УДК 517.98

Adnan Al-Bsoul, Ali Fora, Azzeddine Bellour

SOME PROPERTIES OF GENERALIZED METRIC SPACES AND FIXED POINT THEORY

Adnan Al-Bsoul, Ali Fora, Azzeddine Bellour. *Some properties of generalized metric spaces and fixed point theory*, Mat. Stud. **33** (2010), 85–91.

In this paper we study some properties of generalized metric spaces. We shall give necessary and sufficient conditions for the generalized metric to be a metric. Also, we construct a pseudometric $\widetilde{d}$ from a generalized metric d. Moreover, we generalize Kannan's fixed point theorem.

А. Аль-Бсул, А. Фора, А. Белур. *Некоторые свойства обобщенных метрических пространств и теорема о неподвижной точке* // Мат. Студії. – 2010. – Т.33, №1. – С.85–91.

В статье исследуются свойства обобщенных метрических пространств, устанавливаются условия на обобщенную метрику необходимые и достаточные для того, чтобы она стала метрикой. Для обобщенной метрики d строится псевдометрика $\widetilde{d}$. Доказывается некоторое обобщение теоремы Кенана (Kannan) о неподвижной точке.

1. Introduction. In 2000, A. Branciari [1] introduced the concept of a generalized metric space where the triangle inequality of a metric space has been replaced by an inequality involving three terms instead of two (tetrahedral inequality). So, every metric space is a generalized metric space (gms), but the converse need not be true as we shall see in Section 2. In this paper we shall give a generalization of Kannan's fixed point theorem.

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+$ denotes the set of all nonnegative real numbers, $\mathbb{N}$ denotes the set of all natural numbers, and X denotes a nonempty set.

Let us start with the main definition.

Definition 1. Let X be a nonempty set and let $d\colon X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$, and for all $w, z \in X$ with $z \neq x$, $z \neq w$, $w \neq y$, we have the following properties:

$$d(x,y) = 0 \ \text{ iff } \ x = y, \tag{1}$$

$$d(x,y) = d(y,x) \quad \text{(symmetry)}, \tag{2}$$

$$d(x,y) \leq d(x,z) + d(z,w) + d(w,y) \quad \text{(tetrahedral inequality)}, \tag{3}$$

then we say that $(X,\, d)$ is *a generalized metric space or gms.*

Definition 2. Let (X,d) be a gms. A sequence (x_n) in X is said to be *a Cauchy sequence* if for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}, n \geq n_\varepsilon$, one has $d\,(x_n, x_{n+m}) < \varepsilon$. (X,d) is called complete if every Cauchy sequence in X is convergent.

2000 *Mathematics Subject Classification:* 47H10, 54E99, 54H25.

© Adnan Al-Bsoul, Ali Fora, Azzeddine Bellour, 2010

Let $T\colon X \to X$ be a mapping where (X,d) is a gms. For each $x \in X$, the set

$$O(x) = \{x, Tx, T^2x, T^3x, ...\}$$

will be called the *orbit* at x. $O(x)$ is called T-orbitally complete if and only if every Cauchy sequence in $O(x)$ converges to a point in X.

2. Some Properties of Generalized Metric Spaces. At the beginning we present an example to show that not every generalized metric on a set X is a metric on X.

Example 1. Let $X = \{a_1, a_2, a_3, a_4, , a_5\}$. Define a generalized metric d on X as follows:

$$d(a_i, a_i) = 0 \ (\forall\ i \in \{1,2,3,4,5\}), \quad d(a_i, a_{i+1}) = 1 \ (\forall\ i \in \{1,2,3,4\}),$$
$$d(a_1, a_5) = 5, \quad d(a_1, a_3) = d(a_2, a_4) = d(a_1, a_4) = d(a_2, a_5) = d(a_3, a_5) = 3.$$

Then d is a gm on X, but d is not a metric on X, because

$$d(a_2, a_4) = 3 > 2 = d(a_2, a_3) + d(a_3, a_4).$$

The set $B(x,r) = \{y \in X\colon d(y,x) < r\}$ will denotes the r-ball of x in the gms (X,d). P. Das ([2]) assumed that the collection $\{B(x,r) : x \in X, r > 0\}$ formed a basis for some topology on X, in the next example we see that such collection need not be a basis for any topology on X.

