УДК 517.5

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ON EQUIVALENCE FOR SUBSPACES OF ONE WEIGHTED HARDY SPACE

V. Dilnyi. *On equivalence for subspaces of one weighted Hardy space*, Mat. Stud. **33** (2010), 71–77.

Equivalence is established of subspaces of one exponential-weighted space Hardy in half-plane and of classes of entire functions belonging to L^2 on all half-lines with the exception of a semi-strip.

В. Н. Дильный. *Об эквивалентности подпространств одного весового пространства Харди* // Мат. Студії. – 2010. – Т.33, №1. – С.71–77.

Установлена эквивалентность между подпространствами одного експоненциально-весового пространства Харди в полуплоскости и классами целых функций, принадлежащими L^2 на всех полупрямых, которые не пересекают полуполосу.

Let the Hardy space $H^p(\mathbb{C}_+), 1 \le p < +\infty$, be the class of analytic in $\mathbb{C}_+ = \{z\colon \mathrm{Re} z > 0\}$ functions f for which

$$\sup_{x>0}\left\{\int_{-\infty}^{+\infty} |f(x+iy)|^p dy\right\} < +\infty.$$

The following Paley-Wiener theorem [7] plays the fundamental role in the Hardy space theory.

Theorem P.-W. *The function*

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{0} g(w) e^{zw} dw$$

belongs to $H^2(\mathbb{C}_+)$ if and only if $g \in L^2(-\infty; 0)$.

Let $H^p_\sigma(\mathbb{C}_+), 1 \le p < +\infty, \sigma \ge 0$, be the class of analytic in $\mathbb{C}_+$ functions f for which

$$\sup_{|\varphi|<\frac{\pi}{2}}\left\{\int_{0}^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r|\sin\varphi|} dr\right\} < +\infty.$$

In the case $\sigma = 0$ A. Sedletskii proved [10] that the space $H^p_\sigma(\mathbb{C}_+)$ is equal to the Hardy space $H^p(\mathbb{C}_+)$. Also let the Wiener space be the class of entire functions f of exponential

2000 *Mathematics Subject Classification:* 30D15, 30D10, 30D55.

Keywords: one-exponential weighted Hardy space, entire function, half-line, semi-strip
doi:10.30970/ms.33.1.71-77

type $\le \sigma$ for which $f \in L^2(R)$. The Wiener space is a subset (see [4]) of $H^2_\sigma(\mathbb{C}_+)$. The space $H^p_\sigma(\mathbb{C}_+)$ was studied in [13], [14]. In this papers it is shown that the functions $f \in H^p_\sigma(\mathbb{C}_+)$ have almost everywhere (a.e.) on $\partial\mathbb{C}_+$ the angular boundary values which will be denoted by f and $f(iy)e^{-\sigma|y|} \in L^p(-\infty;+\infty)$. Further, let $E^p[D_\sigma]$ and $E^p_*[D_\sigma], 1 \le p < +\infty, \sigma > 0$, be the spaces of analytic functions respectively in the domains $D_\sigma = \{z\colon \mathrm{Im} z| < \sigma, \mathrm{Re} z < 0\}$ and $D^*_\sigma = C\backslash\overline{D}_\sigma$, for which

$$\sup\left\{\int\limits_\gamma |f(z)|^p |dz|\right\}^{1/p} < +\infty,$$

where supremum is taken over all segments γ that lay respectively in D_σ and D^*_σ. The spaces $E^p[D_\sigma]$ and $E^p_*[D_\sigma]$ were studied in [12]. In this article it has been shown that the functions f in these spaces have a. e. on ∂D_σ the angular boundary values which will be denoted by $f(z)$ and $f \in L^p[\partial D_\sigma]$. It is obvious that $f \in E^p_*[D_\sigma]$ if and only if f belongs to the Hardy spaces H^p in the half-planes $\mathbb{C}_+$, $\{z\colon \mathrm{Im} z > \sigma\}$ and $\{z\colon \mathrm{Im} z < -\sigma\}$. The following result was proved by B. Vynnytskyi (=Vinnitskii=Vinnitsky) (see [12]).

