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DIBANDS OF SUBDIMONOIDS

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We prove that every diband of subdimonoids of type Γ is a semilattice of subdimonoids each of which is a rectangular diband of subdimonoids of type Γ.

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Доказано, что каждая дисвязка поддимоноидов типа Г является полурешеткой поддимоноидов, каждый из которых есть прямоугольная дисвязка поддимоноидов типа Г.

1. Introduction. Jean-Louis Loday introduced the notion of a dimonoid [1]. This notion is a standard tool in the theory of Leibniz algebras. One of the first results about dimonoids is the description of the free dimonoid generated by a given set [1]. The notion of a diband of subdimonoids was introduced in [2]. This notion generalizes the notion of a band of semigroups [3] and is effective to describe structural properties of dimonoids. In terms of dibands of subdimonoids, in particular, it was proved that every commutative dimonoid is a semilattice of archimedean subdimonoids [2]. In [4] it was formulated that every idempotent dimonoid is a semilattice of rectangular subdimonoids.

In this paper we study the notion of a diband of subdimonoids. In section 2 we give necessary definitions and an auxiliary result (Theorem 1). In section 3 we describe the necessary and sufficient conditions under which a dimonoid is a diband of subdimonoids (Theorem 2). Clifford [5] proved that a band of semigroups of type Γ is a semilattice of semigroups each of which is a matrix of semigroups of type Γ. The main result of this paper is a generalization of Clifford's theorem (Theorem 3): every diband of subdimonoids of type Γ is a semilattice of subdimonoids each of which is a rectangular diband of subdimonoids of type Γ. In section 4 we construct examples of dimonoids which are decomposed into a diband of subdimonoids.

2. Preliminaries. A set D equipped with two binary associative operations $\prec$ and $\succ$ satisfying the following axioms

$$(x \prec y) \prec z = x \prec (y \succ z), (x \succ y) \prec z = x \succ (y \prec z), (x \prec y) \succ z = x \succ (y \succ z)$$

for all $x, y, z \in D$, is called a dimonoid. If the operations of a dimonoid coincide, then the dimonoid becomes a semigroup. Examples of dimonoids were given in [1,2,6].

A map f from a dimonoid D_1 to a dimonoid D_2 is a homomorphism, if $(x \prec y)f = xf \prec yf$, $(x \succ y)f = xf \succ yf$ for all $x, y \in D_1$. A subset T of a dimonoid $(D, \prec, \succ)$ is called a subdimonoid, if for any $a, b \in D$, a, $b \in T$ implies $a \prec b$, $a \succ b \in T$.

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A dimonoid $(D, \prec, \succ)$ will be called an idempotent dimonoid or a diband, if $x \prec x = x = x \succ x$ for all $x \in D$. A dimonoid $(D, \prec, \succ)$ will be called a diband of subdimonoids $D_\alpha, \alpha \in I$, if

1) $D = \cup_{\alpha \in I} D_\alpha$, 2) $D_\alpha \cap D_\beta = \varnothing$ for $\alpha \neq \beta$,

3) for any $\alpha, \beta \in I$ there exist $\gamma, \gamma' \in I$ such that $D_\alpha \prec D_\beta \subseteq D_\gamma$, $D_\alpha \succ D_\beta \subseteq D_{\gamma'}$.

If $\gamma = \gamma'$, then a diband is a band. If the operations of a dimonoid $(D, \prec, \succ)$ coincide, then we obtain the ordinary band of semigroups [3]. If for every α and β in I, $D_\alpha \prec D_\beta$, $D_\beta \prec D_\alpha$, $D_\alpha \succ D_\beta$, $D_\beta \succ D_\alpha$ are contained in the same D_γ, then we will call $(D, \prec, \succ)$ a semilattice of subdimonoids $D_\alpha, \alpha \in I$. If each D_α belongs to some more or less restrictive type Γ of dimonoid, then we will say that $(D, \prec, \succ)$ is a diband of subdimonoids $D_\alpha, \alpha \in I$ of type Γ.

We say that a dimonoid is rectangular, if its both semigroups are rectangular bands. We will call a rectangular dimonoid also a rectangular diband.

Theorem 1 ([4]). *Every idempotent dimonoid $(D, \prec, \succ)$ is a semilattice Y of rectangular subdimonoids $D_i, i \in Y$.*

3. Main results. The following theorem gives the necessary and sufficient conditions under which a dimonoid is a diband of subdimonoids.