Example 2. Let $X = [-1,1]$ and let $\mathbb{Q}_1 = \mathbb{Q} \cap [-1,1]$. Define a gms d on X as follows

$$d(x,y) = \begin{cases} 0, \text{ if } (x,y) \in (\mathbb{Q}_1 \times \mathbb{Q}_1) \cup (X - \mathbb{Q}_1) \times (X - \mathbb{Q}_1), \\ |x-y|, \quad \text{if } (x,y) \in \mathbb{Q}_1 \times (X - \mathbb{Q}_1) \cup (X - \mathbb{Q}_1) \times \mathbb{Q}_1. \end{cases}$$

Then, (1) and (2) are obvious. For Condition (3), let $x, y, z, w \in X$ such that $x \neq z, z \neq w, w \neq y$, we have the following cases:
Case 1. If $d(x,z) + d(z,w) + d(w,y) \geq 2$, then Condition (3) holds.
Case 2. If $d(x,z) + d(z,w) + d(w,y) < 2$, then $\begin{cases} d(x,z) = |x-z|, \\ d(z,w) = |z-w|, \\ d(w,y) = |w-y|. \end{cases}$
This implies that $(x,y) \in \mathbb{Q}_1 \times (X - \mathbb{Q}_1) \cup (X - \mathbb{Q}_1) \times \mathbb{Q}_1$, thus $d(x,y) = |x-y|$, hence $d(x,y) \leq d(x,z) + d(z,w) + d(w,y)$, therefore, d is a gm on X.
Since $B(0,1) = (X - \mathbb{Q}_1) \cup \{0\}$ and $B(t,\varepsilon) = ((t-\varepsilon, t+\varepsilon) \cap \mathbb{Q}_1) \cup \{t\}$ for $t \in B(0,1) - \{0\}$, we see that $B(t,\varepsilon) \nsubseteq B(0,1)$, for all $\varepsilon > 0$. Therefore, the collection $\{B(x,r)\colon x \in X,\ r > 0\}$ is not a basis for any topology on X.

Remark 1. In Example 2, consider the sequence $x_n = \frac{\sqrt{2}}{n},\ n \geq 1,$ thus, (x_n) converges to 0, but (x_n) is not a Cauchy sequence, because; $d\left(\frac{\sqrt{2}}{n}, \frac{\sqrt{2}}{m}\right) = 2$ for all $n \neq m$. Consequently, the concepts *convergent sequence*, *Cauchy sequence* are independent in gms.

The following result will be used in the construction of a pseudometric from a generalized metric in Section 3.

Proposition 1. *Let (X,d) be a gms, let $x_i \in X,\ 0 \leq i \leq n,\ n \in \mathbb{N},\ x_0 = x,\ x_n = y,\ x \neq x_1$. Then, we have either $\sum_{i=1}^{n} d(x_{i-1}, x_i) \geq d(x,y)$, or $\sum_{i=1}^{n} d(x_{i-1}, x_i) \geq d(x, x_1) + d(x_1, y)$.*

Proof. We may assume that for all $i \in \{2,3,...,n\},\ x_{i-1} \neq x_i$. The cases $n = 1,2,3$, obviously hold. Assume that $n > 3$. By the tetrahedral inequality of d, we have $\sum_{i=1}^{n} d(x_{i-1}, x_i) = d(x,x_1) + ... + d(x_{n-3}, x_{n-2}) + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, y) \geq d(x,x_1) + ... + d(x_{n-3}, y)$. Then, by the induction on n, we conclude that $\sum_{i=1}^{n} d(x_{i-1}, x_i) \geq d(x,y)$, or $\sum_{i=1}^{n} d(x_{i-1}, x_i) \geq d(x,x_1) + d(x_1, y)$. □

Consider Example 1, we have $d(a_1,a_2)+d(a_2,a_3)+d(a_3,a_4)+d(a_4,a_5)=4<5=d(a_1,a_5)$, but, $d(a_1,a_2)+d(a_2,a_3)+d(a_3,a_4)+d(a_4,a_5)\geq d(a_1,a_2)+d(a_2,a_5)$. Moreover, $d(a_1,a_2)+d(a_2,a_3)+d(a_3,a_4)\geq d(a_1,a_4)$ but, $d(a_1,a_2)+d(a_2,a_3)+d(a_3,a_4)<d(a_1,a_2)+d(a_2,a_4)$. Thus, only one of the inequalities holds.

