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I. V. Protasov

PACKINGS AND COVERINGS OF GROUPS: SOME RESULTS AND OPEN PROBLEMS

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For a group G, we introduce and study the packing and covering spectra $\mathrm{Pack}(G)$ and $\mathrm{Cov}(G)$ as the sets of all possible packing and covering numbers of subsets of G. We show that $|S| \in \mathrm{Cov}(G)$ for every finite subgroup S of G, and $\kappa \in \mathrm{Cov}(G)$ for every infinite group G and every cardinal κ such that $1 \leqslant \kappa \leqslant |G|$ and $|[G]_{<\kappa}| \leqslant |G|$. We calculate or evaluate the packing and covering numbers of subsets of certain combinatorial size. The paper contains also six open problems.

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Для группы G мы вводим и изучаем спектр упаковки $\mathrm{Pack}(G)$ и спектр покрытий $\mathrm{Cov}(G)$ как множества всех возможных чисел упаковки и чисел покрытий подмножеств в G. Мы показываем, что $|S| \in \mathrm{Cov}(G)$ для каждой конечной подгруппы S в G, и $\kappa \in \mathrm{Cov}(G)$ для каждой бесконечной группы G и каждого кардинала κ такого, что $1 \leqslant \kappa \leqslant |G|$ и $|[G]_{<\kappa}| \leqslant |G|$. Мы вычисляем или оцениваем числа упаковки или числа покрытий подмножеств определенного комбинаторного размера. Статья содержит также шесть открытых проблем.

Given a subset A of a group G, the *packing number* $\mathrm{pack}(A)$ and the *covering number* $\mathrm{cov}(A)$ are defined as

$$\mathrm{pack}(A) = \sup\{|S|\colon S \subseteq A, \{sA\colon s \in A\} \text{ is disjoint}\},$$
$$\mathrm{cov}(A) = \min\{|S|\colon S \subseteq A, G = SA\}.$$

We introduce the *packing spectrum* $\mathrm{Pack}(G)$ and the *covering spectrum* $\mathrm{Cov}(G)$:

$$\mathrm{Pack}(G) = \{\mathrm{pack}(A)\colon A \subseteq G\},$$
$$\mathrm{Cov}(G) = \{\mathrm{cov}(A)\colon A \subseteq G\}.$$

For every infinite Abelian group G, the set $\mathrm{Pack}(G)$ was described by N. Lyaskovska [7]. By this description, $\kappa \in \mathrm{Pack}(G)$ for every infinite cardinal κ, $1 \leqslant \kappa \leqslant |G|$ and $\kappa \notin \{2, 3\}$. For the packings of a group by translation of its subset see papers [1-3] or the survey [9].

The paper consists of three sections. In the first section, for a subgroup S of G, we study the packing and covering numbers of S-transversals, and show that $|S| \in \mathrm{Pack}(G)$ and $|S| \in \mathrm{Cov}(G)$ for every finite subgroup S of G. In the second section, we prove that

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$\kappa \in \mathrm{Cov}(G)$ for every infinite group G and every cardinal κ such that $1 \leqslant \kappa \leqslant |G|$ and $|[G]_{<\kappa}| \leqslant |G|$, where $|[G]_{<\kappa}| = \{X \subset G\colon |X| < \kappa\}$. In the third section, we calculate or evaluate the packing and covering numbers of subsets of certain combinatorial size. We put also six open questions uniformly distributed between the sections.

1. Packing and covering numbers of transversals. Let S be a subgroup of a group G. It is easy to see that $\mathrm{pack}(S) = \mathrm{cov}(S) = |G\colon S|$ so $|G\colon S| \in \mathrm{Pack}(G)$ and $|G\colon S| \in \mathrm{Cov}(G)$. In this section we consider the question whether $|S| \in \mathrm{Pack}(G)$ and $|S| \in \mathrm{Cov}(G)$.

For a subgroup S of a group G, by left(right) S-transversal, we mean a system of representatives of the family of all left(right) cosets of G by S.

