t) \le \nu((G_t)_{\delta_0+\varepsilon})$. Next, it is easy to see that

$$(\forall\, t \in \mathbb{R}^k):\ \ (G_t)_{\delta_0+\varepsilon} \subset G_{t-\omega_\varphi(\delta_0+\varepsilon)}.$$

Using [6, p.59], we get

$$\int_{\mathbb{R}^k} \varphi\ d\mu = \int_0^{\max\varphi} \mu(G_t)dt, \quad \int_{\mathbb{R}^k} \varphi\ d\nu = \int_0^{\max\varphi} \nu(G_t)dt,$$

Hence,

$$\int_{\mathbb{R}^k} \varphi d\mu = \int_0^{\omega_\varphi(\delta_0+\varepsilon)} \mu(G_t)dt + \int_{\omega_\varphi(\delta_0+\varepsilon)}^{\max\varphi} \mu(G_t)dt \le \int_0^{\omega_\varphi(\delta_0+\varepsilon)} \mu(G_t)dt +$$
$$+ \int_{\omega_\varphi(\delta_0+\varepsilon)}^{\max\varphi} \nu(G_{t-\omega_\varphi(\delta_0+\varepsilon)})dt \le \mu(\mathrm{supp}\ \varphi)\cdot\omega_\varphi(\delta_0+\varepsilon) + \int_0^{\max\varphi-\omega_\varphi(\delta_0+\varepsilon)} \nu(\mathrm{G_y})\mathrm{dy} \le$$
$$\le \mu(\mathrm{supp}\ \varphi)\cdot\omega_\varphi(\delta_0+\varepsilon) + \int_{\mathbb{R}^\mathrm{k}} \varphi\ \mathrm{d}\nu.$$

□

As a consequence we obtain

Theorem 9. *Let* $(\mu_n), \mu$ *be positive measures on* $\mathbb{R}^k$ *such that* $\mathrm{Di}(\mu_n,\mu) \to 0$ $(n\to\infty)$. *Then* μ_n *converge weakly to the measure* μ.

Proof. Let φ be a continuous compactly supported positive function on $\mathbb{R}^k$. Since $\mathrm{Di}(\mu_n,\mu) \to 0$, there exists N such that $(\forall\ n > N)$:

$$\mu_n(\mathrm{supp}\ \varphi) \le \mu((\mathrm{supp}\ \varphi)_1).$$

By Theorem 8, we get for all $n > N$

$$\left|\int_{\mathbb{R}^k} \varphi d\mu_n - \int_{\mathbb{R}^k} \varphi d\mu\right| \le \mu((\mathrm{supp}\ \varphi)_1)\cdot\omega_\varphi(\mathrm{Di}(\mu_\mathrm{n},\mu)).$$

Hence, $\displaystyle\int_{\mathbb{R}^k} \varphi d\mu_n \underset{n\to\infty}{\to} \int_{\mathbb{R}^k} \varphi d\mu.$ □

Also, the following result is valid.

Theorem 10. *In the class of measures* μ *with the property: for any compact set* $G \subset \mathbb{R}^k$ *there is a constant* $K = K(G)$ *such that*

$$(\forall\ c \in \mathbb{R}^k): \quad \mu(G+c) < K,$$

the convergence in the Sodin-Tsirelson metric implies the weakly uniform convergence.

The proof follows immediately from Theorem 8.

However, Theorem 10 is wrong for arbitrary measures.