In the following result we give a necessary and sufficient conditions for the generalized metric to be a metric.

Theorem 1. *Let (X,d) be a gms, then the following are necessary and sufficient conditions that d is a metric without an isolated point:*

(i) every convergent sequence is a Cauchy sequence;

(ii) for all $x\in X$, there exists $(x_n)\in X-\{x\}$, such that (x_n) converges to x.

Proof. Necessity. If d is a metric without an isolated point, then (i) holds. On the other hand, $\overline{X-\{x\}}=X$, so, $x\in\overline{X-\{x\}}$. Since (X,d) is first countable, (ii) holds.

Sufficiency. Assume that (X,d) is a gms and satisfies (i) and (ii). Let $x,\ y,\ z$ be three distinct points in X. By (ii) there exists a sequence $(x_n)\in X-\{x\}$ convergent to x, thus the set $\{x_n\colon n\in\mathbb{N}\}$ is not finite. Hence (x_n) has a subsequence (x_{n_k}) such that $x_{n_k}\neq x_{nj}$ for all $j,\ k\in\mathbb{N}$. Also, there exists $k_0\in\mathbb{N}$ such that $x_{n_k}\neq z$ for all $k\geq k_0$, hence we have

$$d(x,y)\leqslant d(x,x_{n_k})+d(x_{n_k},z)+d(z,y),\ \text{for all } k\geq k_0. \tag{4}$$

Also, we have

$$d(x_{n_k},z)\leq d\left(x_{n_k},x_{n_{k+1}}\right)+d\left(x_{n_{k+1}},x\right)+d(x,z), \tag{5}$$

but (x_{n_k}) converges, then by (i), $\lim\limits_{k\to\infty}d\left(x_{n_k},x_{n_{k+1}}\right)=0$. Now, as $k\to\infty$ in (4) and (5), we get $d(x,y)\leqslant 0+\overline{\lim\limits_{k\to\infty}}\,d(x_{n_k},z)+d(z,y)\leq d(x,z)+d(z,y)$. Hence, d is a metric. Since d satisfies (ii), we see that $\{x\}$ is not open for all $x\in X$. □

3. A Construction of a Pseudometric from a Generalized Metric. Let us start with a generalized metric space (X,d). We construct a pseudometric $\widetilde{d}$ from d as follows. Set

$$\widetilde{d}(x,y)=\inf\left\{\sum\nolimits_{i=1}^{n}d(x_{i-1},x_i)\colon x_i\in X,\ x_0=x,\ x_n=y,\ n\in\mathbb{N}\right\}. \tag{6}$$

Clearly $\widetilde{d}$ is well defined and nonnegative.

To see that $\widetilde{d}$ is a pseudometric on X, let $x\in X$. In (6), set $n=1$, $x_1=x$, we get $\widetilde{d}(x,x)\leq d(x,x)=0$. Also, the symmetry of d gives us the symmetry of $\widetilde{d}$. Now, let $x,\ y,\ z\in X$ and let $\epsilon>0$, there exist $\{u_i\colon u_0=x,u_n=z;i\in\{0,1,...,n\}\}\subseteq X$ and $\{v_j\colon v_0=z,v_m=y;j\in\{0,1,...,m\}\}\subseteq X$ such that $\widetilde{d}(x,z)\leq\sum_{i=1}^{n}d(u_{i-1},u_i)<\widetilde{d}(x,z)+\frac{\epsilon}{2}$ and $\widetilde{d}(z,y)\leq\sum_{j=1}^{m}d(v_{j-1},v_j)<\widetilde{d}(z,y)+\frac{\epsilon}{2}$.
Now, choose the finite set $\{s_k\colon k\in\{0,1,...,n+m\}\}\subseteq X$ as $s_k=u_k$ for $k\in\{0,1,...,n\}$, and $s_{k+n}=v_k$ for $k\in\{1,...,m\}$. Then, $\widetilde{d}(x,y)\leq\sum_{k=1}^{n+m}d(s_{k-1},s_k)<\widetilde{d}(x,z)+\widetilde{d}(z,y)+\epsilon$, taking $\epsilon\to0^+$, we get $\widetilde{d}(x,y)\leq\widetilde{d}(x,z)+\widetilde{d}(z,y)$. Therefore, $\widetilde{d}$ is a pseudometric on X.