Proposition 1. *Let S be a subgroup of a group G, L and R be left and right transversals. Then*

$$\mathrm{pack}(L) \leqslant |S|, \quad \mathrm{cov}(L) \geqslant |S|, \quad \mathrm{pack}(R) \geqslant |S|, \quad \mathrm{cov}(R) \leqslant |S|.$$

Proof. Since $|gL \cap S| = 1$ for every $g \in G$, we have $\mathrm{pack}(L) \leqslant |S|$, $\mathrm{cov}(L) \geqslant |S|$. Since the family $\{sR\colon s \in S\}$ is disjoint and $G = \bigcup\{sR\colon s \in S\}$, we have $\mathrm{pack}(R) \geqslant |S|$, $\mathrm{cov}(R) \leqslant |S|$. □

The following theorem shows that $\mathrm{pack}(L)$ and $\mathrm{cov}(L)$ ($\mathrm{pack}(R)$ and $\mathrm{cov}(R)$) could take the opposite values.

Theorem 1. *Let G be a free group in the alphabet A, $|A| > 1$, B be a nonempty subset of A, $B \neq A$, S be a subgroup of G generated by B. Then there exist the left and right S-transversals L and R such that*

$$\mathrm{pack}(L) = 1, \quad \mathrm{cov}(L) = |G|, \quad \mathrm{pack}(R) = |G|, \quad \mathrm{cov}(R) = 2.$$

Proof. We denote by R' the set of all (reduced) group words in A with the first letter from $(A \setminus B)^{\pm 1}$, and put $R = R' \cup \{\varnothing\}$, where $\varnothing$ is the empty word. Clearly, R is an S-transversal. We choose $a \in A \setminus B$, $b \in B$. Then $G = R \cup aR$, thus $\mathrm{cov}(R) = 2$. Let X be a subgroup of G generated by the subset bAb^{-1}. Then $X \cap RR^{-1} = \{\varnothing\}$ because either first or last letter of every non-empty word from RR^{-1} belongs to $(A \setminus B)^{\pm 1}$, but the first and the last letters of every word from X are in $\{b, b^{-1}\}$. It follows that the family $\{xR\colon x \in X\}$ is disjoint, therefore $\mathrm{pack}(R) \geqslant |X| = |G|$.

We put $L = R^{-1}$ and note that L is a left S-transversal. Denote by G_b the set of all group words whose last letter is b. Since the last letter of every non-empty word from L belongs to $(A \setminus B)^{\pm 1}$, necessary condition for $gl, l \in L$, to have last letter b is that g can be presented as $g = g'bl^{-1}$, which can be done in finite number of ways, precisely not more than length of g. Thus, $gL \cap G_b$ is finite for every $g \in G$. Hence, $\mathrm{cov}(L) \geqslant |G_b| = |G|$. Let g_1, g_2 be distinct elements from G. We choose a non-empty word $g \in L$ such that the first letter of g differs from the last letters of g_1^{-1} and g_2^{-1}. Then $g_1^{-1}g \in L$, $g_2^{-1}g \in L$ so $g_1L \cap g_2L \neq \varnothing$ and $\mathrm{pack}(L) = 1$. □

Theorem 2. *Let S be a subgroup of a group G. If S is either normal or finite then $|S| \in \mathrm{Pack}(G)$ and $|S| \in \mathrm{Cov}(G)$.*

Proof. In view of Proposition 1, it suffices to point out a joint left and right S-transversal. If S is normal then every left S-transversal is also a right S-transversal. By [8, Theorem 7.4.4], for every finite subgroup S, there is a joint S-transversal. □

Problem 1. *Is it true that* $[\aleph_0, G] \subseteq \mathrm{Pack}(G)$ *for every infinite group* G*?*

Problem 2. *Let* G *be an infinite group. Does* $\mathrm{Pack}(G)$ *contain all but finitely many finite cardinals?*

By [7], for Abelian groups, the answers to both questions are positive.
The covering variants of these questions will be considered in next section.

2. Covering spectrum.

Theorem 3. *Let* G *be an infinite group,* κ *be a cardinal such that* $1 \leqslant \kappa \leqslant |G|$ *and* $|[G]_{<\kappa}| \leqslant |G|$*. Then* $\kappa \in \mathrm{Cov}(G)$.