Theorem V. *The function*

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int\limits_{\partial D_\sigma} g(w)e^{zw}dw, \tag{1}$$

belongs to $H^2_\sigma(\mathbb{C}_+)$ if and only if $g \in E^2_*[D_\sigma]$ and the dual formula

$$g(w) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} G(x)e^{-xw}dx, \qquad Re\ w > 0 \tag{2}$$

is valid.

The purpose of this paper is to prove two theorems about $H^2_\sigma(\mathbb{C}_+)$ and some spaces of entire functions.

Theorem 1. *The function G defined by (1) belongs to $H^2_\sigma(\mathbb{C}_+), \sigma > 0$, and*

$$\lim_{x\to+\infty} \frac{\ln|G(x)|}{x} = -\infty \tag{3}$$

if and only if $g \in E^2_*[D_\sigma]$ and g is an entire function.

Theorem 2. *The function G defined by (1) belongs to $H^2_\sigma(\mathbb{C}_+), \sigma > 0$, and*

$$(\exists c_1 \in R) : G(z)\exp\left(\frac{2\sigma}{\pi} z\ln z - c_1 z\right) \in H^2(\mathbb{C}_+) \tag{4}$$

if and only if $g \in E^2_*[D_\sigma]$, g is an entire function and

$$(\exists c \in R) : g(w)\exp\left(-ce^{-\frac{w\pi}{2\sigma}}\right) \in E^2[D_\sigma]. \tag{5}$$

We remark that formula (4) is received in [11] as a condition of completeness of a functional system in $H^2_\sigma(\mathbb{C}_+)$.

Proof of Theorem 1. Necessity. Let $g \in E^2_*[D_\sigma]$ be an entire function. Then g is an analytic function in each closed rectangle $\overline{M}_k, k < 0$, where $M_k = \{z\colon z \in D_\sigma, \operatorname{Re} z > k\}$. By the Cauchy formula we obtain

$$\int\limits_{\partial M_k} g(w)e^{zw}dw = 0, k < 0,$$

then by Theorem V

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int\limits_{\partial D_\sigma \setminus \overline{M}_k} g(w)e^{zw}dw. \tag{6}$$

From the last formula we obtain

$$|G(x)| = \frac{1}{\sqrt{2\pi}} \int\limits_{\partial D_\sigma \setminus \overline{M}_k} |g(w)|e^{xu}|dw| = \frac{1}{\sqrt{2\pi}}(I_1 + I_2 + I_3),\ z = x + iy,\ w = u + iv,$$

for $x > 0$. Then, by the Schwarz inequality,

$$I_1 = \int\limits_{-\infty}^{k} |g(u - i\sigma)|e^{xu}du \leq \left(\int\limits_{-\infty}^{k} |g(u - i\sigma)|^2 du \cdot \int\limits_{-\infty}^{k} e^{2xu}\right)^{1/2} du \leq$$

$$\leq \left(\int\limits_{-\infty}^{0} |g(u - i\sigma)|^2 du \cdot \frac{e^{2xk}}{2x}\right)^{1/2} \leq c_2 \frac{e^{xk}}{\sqrt{x}},$$

analogously

$$I_3 = \int\limits_{-\infty}^{k} |g(u + i\sigma)|e^{xu}du \leq c_3 \frac{e^{xk}}{\sqrt{x}}.$$