Let us give some properties of the pseudometric $\widetilde{d}$.

Let (X,d) be a gms. Since $\widetilde{d}(x,y)\leq d(x,y)$ for all $x,\ y\in X$, so, if (x_n) is a Cauchy sequence in (X,d), then (x_n) is a Cauchy sequence in $(X,\widetilde{d})$, also, if (x_n) converges to x_0 in (X,d), then (x_n) converges to x_0 in $(X,\widetilde{d})$.

Proposition 2. *Let X be a nonempty set, and let d be a metric on X. Then, (X,d) is a gms and $\widetilde{d}\,(x,y)=d(x,y)$ for all $x,\ y\in X$.*

Proof. Let $x,\ y\in X$, and let $x_i\in X,\ x_0=x,\ x_n=y,\ i\in\{0,1,...,n\},\ n\in\mathbb{N}$. Thus, we have $\sum_{i=1}^{n}d(x_{i-1},x_i)\geq d(x_0,x_2)+d(x_2,x_3)+\ldots+d(x_{n-1},x_n)\geq\ldots\geq d(x_0,x_n)=d(x,y)$. Then, $\widetilde{d}\,(x,y)\geq d(x,y)$. Since $\widetilde{d}\,(x,y)\leq d(x,y)$, we see that, $\widetilde{d}\,(x,y)=d(x,y)$ for all x, $y\in X$. □

Example 3. Let (X,d) be a gms. Then, for the following generalized metrics: $d_1=\min\{1,d\}$, and $d_2=\frac{d}{d+1}$, we have $\widetilde{d_1}(x,y)=\min\{1,\widetilde{d}\,(x,y)\}$, and $\widetilde{d_2}(x,y)=\widetilde{d}\,(x,y)/(\widetilde{d}\,(x,y)+1)$ for all $x,\ y\in X$.

In the next proposition we give another expression of $\widetilde{d}\,(x,y)$.

Proposition 3. *Let (X,d) be a gms. Then $\widetilde{d}\,(x,y)=\inf\{d\,(x,t)+d\,(t,y):t\in X\}$.*

Proof. Let $x,\ y\in X$. If $x=y$, then $\widetilde{d}\,(x,y)=\inf\{d\,(x,t)+d\,(t,y):t\in X\}$. If $x\neq y$, let $x_i\in X,\ i\in\{0,1,...,n\},\ n\in\mathbb{N}$, such that $x_0=x$ and $x_n=y$, and let i_0 be the smallest i such that $d\,(x_{i-1},x_i)\neq 0$, then $\sum_{i=1}^{n}d\,(x_{i-1},x_i)=\sum_{i=i_0}^{n}d\,(x_{i-1},x_i)$, by Proposition 1, we get $\sum_{i=1}^{n}d\,(x_{i-1},x_i)\geq d\,(x,y)$ or $\sum_{i=1}^{n}d\,(x_{i-1},x_i)\geq d\,(x,x_{i_0})+d\,(x_{i_0},y)$ we get,

$$\sum_{i=1}^{n}d\,(x_{i-1},x_i)\geq\inf\{d\,(x,t)+d\,(t,y):t\in X\}. \tag{7}$$

By taking the infimum in (7), we get $\widetilde{d}\,(x,y)\geq\inf\{d\,(x,t)+d\,(t,y):t\in X\}$. On the other hand, by the definition of $\widetilde{d}$, clearly we have $\widetilde{d}\,(x,y)\leq\inf\{d\,(x,t)+d\,(t,y):t\in X\}$. Then, we conclude that $\widetilde{d}\,(x,y)=\inf\{d\,(x,t)+d\,(t,y):t\in X\}$. □

Remark 2. Let (X,d) be a gms, then for every $x,\ y\in X$ there exists a sequence (x_n) such that $\widetilde{d}\,(x,y)=\lim_{n\to\infty}(d\,(x,x_n)+d\,(x_n,y))$, moreover if $\widetilde{d}\,(x,y)=0$, then there exists a sequence (x_n) that converges in (X,d) to x and y.

By the above representation, (X,d) is a topological space with the neighborhood basis given by

$$\mathcal{B}=\{B(x,r)\colon x\in X,r>0\},$$

where $B(x,r)=\{y\in X\colon\ \inf\{d\,(x,t)+d\,(t,y):t\in X\}<\ r\}$ is the open ball with center x and radius r.