Example 2. Let μ be a positive measure such that

$$\mu(A) = \int_{A\cap\bigcup_{m\in\mathbb{N}}[2m-2,2m-1]} xdx$$

for any Borel set $A \subset \mathbb{R}^k$. Define the discrete measures

$$\mu_n(\{p/n\}) = \mu\left(\left[\frac{p-1}{n}, \frac{p}{n}\right)\right), \quad p, n \in \mathbb{N}.$$

Consider the measure

$$\gamma = \sum_{p\in\mathbb{N}} \delta(x - p/n) \otimes \mu\Big|_{[\frac{p-1}{n},\frac{p}{n})}(y).$$

For any continuous compactly supported function φ on $\mathbb{R}$ we have

$$\int\int_{\mathbb{R}\times\mathbb{R}} \varphi(x)d\gamma(x,y) = \sum_p \varphi(p/n)\mu\left(\left[\frac{p-1}{n}, \frac{p}{n}\right)\right) = \int_{\mathbb{R}} \varphi(x)d\mu_n(x),$$

$$\int\int_{\mathbb{R}\times\mathbb{R}} \varphi(y)d\gamma(x,y) = \sum_k \int_{\frac{p-1}{n}}^{\frac{p}{n}} \varphi(y)d\mu(y) = \int_{\mathbb{R}} \varphi(y)d\mu(y).$$

Moreover,

$$\sup\{|x-y| : (x,y) \in \mathrm{supp}\gamma\} \le 1/\mathrm{n}.$$

Hence, $\mathrm{Tra}(\mu_n, \mu) \le 1/n$. Thus μ_n converge to μ in the Sodin-Tsirelson metric.

Furthermore, put

$$\varphi(x) = \begin{cases} x+1, \ x \in [-1,0); \\ 1-x, \ x \in [0,1); \\ 0, \ x \notin [-1,1). \end{cases}$$

Fix $n \in \mathbb{N}$. Take any even $h \in \mathbb{N}$. We get

$$\left|\int_{\mathbb{R}} \varphi(x)d\mu^h(x) - \int_{\mathbb{R}} \varphi(x)d\mu_n^h(x)\right| =$$

$$= \sum\nolimits_{p=1}^{n} \left(\int_{\frac{p-1}{n}}^{\frac{p}{n}} (1-x)(x+h)dx - \left(1-\frac{p}{n}\right)\int_{\frac{p-1}{n}}^{\frac{p}{n}} (x+h)dx\right) =$$

$$= \frac{3n-1}{12n^2} + \frac{h}{2n} \ge 1,$$

when h is large enough. Hence, the measures μ_n do not converge uniformly weakly to μ.

Theorem 11. *The set of positive measures on $\mathbb{R}^k$ is complete in the Sodin-Tsirelson metric.*

Proof. Let (μ_n) be a sequence of positive measures on $\mathbb{R}^k$ such that $\mathrm{Di}(\mu_n, \mu_m) \to 0$ for $n, m \to \infty$. There exists N' such that for all $n, m > N'$ we have $\mathrm{Di}(\mu_n, \mu_m) < 1/2$. Hence, for all $n > N'$ for any ball $B \subset \mathbb{R}^k$

$$\mu_n(B) \le \mu_{N'}(B_1).$$

Therefore, one can find a subsequence (μ_{n_k}) of (μ_n) such that the measures μ_{n_k} converge weakly to some measure μ. Take any bounded Borel set $A \in \mathbb{R}^k$. We get

$$\overline{\lim_{k\to\infty}}\, \mu_{n_k}(\overline{A}) \le \mu(\overline{A}), \tag{8}$$

$$\underline{\lim}_{k\to\infty}\, \mu_{n_k}(\mathrm{Int}A) \ge \mu(\mathrm{Int}A). \tag{9}$$

Take arbitrary $\varepsilon > 0$. There exists $N = N(\varepsilon)$ such that for all $n > N$

$$\mu_N(A) \le \mu_n(A_{\varepsilon/2}) \le \mu_n(\overline{A}_{\varepsilon/2}), \tag{10}$$

$$\mu_n(A_{\varepsilon/2}) \le \mu_N(A_\varepsilon). \tag{11}$$

From (8) and (10) we get

$$\mu_N(A) \le \overline{\lim}_{k\to\infty}\, \mu_{n_k}(\overline{A}_{\varepsilon/2}) \le \mu(\overline{A}_{\varepsilon/2}) \le \mu(A_\varepsilon).$$