Further

$$I_2 = \int\limits_{-\sigma}^{\sigma} |g(k + iv)|e^{xk}dv \leq \max_{v \in [-\sigma;\sigma]} \{|g(k + iv)|\}\, e^{xk} \int\limits_{-\sigma}^{\sigma} dv \leq J(k)e^{kx},$$

where $J(k) = 2\sigma \max\{|g(t + iv)|\colon v \in [-\sigma;\sigma], t \in [k;0]\}, k < 0$. If $\sup_{k<0}\{J(k)\} < +\infty$, then the function g belongs to the Hardy spaces in the both domains D_σ and D^*_σ. Then $g \equiv 0$ and by (1) $G \equiv 0$, hence the theorem is proved. On the other hand from the nonincreasity of J we have $\lim_{k \to -\infty} J(k) = +\infty$. Let J_1 be defined on the intervals of decreasing of J as $J_2 = J$. Then the inverse function J_1 of the function $-J$ increases on $(-\infty;0)$ and $\lim\limits_{s \to -\infty} J_1(s) = -\infty$. Since k in (6) is an arbitrary negative number, we can choose $k = J_1(-x)$, then $I_2 \leq J(J_1(-x))e^{xJ_1(-x)} = xe^{xJ_1(-x)}$. Hence $|G(x)| \leq c_4 xe^{xJ_1(-x)}, x > 1$, and we obtain

$$\lim_{x \to +\infty} \frac{\ln|G(x)|}{x} = \lim_{x \to +\infty} J_1(-x) = -\infty.$$

Sufficiency. Let $G \in H^2_\sigma(\mathbb{C}_+)$. Then from Theorem V the function g defined by (2) belongs to $E^2_*[D_\sigma]$. From formula (3) we obtain that the integral in the right-hand side of (2) converges uniformly on any compact subset of $\mathbb{C}_+$, hence g is an entire function.

Theorem 1 is proved. □

For the proof of Theorem 2 we need some auxiliary results.

Lemma 1. *If f is an analytic function in D_σ, has a.e. on ∂D_σ the angular boundary values, $f \in L^p[\partial D_\sigma], 1 \le p < +\infty$, and*

$$\sup_{u\in(-\infty;0)} \left\{ \int_{-\sigma}^{\sigma} |f(u+iv)|^p dv \right\} < +\infty,$$

then $f \in E^p[D_\sigma]$.

Proof. Function f has the angular boundary values a. e. on $\partial M_k, k < 0$ from inside M_k, $f \in L^p[\partial M_k]$. Hence [8, 7.1 and 6.4] by the Cauchy integral formula we obtain

$$f(w) = \frac{1}{2\pi i} \int_{\partial M_k} \frac{f(t)}{t-w} dt, \; w \in M_k.$$

From the estimate

$$\left| \int_{-\sigma}^{\sigma} \frac{f(k+is)}{k+is-w} ds \right| \le \int_{-\sigma}^{\sigma} \frac{|f(k+is)|}{\sqrt{(k-u)^2+(v-s)^2}} ds \le \frac{1}{|k-u|} \int_{-\sigma}^{\sigma} |f(k+is)| ds \le$$

$$\le \frac{c_4}{|k-u|} \left(\int_{-\sigma}^{\sigma} |f(k+is)|^p ds \right)^{1/p} \le \frac{c_5}{|k-u|}$$

by tending k to $-\infty$ we obtain

$$f(w) = \frac{1}{2\pi i} \int_{\partial D_\sigma} \frac{f(t)}{t-w} dt, \; w \in D_\sigma.$$

It means (see [12]) that $f \in E^p[D_\sigma]$.
The lemma is proved. □

The following result can be considered as an analogue to the Phragmen-Lindelof theorem for the half-strip.

Lemma 2. *If f is an analytic function in D_σ, has a.e. on ∂D_σ the angular boundary values, $f \in L^p[\partial D_\sigma], 1 \le p < +\infty$, and*

$$(\forall \varepsilon > 0): \sup_{u\in(-\infty;0)} \left\{ \int_{-\sigma}^{\sigma} |f(u+iv)|^p \exp\left(-\varepsilon e^{-\frac{\pi u}{2\sigma}} \cos\frac{\pi v}{2\sigma}\right) dv \right\} < +\infty,$$

then $f \in E^p[D_\sigma]$.