Corollary 1. *Let (X,d) be a gms, and let $y,\ t\in X$, then $d\,(t,z)+d\,(z,y)\geq d\,(t,y)$ $(\forall\ z\in X)$ or $d\,(x,t)+d\,(t,y)>d\,(x,y)$ and $\ d\,(x,y)+d\,(t,y)>d\,(x,t)$ $(\forall\ x\in X-\{t,y\})$.*

Proof. For all $x,z\in X-\{t,y\}$, we have

$$d\,(x,y)\leq d\,(x,t)+d\,(t,z)+d\,(z,y) \tag{8}$$

and,

$$d\,(x,t)\leq d\,(x,y)+d\,(y,z)+d\,(z,t)\,. \tag{9}$$

So, we have two cases.

I. If $\widetilde{d}(t,y) = d(t,y)$, then $\inf\{d(t,z)+d(z,y) : z \in X\} = d(t,y)$, hence $d(t,z)+d(z,y) \geq d(t,y)$, for all $z \in X$.
II. If $\widetilde{d}(t,y) < d(t,y)$, then $\inf\{d(t,z)+d(z,y) : z \in X\} < d(t,y)$, hence there exists $z \notin \{t,y\}$, such that $d(t,y) > d(t,z)+d(z,y)$

By (8), we get $d(x,y) < d(x,t)+d(t,y)$, for all $x \in X - \{t,y\}$, and by (9), we get $d(x,t) < d(x,y)+d(y,t)$, for all $x \in X - \{t,y\}$. □

Corollary 2 (Contraction of $\widetilde{d}$). *Let (X,d) be a gms, and let $T\colon X \to X$ be a contraction in (X,d), that is $d(Tx,Ty) \leq q\, d(x,y)$, for all $x,y \in X$, where $0 \leq q < 1$, then T is a contraction in $(X,\widetilde{d})$.*

Proof. Let $x,\ y \in X$, and let $x_i \in X,\ i = 0,1,...,n$, such that $x_0 = x,\ x_n = y,\ n \in \mathbb{N}$. Hence, $q\sum_{i=1}^{n} d(x_{i-1},x_i) \geq \sum_{i=1}^{n} d(Tx_{i-1},Tx_i) \geq \widetilde{d}(Tx,Ty)$ Thus,

$$\inf\left\{\sum_{i=1}^{n} d(x_{i-1},x_i), x_i \in X,\ i \in \{1,...,n\},\ x_0 = x,\ x_n = y,\ n \in \mathbb{N}\right\} \geq \tfrac{1}{q}\widetilde{d}(Tx,Ty).$$

Which implies that $\widetilde{d}(Tx,Ty) \leq q\widetilde{d}(x,y)$ for all $x,\ y \in X$. Then, T is a contraction in $(X,\widetilde{d})$. □

Corollary 3. *Let (X,d) be a gms, and let $x,y \in X$. If $\widetilde{d}\ (x,y) = 0$, then $d(x,z) = d(y,z)$, for all $z \in X - \{x,y\}$.*

Proof. Assume that $\widetilde{d}\ (x,y) = 0$, by Remark 2 there exists a sequence (x_n) that converges to x and y in (X,d). If $x = y$, then $d(x,z) = d(y,z)$ for all $z \in X$. If $x \neq y$, let $z \in X - \{x,y\}$, then there exists a subsequence (x_{n_k}) such that $x_{n_k} \in X - \{x,y,z\}$ for all $k \in N$, hence $d(x,z) \leq d(x,x_{n_k}) + d(x_{n_k},y) + d(y,z)$, as $k \to \infty$, and we get $d(x,z) \leq d(y,z)$. Similarly, we get $d(x,z) \leq d(y,z)$, which implies that $d(x,z) = d(y,z)$, for all $z \in X - \{x,y\}$. □

Consider the following two statements in a gms (X,d) :

(A) every convergent sequence is a Cauchy sequence;

(B) every convergent sequence has a unique limit.

Then we have the following result.