From (9) and (11) we get

$$\mu(A) \le \mu(A_{\varepsilon/2}) \le \underline{\lim}_{k\to\infty}\, \mu_{n_k}(A_{\varepsilon/2}) \le \mu_N(A_\varepsilon).$$

Hence,

$$\mathrm{Di}(\mu_n, \mu) \le \mathrm{Di}(\mu_n, \mu_N) + \mathrm{Di}(\mu_N, \mu) \le \tfrac{3\varepsilon}{2}.$$

Consequently, the measures μ_n converges to the measure μ in the Sodin-Tsirelson metric. □

3. Almost periodicity with respect to the Sodin-Tsirelson metric.

Definition 3. Let μ be a positive measure on $\mathbb{R}^k$. We will say that μ is *almost periodic with respect to the Sodin-Tsirelson metric* (or, equivalently, *ST-almost periodic*) if for each $\varepsilon > 0$ there exists a relatively dense in $\mathbb{R}^k$ set E_ε (the set of ε-almost periods) such that for all $\tau \in E_\varepsilon$

$$\mathrm{Di}(\mu, \mu^\tau) < \varepsilon.$$

In other words, a positive measure is *ST-almost periodic* if for each $\varepsilon > 0$ there exists a relatively dense set $E_\varepsilon \subset \mathbb{R}^k$ such that for all $\tau \in E_\varepsilon$ the inequalities

$$\mu(A) \le \mu^\tau(A_\varepsilon), \quad \mu^\tau(A) \le \mu(A_\varepsilon)$$

hold for any bounded Borel set $A \subset \mathbb{R}^k$.

Theorem 12. *Let μ be an ST-almost periodic measure. Then there exists $M' < \infty$ such that the condition*

$$\mu(B(c,1)) \le M' \quad \text{for all } c \in \mathbb{R}^k \tag{12}$$

is fulfilled.

Proof. By assumption, for some $L < \infty$ any ball $B(-c, L)$ contains a vector τ such that

$$\mu(A) \le \mu(A_1 + \tau) \tag{13}$$

for any bounded Borel set A. Recall that here $A_1 = \cup_{x\in A} B(x,1)$. Since $c + \tau$ belongs to $B(\mathbf{0}, L)$, we have $B(c+\tau, 2) \subset B(\mathbf{0}, L+2)$. Consequently, inequality (13) with $A = B(c,1)$ yields

$$\mu(B(c,1)) \le \mu(B(\mathbf{0}, L+2)).$$

The latter inequality proves (12). □

Corollary 1. *Let μ be an ST-almost periodic measure. Then for any $R > 0$ there exists $C_R < \infty$ such that*

$$\mu(B(c,R)) \le C_R M' R^k \quad \text{for all } c \in \mathbb{R}^k,$$

where M' satisfies (12).

Theorem 13. *If ST-almost periodic measures converge to some measure μ with respect to the Sodin-Tsirelson metric, then μ is ST-almost periodic as well.*

Proof. Let (μ_n) be a sequence of ST-almost periodic measures such that $\mathrm{Di}(\mu_n, \mu) \to 0$ $(n \to \infty)$. Take $\varepsilon > 0$. Let n_0 be a number such that $\mathrm{Di}(\mu_{n_0}, \mu) < \varepsilon/3$. Let $E_{\varepsilon/3}$ be a relatively dense set of $\varepsilon/3$-almost periods of measure μ_{n_0}. For $\tau \in E_{\varepsilon/3}$ we have $\mathrm{Di}(\mu_{n_0}, \mu_{n_0}^\tau) < \varepsilon/3$. Therefore, (6), (7), and the triangle inequality yield

$$\mathrm{Di}(\mu, \mu^\tau) \le \mathrm{Di}(\mu, \mu_{n_0}) + \mathrm{Di}(\mu_{n_0}, \mu_{n_0}^\tau) + \mathrm{Di}(\mu_{n_0}^\tau, \mu^\tau) < \varepsilon.$$

Hence, τ is an ε-almost period of measure μ, and the set of almost periods is relatively dense in $\mathbb{R}^k$. □

There is an analogue of the Bochner criterion for ST-almost periodic measures.