Proof. For $\kappa = 1$, the statement is trivial so let $\kappa > 1$. We put $\nu = |[G]_{<\kappa}|$ and enumerate the set $[G]_{<\kappa} = \{X_\alpha\colon \alpha < \nu\}$. We fix a subset S of G such that $S = \kappa$ end $e \in S$ where e is the identity of G. Since $\nu \leqslant |G|$, we can choose inductively the elements $\{g_\alpha\colon \alpha < \kappa\}$ of G such that the family $\{SX_\alpha g_\alpha\colon \alpha < \nu\}$ is disjoint. We put

$$A = G \setminus \bigcup_{\alpha<\nu} X_\alpha g_\alpha,$$

and show that $\mathrm{cov}(A) = \kappa$.

Let us suppose that $\mathrm{cov}(A) < \kappa$, therefore $G = YA$ for some $Y \in [G]_{<\kappa}$. Then there exists $\alpha < \nu$ such that $Y = X_\alpha^{-1}$. Since $g_\alpha \in YX_\alpha g_\alpha$, we see that $g_\alpha \notin YA$. Hence, $\mathrm{cov}(A) \geqslant \kappa$.

On the other hand, we fix $\alpha < \nu$ and take an arbitrary element $x \in X_\alpha$. Since $|S| > |X_\alpha|$, we have $Sxg_\alpha \setminus X_\alpha g_\alpha \subseteq A$ and $xg_\alpha \in S^{-1}A$. Since $e \in S^{-1}$, we obtain $A \subseteq S^{-1}A$. It follows that $G = S^{-1}A$ and $\mathrm{cov}(A) \leqslant \kappa$. □

Theorem 4. *For every infinite Abelian group* G*, we have*

$$\mathrm{Cov}(G) = [1, |G|].$$

Proof. We take an arbitrary cardinal $\kappa \in [1, |G|]$. If κ is finite, we apply Theorem 3. If κ is infinite, we choose a subgroup S of G of cardinality κ, and apply Theorem 2. □

Problem 3. *Is it true that* $\mathrm{Cov}(G) = [1, |G|]$ *for every infinite group* G*?*

Let S be a subset of an amenable group G with finite $\mathrm{cov}(S)$, μ be a Banach measure on G. Then $\mu(S) \geqslant \frac{1}{\mathrm{cov}(S)}$, thus $\mathrm{pack}(S) \leqslant \mathrm{cov}(S)$. By Theorem 3, the packing spectrum of any infinite group contains all finite cardinals.

Problem 4. *Is a group* G *amenable provided that* $\mathrm{pack}(S) \leqslant \mathrm{cov}(S)$ *for every subset* S *of* G *with finite* $\mathrm{pack}(S)$*?*

Remark 1. Let G be a countable amenable group. By [5, Theorems 5.2 and 5.3], there exists a subset of S of G such that $\mu(S) > \frac{1}{2}$ and $\mu(KS) < 1$ for some Banach measure μ on G and any finite subset K of G. Then $\mathrm{pack}(S) = 1$ but $\mathrm{cov}(S) = \aleph_0$.

3. Packings, coverings and combinatorial size. By the combinatorial size of a subset A of a group G we mean (see [9] or [11, Chapter 9]) some (mainly, cardinal) characteristic which reflects the arrangement of A in G.

Let G be an infinite group with the identity e, κ be an infinite cardinal, $\kappa \leqslant |G|$. We say that a subset A of G is

- κ-*large* if there exists $F \in [G]_{<\kappa}$ such that $G = FA$;
- κ-*small* if $L \setminus A$ is κ-large for every κ-large subset L;
- κ-*thick* if, for every $F \in [G]_{<\kappa}$, there exists $a \in A$ such that $Fa \subseteq A$;
- κ-*thin* if $|gA \cap A| < \kappa$ for every $g \in G$, $g \neq e$;

In the case $\kappa = \aleph_0$, we omit κ and write, say, thin instead of $\aleph_0$-thin.

Clearly, A is κ-large if and only if $\operatorname{cov}(A) < \kappa$.