Theorem 2. *Let (X,d) be a gms, then $(A) \to (B)$. Moreover, if (X,d) satisfies (B) then $\widetilde{d}$ is a metric.*

Proof. Assume that (X,d) satisfies (A). Suppose that (x_n) is a sequence which converges to x and y with $x \neq y$. Then, there exists a subsequence (x_{n_k}) such that $x_{n_k} \in X - \{x,y\}$ for all $k \in \mathbb{N}$, hence there exists a subsequence (y_j) from (x_{n_k}) such that $y_n \neq y_m$, for all $n \neq m$, then

$$d(x,y) \leq d(x,y_n) + d(y_n,y_{n+1}) + d(y_{n+1},y). \tag{10}$$

Since (y_n) converges, so (y_n) is a Cauchy sequence, hence as $n \to \infty$ in (10) we get $d(x,y) = 0$, a contradiction. Then (B) holds.

Now, assume that (X,d) satisfies (B). Let $x,y \in X$ be such that $\widetilde{d}\ (x,y) = 0$, then by Remark 2, there exists a sequence (x_n) that converges to both x and y in (X,d), and therefore $x = y$, which implies that $\widetilde{d}$ is a metric. □

4. A Fixed Point Theorem in Generalized Metric Spaces. In this section we shall give a generalization of the following Kannan's fixed point theorem.

Theorem (Kannan's fixed point theorem). *Let (X,d) be a complete metric space and let $T\colon X\to X$ be a mapping such that $d(Tx,Ty)\le\beta\left[d(x,Tx)+d(y,Ty)\right]$, where $0\le\beta<\frac{1}{2}$ and $x,\ y\in X$. Then T has a unique fixed point in X.*

In the following theorem we extend this result.

Let Φ denote the class of all functions $\phi\colon\mathbb{R}_+^4\to\mathbb{R}_+$, which are continuous, nondecreasing and $\phi(t,t,t,t)<t$, $\sum_{k=1}^{\infty}\phi^k(t,t,t,t)<\infty$ for all $t>0$, where ϕ^n is the function $\phi^n\colon\mathbb{R}_+^4\to\mathbb{R}_+$ such that, for $n\ge 2$,

$$\phi^n(t_1,t_2,t_3,t_4)=\phi\big(\phi^{n-1}(t_1,t_1,t_1,t_1),\phi^{n-1}(t_2,t_2,t_2,t_2),\phi^{n-1}(t_3,t_3,t_3,t_3),\phi^{n-1}(t_4,t_4,t_4,t_4)\big).$$

Theorem 3. *Let (X,d) be a gms, let $T\colon X\to X$ be a mapping such that*

$$d(Tx,Ty)\le\phi(d(x,y),d(x,Tx),d(y,Ty),d(y,Tx)),\tag{11}$$

where $\phi\in\Phi$. If there exists $x\in X$ such that $O(x)$ is orbitally complete, then T has a unique fixed point in X.

Proof. Define the sequence (x_n) as follows: $x_0=x$, $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$. We have

$$\begin{aligned}d(Tx,T^2x)&\le\phi(d(x,Tx),d(x,Tx),d(Tx,T^2x),d(Tx,Tx))=\\&=\phi(d(x,Tx),d(x,Tx),d(Tx,T^2x),0),\end{aligned}$$

which implies that $d(Tx,T^2x)\le\phi(d(x,Tx),d(x,Tx),d(x,Tx),d(x,Tx))$. Thus, in general, if n is a positive integer, then

$$d\left(T^nx,T^{n+1}x\right)\le\phi^n(d(x,Tx),d(x,Tx),d(x,Tx),d(x,Tx)).\tag{12}$$

If there exists $m>n$ such that $x_n=x_m$, let $y=T^nx$, then $T^ky=y$ for $k=m-n$. Since $k\ge 1$, we have $d(y,Ty)=d\left(T^ky,T^{k+1}y\right)\le\phi^k(d(y,Ty),d(y,Ty),d(y,Ty),d(y,Ty))$. Hence, $d(y,Ty)=0$, and thus y is a fixed point of T.