Theorem 14. *A positive measure μ is ST-almost periodic if and only if for every sequence $(h_n) \subset \mathbb{R}^k$ there is a subsequence $(h_{n'})$ such that measures $\mu^{h_{n'}}$ form a fundamental sequence with respect to the Sodin-Tsirelson metric.*

Proof. Let μ be an ST-almost periodic measure. Let $(h_n) \subset \mathbb{R}^k$ be an arbitrary sequence. Take $\varepsilon = 1$. There exists a number L_ε such that every ball of radius L_ε contains an ε-almost period of μ. Hence, for all $n \in \mathbb{N}$ we can take $\tau_n \in \mathbb{R}^k$ such that $h_n - \tau_n \in B(\mathbf{0}, L_\varepsilon)$. Take a subsequence $(h_{n'} - \tau_{n'})$ of the sequence $(h_n - \tau_n)$ such that

$$h_{n'} - \tau_{n'} \to h \quad (n' \to \infty), \tag{14}$$

where h belongs to $\overline{B(\mathbf{0}, L_\varepsilon)}$. From (7) and (14) we get that there exists N such that for all $n > N$

$$\mathrm{Di}(\mu^{h_{n'}-\tau_{n'}}, \mu^h) = \mathrm{Di}(\mu^{h_{n'}-\tau_{n'}-h}, \mu) \le |h_{n'} - \tau_{n'} - h| < \varepsilon.$$

Since any τ_n is an ε-almost period, we get for all $n' \in \mathbb{N}$

$$\mathrm{Di}(\mu^{h_{n'}-\tau_{n'}}, \mu^{h_{n'}}) < \varepsilon.$$

Therefore, for any $n', m' > N$ we have

$$\mathrm{Di}(\mu^{h_{n'}}, \mu^{h_{m'}}) < \mathrm{Di}(\mu^{h_{n'}}, \mu^{h_{n'}-\tau_{n'}}) + \mathrm{Di}(\mu^{h_{n'}-\tau_{n'}}, \mu^h) +$$
$$+ \mathrm{Di}(\mu^h, \mu^{h_{m'}-\tau_{m'}}) + \mathrm{Di}(\mu^{h_{m'}-\tau_{m'}}, \mu^{h_{m'}}) < 4\varepsilon.$$

Repeat this procedure for $\varepsilon = 1/2, 1/2^2, \dots$ and choose a diagonal subsequence (h_{n_s}). We get that the sequence $(\mu^{h_{n_s}})$ is fundamental in the Sodin-Tsirelson metric.

Conversely, suppose that for every sequence $(h_n) \subset \mathbb{R}^k$ there is a subsequence $(h_{n'})$ such that measures $\mu^{h_{n'}}$ form a fundamental sequence with respect to the Sodin-Tsirelson metric. If measure μ is not ST-almost periodic, then for some $\varepsilon_0 > 0$ there exists a sequence of balls $(B_p)_{p=1}^\infty$ with infinitely increasing radii l_p, such that no ball contains an ε_0-almost period of the measure μ.

Take $h_0 = 0$ and a number ν_1 such that $l_{\nu_1} > 1$. Let h_1 be the center of B_{ν_1}. Let ν_2 be a number such that $l_{\nu_2} > \max\{2, |h_1|\}$. Let h_2 be the center of B_{ν_2}. Generally, take a number ν_n such that $l_{\nu_n} > \max\{n, |h_1|, |h_2|, \dots, |h_{n-1}|\}$. If h_n be the center of B_{ν_n}, then the differences $h_n - h_1, h_n - h_2, \dots, h_n - h_{n-1}$ belong to B_{ν_n}.