It is easy to see that a subset A of G is κ-small if and only if $G \setminus FA$ is κ-large for every $F \in [G]_{<\kappa}$. Thus, a subgroup A is κ-small if and only if $\kappa < |G\colon A|$.

If κ is regular then every κ-thin subset is κ-small but (in contrast to κ-small subsets) $\operatorname{cov}(A) \geqslant cf|G|$ for every κ-thin subset A of a group G.

Theorem 5. *Let G be an infinite group, κ be a cardinal such that $\kappa \leqslant |G|$, A be a κ-thin subset of G. If $\kappa < |G|$ then $\operatorname{cov}(A) = |G|$. If $\kappa = |G|$ then $\operatorname{cov}(A) \geqslant cf|G|$.*

Proof. Let $\kappa < |G|$ but there exists a subset S of G such that $G = SA$ and $|S| < |G|$. On one hand, $|A| = |G|$. On the other hand, we take an arbitrary element $g \in G \setminus S$. Since $gA = \cup\{gA \cap sA\colon s \in S\}$ and $|s^{-1}gA \cap A| \leqslant \kappa$, we see that $|A| \leqslant \max\{|S|, \kappa\} < |G|$, hence we get a contradiction.

Let $\kappa = |G|$ but there exists a subset S of G such that $|S| < cf|G|$ and $G = SA$. Then $|A| = |G|$. We take an arbitrary element $g \in G \setminus S$. Since $gA = \cup\{gA \cap sA\colon s \in S\}$, $|gA \bigcap sA| < \kappa$ and $|S| < cf|G|$, we have $|A| < |G|$. □

For a natural number k, it is more convenient to say that a subset A of G is k-*thin* if $|gA \cap A| \leqslant k$ for every $g \in G$, $g \neq e$.

A subset A of a group G is called self–linked if $\operatorname{pack}(A) = 1$, equivalently, $gA \cap A \neq \varnothing$ for every $g \in G$ (or $G = AA^{-1}$).

Problem 5. *Has every infinite group G a 2-thin self–linked subset? Under some additional assumptions on G (in particular, if G is Abelian), this is so [4], [6].*

Can $\mathbb{R}$ be partitioned in $\aleph_0$ thin subsets? By [10], Yes under CH, and No under $\neg CH$.

Problem 6. *Can $\mathbb{R}$ be partitioned in $\aleph_0$ subsets such that each subset of the partition is k-thin for some k? Does there exist $k \in \mathbb{N}$ such that $\mathbb{R}$ can be partitioned in $\aleph_0$ k-thin subsets? Can $\mathbb{R}$ be partitioned in $\aleph_0$ subsets linearly independent over $\mathbb{Q}$ (each such subset is 1-thin)?*

It follows from corresponding definition, that every κ-thick subset A of a group G is self–linked, i.e. $\operatorname{pack}(A) = 1$.

Theorem 6. *For every infinite group G of regular cardinality κ, there exists a κ-thick subset A such that $\operatorname{cov}(A) = \kappa$.*

Proof. Let $\{g_\alpha\colon \alpha < \kappa\}$ be a numeration of G, g_0 be the identity of G, $G_\alpha = \{g_\gamma\colon \gamma < \alpha\}$. Then we choose inductively elements $\{x_\alpha\colon \alpha < \kappa\}$ and $\{y_\alpha\colon \alpha < \kappa\}$ such that

$$y_\alpha \notin \bigcup\{G_\alpha G_\gamma x_\gamma\colon \gamma < \kappa\}$$

for every $\alpha < \kappa$. We put

$$A = \bigcup_{\alpha<\kappa} G_\alpha x_\alpha.$$

Let F be a subset of G such that $|F| < \kappa$. Then $F \subseteq G_\alpha$ for some $\alpha < \kappa$ so $Fx_\alpha \subseteq A$ and A is κ-thick.

Let us assume that $\mathrm{cov}(A) < \kappa$ and choose a subset F, $|F| < \kappa$ such that $G = FA$. We pick $\alpha < \kappa$ such that $F \subseteq G_\alpha$. By the choice of y_α, $y_\alpha \notin FA$ and we get a contradiction. □

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Kyiv National University
Department of Cybernetics,
I.V.Protasov@gmail.com

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