Therefore, we may assume that $x_n\ne x_m$ for all $n\ne m$. Also, if n is a positive integer, then

$$d\left(T^nx,T^{n+2}x\right)\le\phi^n(M,M,M,M),\tag{13}$$

where $M=\max\{d(x,T^2x),d(x,Tx)\}$. Now, let $n,m\in\mathbb{N}$, with $m\ge n+3$, so,

$$d(T^nx,T^mx)\le d(T^nx,T^{n+1}x)+d(T^{n+1}x,T^{n+2}x)+\ldots+d(T^{m-1}x,T^mx)$$

or $d(T^nx,T^mx)\le d(T^nx,T^{n+1}x)+d(T^{n+1}x,T^{n+2}x)+\ldots+d(T^{m-2}x,T^mx)$. Then either

$$\begin{aligned}d(T^nx,T^mx)&\le\phi^n(M,M,M,M)+\phi^{n+1}(M,M,M,M)+\ldots\\&\quad\ldots+\phi^{m-1}(M,M,M,M)\le\sum_{k=n}^{\infty}\phi^k(M,M,M,M)\end{aligned}\tag{14}$$

or

$$\begin{aligned}d(T^nx,T^mx)&\le\phi^n(M,M,M,M)+\phi^{n+1}(M,M,M,M)+\ldots\\&\quad\ldots+\phi^{m-2}(M,M,M,M)\le\sum_{k=n}^{\infty}\phi^k(M,M,M,M).\end{aligned}\tag{15}$$

Thus, by (12)–(15), we have $d(T^nx,T^mx)\le\sum_{k=n}^{\infty}\phi^k(M,M,M,M)$ for all $m,n\in\mathbb{N},m\ge n$. Since $\sum_{k=n}^{\infty}\phi^k(M,M,M,M)\to 0$ $(n\to\infty)$, we see that (x_n) is a Cauchy sequence, but $O(x)$ is T-orbitally complete, hence (x_n) converges to $p\in X$.

To prove that p is a fixed point of T, we consider two cases.

Case 1. If (x_n) does not converge to Tp. Then, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq Tp$, for all $k \in \mathbb{N}$. Hence $d(p,Tp) \leq d(p,x_{n_k-1}) + d(x_{n_k-1},x_{n_k}) + d(x_{n_k},Tp)$ ($p \neq x_{n_k-1}$ because $Tp \neq x_{n_k}$), then if $k \to \infty$, we get

$$d(p,Tp) \leq \overline{\lim_{k\to+\infty}} d(x_{n_k},Tp). \tag{16}$$

On the other hand, from (11) we have

$$d(x_n,Tp) = d(Tx_{n-1},Tp) \leq \phi(d(x_{n-1},p), d(x_{n-1},x_n), d(p,Tp), d(p,x_n)).$$

Let $n \to \infty$, we get

$$\overline{\lim_{n\to+\infty}} d(x_n,Tp) \leq \phi(0,0,d(p,Tp),0). \tag{17}$$

Hence, by (16) and (17), we have $d(p,Tp) = 0$, then $p = Tp$.

Case 2. Let (x_n) converge to Tp. Suppose that $p \neq Tp$, then there exists a subsequence (x_{nk}) such that for all $k \in \mathbb{N}$, $x_{n_k} \in X - \{p, Tp\}$. Hence

$$d(p,Tp) \leq d(p,x_{n_k}) + d\left(x_{n_k},x_{n_{k+1}}\right) + d\left(x_{n_{k+1}},Tp\right) \tag{18}$$

But (x_{n_k}) is a Cauchy sequence, hence as $k \to \infty$ in (18), we get $d(p,Tp) = 0$, a contradiction. Therefore, $p = Tp$.

Then in all cases $p = Tp$, which implies that p is a fixed point of T.

For uniqueness, assume that $w \neq p$ is also a fixed point of T. From (11),

$$d(p,w) = d(Tp,Tw) \leq \phi(d(p,w), d(p,Tp), d(w,Tw), d(w,Tp)) < d(p,w),$$

a contradiction, therefore p is the unique fixed point of T. □

Remark 3. If we put $\phi(t_1,t_2,t_3,t_4) = q\max\{t_1,t_2,t_3,t_4\}$, where $0 \leq q < 1$,we get a generalization of Kannan's fixed point theorem.

REFERENCES

1. Branciari A. *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*// Publ. Math. Debrecen. – 2000. – V.57, №1-2. – P.31–37.
2. Pratulananda Das. *A fixed point theorem on a class of generalized metric spaces*// Korean J. Math. Sciences. – V.9, №1. – P.29–33.

Department of Mathematics, Yarmouk University,
Irbid–Jordan,
afora_jo@yahoo.com
albsoulaaa@hotmail.com
bellourazze123@yahoo.com

Received 14.10.2008
Revised 8.10.2009