Take arbitrary $p, m\,(p > m)$. By construction $h_p - h_m \in B_{\nu_p}$, hence $h_p - h_m$ is not an ε_0-almost period of μ. Therefore,

$$\mathrm{Di}(\mu^{h_p}, \mu^{h_m}) = \mathrm{Di}(\mu^{h_p - h_m}, \mu) \geq \varepsilon_0.$$

Thus, there are no fundamental subsequences of the sequence (μ^{h_p}), which contradicts to our assumption. □

Theorem 15. *If a measure is ST-almost periodic then it is almost periodic.*

Proof. Let μ be a ST-almost periodic measure. By Theorems 14 and 11 for every sequence $(h_n) \subset \mathbb{R}^k$ there is a subsequence $(h_{n'})$ such that measures $\mu^{h_{n'}}$ converges with respect to the Sodin-Tsirelson metric. Since measures $\mu^{h_{n'}}$ satisfy the condition of Corollary 1, $\mu^{h_{n'}}$ converge uniformly weakly. Hence, by Theorem 3 μ is almost periodic. □

Theorem 16. *If a discrete positive integer-valued measure is almost periodic, then it is ST-almost periodic.*

For the proof of this theorem we need the notion of almost periodic discrete multiple set, which was introduced in [5].

We say that a discrete set in $\mathbb{R}^k$ is a *discrete multiple set* if any its point a has a finite multiplicity $m(a)$. We will denote it as a sequence (a_n), where every point $a \in \mathbb{R}^k$ appears $m(a)$ times.

For any two discrete multiple sets $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we define a *distance* between them by the formula

$$\mathrm{dist}((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = \inf_{\sigma} \sup_{n\in\mathbb{N}} |\, a_n - b_{\sigma(n)}|,$$

where infimum is taken over all bijections $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$. As shown in [5], Theorem 1, this function satisfies all the axioms of metric except the finiteness.

A vector $\tau \in \mathbb{R}^k$ is called an ε-*almost period* of a discrete multiple set $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^k$, if

$$\mathrm{dist}((a_n)_{n\in\mathbb{N}}, (a_n + \tau)_{n\in\mathbb{N}}) < \varepsilon.$$

A discrete multiple set $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^k$ is called *almost periodic*, if for each $\varepsilon > 0$ the set of its ε-almost periods is relatively dense in $\mathbb{R}^k$.

We associate the measure

$$\mu_D = \sum_{a_n\in D} \delta(x - a_n)$$

with any discrete multiple set $D = (a_n)_{n\in\mathbb{N}}$. Conversely, any discrete positive integer-valued measure can be associated with some discrete multiple set. Note that for every continuous function φ with a compact support we have

$$\varphi * \mu_D(x) = \sum_{n\in\mathbb{N}} \varphi(x - a_n), x \in \mathbb{R}^k.$$

Clearly, the sum is always finite.

It was proved in [2] that any discrete multiple set is almost periodic if and only if the corresponding measure is almost periodic.

Proof of Theorem 16. Let μ be an almost periodic discrete integer-valued measure, and $D = (a_n)_{n\in\mathbb{N}}$ be the corresponding almost periodic discrete multiple set. Take any $\varepsilon > 0$. There exists a relatively dense in $\mathbb{R}^k$ set E_ε such that for every $\tau \in E_\varepsilon$ one can find a bijection $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ with the property

$$\sup_{n\in\mathbb{N}} |\, a_n + \tau - a_{\sigma(n)}| < \varepsilon. \tag{15}$$

For some τ from E_ε and corresponding bijection σ consider the measure

$$\gamma = \sum_n \delta_{a_n+\tau} \otimes \delta_{a_{\sigma(n)}}.$$

In view of (15) we have

$$\sup\{|x-y| : (x,y) \in \operatorname{supp} \gamma\} < \varepsilon.$$

It remains to show that γ satisfies (4) and (5). Let φ be a compactly supported continuous function on $\mathbb{R}^k$. We have

$$\int\int_{\mathbb{R}^k\times\mathbb{R}^k} \varphi(x) d\gamma(x,y) = \sum_n \varphi(a_n+\tau) = \int_{\mathbb{R}^k} \varphi(x) d\mu^\tau(x).$$

Next,

$$\int\int_{\mathbb{R}^k\times\mathbb{R}^k} \varphi(y) d\gamma(x,y) = \sum_n \varphi(a_{\sigma(n)}) = \int_{\mathbb{R}^k} \varphi(y) d\mu(y).$$

□

The claim of Theorem 16 does not hold for arbitrary discrete almost periodic measures.