Theorem 2. *A dimonoid $(D, \prec, \succ)$ is a diband of subdimonoids if and only if there are exist an idempotent dimonoid I and a homomorphism ϕ from $(D, \prec, \succ)$ onto I such that pre-image $\alpha\phi^{-1}$ of every element $\alpha \in I$ is a subdimonoid D_α of the dimonoid $(D, \prec, \succ)$.*

Proof. Let $(D, \prec, \succ)$ be a diband of subdimonoids D_α, $\alpha \in \bar{I}$. Then to each pair of elements α, $\beta \in \bar{I}$ there is a unique pair of elements γ, $\gamma' \in \bar{I}$ such that

$$D_\alpha \prec D_\beta \subseteq D_\gamma, D_\alpha \succ D_\beta \subseteq D_{\gamma'}.$$

Define the operations $\prec'$ and $\succ'$ on $\bar{I}$ by $\alpha \prec' \beta = \gamma, \alpha \succ' \beta = \gamma'$. As

$$(D_\alpha \circ D_\beta) \circ D_\gamma = D_\alpha \circ (D_\beta \circ D_\gamma)$$

for $\circ = \prec$ or $\succ$,

$$(D_\alpha \prec D_\beta) \prec D_\gamma = D_\alpha \prec (D_\beta \succ D_\gamma),$$

$$(D_\alpha \succ D_\beta) \prec D_\gamma = D_\alpha \succ (D_\beta \prec D_\gamma),\ D_\alpha \succ (D_\beta \succ D_\gamma) = (D_\alpha \prec D_\beta) \succ D_\gamma$$

and $D_\alpha \prec D_\alpha \subseteq D_\alpha, D_\alpha \succ D_\alpha \subseteq D_\alpha$, then $I = (\bar{I}, \prec', \succ')$ is an idempotent dimonoid.

Observe that each element $a \in D$ belongs to exactly one dimonoid D_α and define a map

$$\phi\colon a \mapsto a\phi = \alpha.$$

The map ϕ is a homomorphism from $(D, \prec, \succ)$ onto I. Indeed, for $a \in D_\alpha$, $b \in D_\beta$ we have $(a \prec b)\phi = \gamma$ as $a \prec b \in D_\gamma$ and $(a \succ b)\phi = \gamma'$ as $a \succ b \in D_{\gamma'}$. On the other hand

$$a\phi \prec' b\phi = \alpha \prec' \beta = \gamma,\ a\phi \succ' b\phi = \alpha \succ' \beta = \gamma'.$$

If ϕ is a homomorphism from the dimonoid $(D, \prec, \succ)$ onto the idempotent dimonoid I, then the classes D_α, $\alpha \in I$ of the congruence on $(D, \prec, \succ)$ which corresponds to ϕ are dimonoids and the dimonoid $(D, \prec, \succ)$ is a diband of subdimonoids D_α, $\alpha \in I$. □

We will use the expression, "$(D, \prec, \succ)$ is a diband I of subdimonoids $D_\alpha\,(\alpha \in I)$ to indicate the situation described in Theorem 2. The homomorphism ϕ from the dimonoid $(D, \prec, \succ)$ onto the idempotent dimonoid I will be called natural.

The main result of this paper is the following.

Theorem 3. *Every diband of subdimonoids of type* Γ *is a semilattice of subdimonoids each of which is a rectangular diband of subdimonoids of type* Γ.

Proof. Let $(D, \prec, \succ)$ be a diband I of subdimonoids D_α, $\alpha \in I$ of type Γ. According to Theorem 1, I is a semilattice P of rectangular subdimonoids I_τ $(\tau \in P)$. Let ϕ be the natural homomorphism from $(D, \prec, \succ)$ onto I and let ψ be the natural homomorphism from I onto P. The map

$$\mu\colon (D, \prec, \succ) \to P\colon a \mapsto (a\phi)\psi$$

is a homomorphism. Indeed, if $a, b \in D$, then

$$(a \prec b)\mu = ((a \prec b)\phi)\psi = (a\phi \prec' b\phi)\psi = (a\phi)\psi * (b\phi)\psi = a\mu * b\mu,$$

$$(a \succ b)\mu = ((a \succ b)\phi)\psi = (a\phi \succ' b\phi)\psi = (a\phi)\psi * (b\phi)\psi = a\mu * b\mu.$$

The pre-image D'_τ of an element $\tau \in P$ is a union of all classes D_α for which $\alpha\psi = \tau$, that is for which $\alpha \in I_\tau$. Thus, D'_τ is a rectangular diband I_τ of subdimonoids $D_\alpha, \alpha \in I_\tau$ of type Γ. □

If the operations of a diband of subdimonoids coincide, then from Theorem 3 we obtain Clifford's theorem [5] about the structure of an arbitrary band of semigroups.

4. Examples of dibands of subdimonoids. In this section we give examples of dimonoids which are decomposed into a diband of subdimonoids.

a) Let X^* be a set of nonempty words in the alphabet X. If $w \in X^*$, then the first (respectively, the last) letter of a word w we denote by $w^{(0)}$ (respectively, by $w^{(1)}$).

Define the operations $\prec$ and $\succ$ on the set X^* by

$$w \prec u = w^{(0)}w^{(1)}, \quad w \succ u = u^{(0)}u^{(1)}$$

for all $w, u \in X^*$.