Proof. Let $f_\varepsilon(w) = f(w)\exp\left(-\varepsilon e^{-\frac{\pi w}{2\sigma}}\right)$. Then $|f_\varepsilon(u+iv)|^p = |f(u+iv)|^p \exp\left(-\varepsilon e^{-\frac{\pi u}{2\sigma}}\cos\frac{\pi v}{2\sigma}\right)$. Hence by the condition of the lemma $f_\varepsilon \in E^p[D_\sigma]$. From this we obtain (see [12])

$$\sup_{u\in(-\infty;0)}\left\{\int_{-\sigma}^{\sigma}|f_\varepsilon(u+iv)|^p dv\right\}^{1/p} \le \left\{\int_{\partial D_\sigma}|f_\varepsilon(w)|^p|dw|\right\}^{1/p} \le$$

$$\le \left\{\int_{\partial D_\sigma}|f(w)|^p|dw|\right\}^{1/p} < c_6 < +\infty.$$

By the Fatou lemma

$$\int_{-\sigma}^{\sigma}|f(u+iv)|^p dv \le \varliminf_{\varepsilon\to 0}\int_{-\sigma}^{\sigma}|f_\varepsilon(u+iv)|^p dv,$$

then from Lemma 1 we have $f \in E^p[D_\sigma]$.
The lemma is proved. □

Proof of Theorem 2. Necessity. Let $g \in E_*^2[D_\sigma]$ be an entire function. Then representation (1) is valid. Suppose that $c > 0$ in (5) (in the other case $g \equiv 0$ as in the proof of Theorem 1), then $g_1(w) := g(w)\exp\left(-ce^{-\frac{w\sigma}{2\sigma}}\right) \in E^2[D_\sigma]$. Hence by (6) for $k < 0$ we obtain

$$|G(x)| = \frac{1}{\sqrt{2\pi}}\left|\int_{\partial\left(D_\sigma\setminus\overline{M}_k\right)} g_1(w)\exp\left(ce^{-\frac{w\pi}{2\sigma}}\right)e^{wx}dw\right| \le$$

$$\le \frac{1}{\sqrt{2\pi}}\int_{\partial\left(D_\sigma\setminus\overline{M}_k\right)}|g_1(w)|\exp\left(ce^{-\frac{u\pi}{2\sigma}}\cos\frac{v\pi}{2\sigma}\right)e^{ux}|dw| =$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{k}|g_1(u-i\sigma)|\,e^{ux}du + \int_{-\sigma}^{\sigma}|g_1(k+iv)|\exp\left(ce^{-\frac{k\pi}{2\sigma}}\cos\frac{v\pi}{2\sigma}\right)e^{kx}dv+\right.$$

$$\left. + \int_{-\infty}^{k}|g_1(u+i\sigma)|\,e^{ux}du\right) \le$$

$$\le \frac{1}{\sqrt{2\pi}}\left(\frac{e^{kx}}{\sqrt{2x}}\left(\int_{-\infty}^{0}|g_1(u-i\sigma)|^2\,du\right)^{1/2} + \exp\left(ce^{-\frac{k\pi}{2\sigma}}\right)e^{kx}\sqrt{2\sigma}\times\right.$$

$$\left.\times\left(\int_{-\sigma}^{\sigma}|g_1(k+iv)|^2\,dv\right)^{1/2} + \frac{e^{kx}}{\sqrt{2x}}\left(\int_{-\infty}^{0}|g_1(u+i\sigma)|^2\,du\right)^{1/2}\right).$$