Example 3. Consider the discrete measure μ on $\mathbb{R}$ such that

$$\mu(n) = \begin{cases} 1, & n = 0; \\ 0, & n \notin \mathbb{Z}; \\ 1 - \dfrac{1}{2^{q+1}}, & n \in \mathbb{Z}\setminus\{0\}, \end{cases}$$

where $q \in \mathbb{N} \cup \{0\}$ is such that $2^q \mid n$ (2^q divides n), $2^{q+1} \nmid n$ (2^{q+1} does not divide n).

Let us check that this measure is almost periodic with respect to the uniform metric but is not with respect to the Sodin-Tsirelson one.

Take a compactly supported continuous function $\varphi(x)$. Since any continuous compactly supported function can be represented as a finite sum of continuous functions with arbitrary small supports, we may suppose that $\operatorname{diamsupp} \varphi < 1$. For any number $\tau \in \mathbb{Z}$ and arbitrary $z \in \mathbb{R}$ we have

$$\left|\int \varphi(x-z) d\mu(x) - \int \varphi(x-z-\tau) d\mu(x)\right| = \\ = |\varphi(n-z)(\mu(n) - \mu(n+\tau))|$$

where n is a unique integer belonging to $z + \operatorname{supp} \varphi$. Take an arbitrary positive integer $\widetilde{q}$ and set

$$E = \{\tau \in \mathbb{Z} : 2^{\widetilde{q}} \mid \tau, 2^{\widetilde{q}+1} \nmid \tau\}.$$

Clearly, E is a relatively dense set. Take any $\tau \in E$. We will show that

$$|\mu(n) - \mu(n+\tau)| < 2^{-\widetilde{q}} \quad \text{for all } n \in \mathbb{Z}. \tag{16}$$

If n is odd, then

$$\mu(n) = \mu(n+\tau) = 1/2.$$

Let n be even and q be a positive integer with the property

$$2^q \mid n, 2^{q+1} \nmid n.$$

Hence,

$$\mu(n) = 1 - 2^{-(q+1)}.$$

If $q < \widetilde{q}$, then $2^q \mid n+\tau, 2^{q+1} \nmid n+\tau$, therefore,

$$\mu(n) = \mu(n+\tau) = 1 - 2^{-(q+1)}.$$

If $q \geq \widetilde{q}$, then $2^{\widetilde{q}} \mid n+\tau$, therefore,

$$\mu(n+\tau) \geq 1 - 2^{-(\widetilde{q}+1)}.$$

Thus, in all cases we get (16) and μ is almost periodic.

Now we will show that μ is not an ST-almost periodic measure. Set $A = \left[-\frac{1}{4}, \frac{1}{4}\right)$. Then for any $\tau \geq 1$ we have

$$\mu(A_{1/4} + \tau) = \mu\left(\left(-\frac{1}{2} + \tau, \frac{1}{2} + \tau\right)\right) = 1 - \frac{1}{2^{q+1}}$$

for some $q \in \mathbb{N} \cup \{0\}$. Since $\mu(A) = 1$, we get $\mathrm{Di}(\mu, \mu^\tau) \geq \varepsilon$ for $\varepsilon = 1/4$ and any $\tau \neq 0$. Thus, μ is not ST-almost periodic.

As a consequence, we obtain that the weakly uniform convergence does not imply the convergence in the Sodin-Tsirelson metric. In fact, if it does, then theorems 3 and 14 imply that almost periodicity of a measure yields its ST-almost periodicity.

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