Proposition 1. ([2], Proposition 3) $(X^*, \prec, \succ)$ *is a dimonoid.*

Let $(X \times X, \prec', \succ')$ be an idempotent dimonoid with the operations

$$(x, y) \prec' (a, b) = (x, y),\ (x, y) \succ' (a, b) = (a, b)$$

for all $(x, y), (a, b) \in X \times X$. Denote this dimonoid by $\tilde{X}$ and for all $i, j \in X$ put

$$A_{(i,j)} = \{w \in X^* | (w^{(0)}, w^{(1)}) = (i, j)\}.$$

The next assertion describes the structure of the dimonoid $(X^*, \prec, \succ)$.

Lemma 1. ([2], Proposition 4) *The dimonoid* $(X^*, \prec, \succ)$ *is a diband* $\tilde{X}$ *of zero semigroups* $A_{(i,j)},\ (i, j) \in \tilde{X}$.

b) Let $(D, \prec, \succ)$ be an arbitrary dimonoid and let I, J be arbitrary nonempty sets. Define a map $p\colon J \times I \to D\colon (j,i) \mapsto (j,i)p = p_{ji}$.

Define the operations $\prec'$ and $\succ'$ on $D' = I \times D \times J$ by

$$(i,g,j) \prec' (k,h,l) = (i, g \prec p_{jk} \prec h, l), (i,g,j) \succ' (k,h,l) = (i, g \succ p_{jk} \succ h, l)$$

for all $(i,g,j), (k,h,l) \in D'$.

Proposition 2. ([6], Section 3.3, Lemma) *The algebra* $(D', \prec', \succ')$ *is a dimonoid.*

The dimonoid obtained will be called a Rees dimonoid and it will be denoted by $M(I, D, J; p)$.

Let $(D, \prec, \succ)$ be an arbitrary dimonoid, a be an arbitrary but fixed element of $(D, \prec, \succ)$. Define the operations $\prec_a$ and $\succ_a$ on D by $x \prec_a y = x \prec a \prec y$, $x \succ_a y = x \succ a \succ y$ for all $x, y \in D$.

Proposition 3. ([6], Section 3.5, Lemma) *The algebra* $(D, \prec_a, \succ_a)$ *is a dimonoid.*

The dimonoid obtained will be called a dimonoid with deformed multiplications.

Let $I \times J$ be a rectangular band. The following lemma describes the structure of the Rees dimonoid $M(I, D, J; p)$.

Lemma 2. *Every Rees dimonoid* $M(I, D, J; p)$ *is a rectangular band* $I \times J$ *of subdimonoids with deformed multiplications* $(D, \prec_{p_{ji}}, \succ_{p_{ji}})$, $(i,j) \in I \times J$.

Proof. Assuming

$$\sigma\colon M(I, D, J; p) \to I \times J\colon (i,a,j) \mapsto (i,j),$$

we obtain, as it is easy to see, the homomorphism from $M(I, D, J; p)$ onto $I \times J$. The classes of the congruence on $M(I, D, J; p)$ which corresponds to σ are subdimonoids

$$T_{(i,j)} = \{(i,a,j) \in M(I, D, J; p) | a \in D\},$$

where $(i,j) \in I \times J$, of $M(I, D, J; p)$. It is immediate to check that the map

$$T_{(i,j)} \to (D, \prec_{p_{ji}}, \succ_{p_{ji}})\colon (i,a,j) \mapsto a$$

is an isomorphism. □

c) Let S be a semigroup, f its idempotent endomorphism. Define the operations $\prec$ and $\succ$ on S by $x \prec y = x(yf)$, $x \succ y = (xf)y$ for all $x, y \in S$.

Proposition 4. ([2], Proposition 1) $(S, \prec, \succ)$ *is a dimonoid.*

The dimonoid obtained will be denoted by S_f.

Lemma 3. *There exists a homomorphism* $S_f \to S$.

Proof. Define a map $\alpha\colon S_f \to S\colon t \mapsto t\alpha = tf$.

For all t, $s \in S_f$ we have

$$(t \prec s)\alpha = (t(sf))\alpha = (t(sf))f = (tf)(sf^2) = (tf)(sf) = (t\alpha)(s\alpha),$$

$$(t \succ s)\alpha = ((tf)s)\alpha = ((tf)s)f = (tf^2)(sf) = (tf)(sf) = (t\alpha)(s\alpha),$$

hence α is a homomorphism. □

Let X be a nonempty set. If $\varphi\colon X \to X$ is a transformation, then $\ker(\varphi) = \{(x, y) \in X \times X | x\varphi = y\varphi\}$.

Let E be an arbitrary idempotent semigroup, f its idempotent endomorphism and let D_x be an arbitrary equivalence class of $\ker(f)$ with the representative $x \in Ef$.

Lemma 4. *The dimonoid E_f is a band Ef of subdimonoids $(D_x, \prec, \succ)$, $x \in Ef$.*

Proof. From Lemma 10 it follows that there exists a homomorphism $\alpha\colon E_f \to E$ such that its image is an idempotent subsemigroup Ef of the semigroup E. It is clear that the classes of the congruence on E_f which corresponds to the homomorphism α are the sets D_x, $x \in Ef$ which are dimonoids with respect to the operations $\prec$ and $\succ$. $\square$

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