The latter is a consequence of the Schwarz inequality. If $k = -\frac{2\sigma}{\pi}\ln x$, then

$$|G(x)| \le \frac{c_7}{\sqrt{x}} \exp\left(-\frac{2\sigma}{\pi} x \ln x\right) + c_8 e^{cx} \exp\left(-\frac{2\sigma}{\pi} x \ln x\right) \le$$

$$\le c_9 e^{cx} \exp\left(-\frac{2\sigma}{\pi} x \ln x\right), x > 1.$$

Let $\psi(z) \colon = G(z) e^{-cz} \exp\left(-\frac{2\sigma}{\pi} z \ln z\right)$. Obviously $\psi \in L^2(\partial \mathbb{C}_+)$ and for all $\varepsilon > 0$ we have $\psi(x)e^{-\varepsilon x} \in L^2(0; +\infty)$ and by Theorem V we obtain $G \in H^2_\sigma(\mathbb{C}_+)$, hence for all $\gamma \in (1; 2]$

$$(\forall \varepsilon > 0) : \sup_{|\varphi| < \frac{\pi}{2}} \left\{ \int_0^{+\infty} \left|\psi\left(re^{i\varphi}\right)\right|^2 \exp\left(-\varepsilon r e^\gamma\right) dr \right\} < +\infty.$$

From a Phragmen-Lindelöf type theorem for half-plane [5], [15] we obtain $\psi \in H^2(\mathbb{C}_+)$. This means

$$G(z) e^{cz} \exp\left(\frac{2\sigma}{\pi} z \ln z\right) \in H^2(\mathbb{C}_+),$$

and formula (4) is valid.
Vice versa, let $G \in H^2_\sigma(\mathbb{C}_+)$ and (4) is valid. Then by Theorem V equality (2) is valid for all $w \in \mathbb{C}_+$. Let $G_1(z) = G(z)\exp\left(\frac{2\sigma}{\pi} z \ln z - cz\right) \in H^2(\mathbb{C}_+)$ for some $c > 0$. Then after change of the line of integration from $\{x \colon x > 0\}$ to $\left\{t \exp\left(-\frac{(w-c)\pi}{2\sigma} : t > 0\right)\right\}$ we obtain

$$|g(w)| = \frac{1}{\sqrt{2\pi}} \left| \int_0^{+\infty} G_1\left(te^{-\frac{(w-c)\pi}{2\sigma}}\right) \exp\left(-\frac{2\sigma}{\pi} t \ln t e^{-\frac{(w-c)\pi}{2\sigma}}\right) e^{-\frac{(w-c)\pi}{2\sigma}} dt \right| \le$$

$$\frac{1}{\sqrt{2\pi}} \left(\int_0^{+\infty} \left| G_1\left(te^{-\frac{(w-c)\pi}{2\sigma}}\right) e^{-\frac{(w-c)\pi}{2\sigma}} \right|^2 dt \cdot \int_0^{+\infty} \left| \exp\left(-\frac{4\sigma}{\pi} t \ln t e^{-\frac{(w-c)\pi}{2\sigma}}\right) \right|^2 dt \right)^{1/2} \le$$

$$\le c_{10} \left(e^{-\frac{(u-c)\pi}{2\sigma}} \exp\left(\frac{1}{2} \ln\left(\frac{4\sigma}{\pi} e^{-\frac{(u-c)\pi}{2\sigma}} \cos\frac{v\pi}{2\sigma}\right)\right) \cdot\right.$$

$$\left. \cdot \exp\left(\exp\left(\ln\left(\frac{4\sigma}{\pi} e^{-\frac{(u-c)\pi}{2\sigma}} \cos\frac{v\pi}{2\sigma}\right) - 1\right)\right)\right)^{1/2}.$$

The latter estimate follows from [2, p.326]. Then we have

$$|g(w)| \le c_{11} e^{-\frac{2u\pi}{\sigma}} \exp\left(\frac{1}{e}\frac{2\sigma}{\pi} e^{-\frac{(u-c)\pi}{2\sigma}} \cos\frac{v\pi}{2\sigma}\right).$$

The function $g_2(w) = g(w)\exp\left(-c_{12} e^{-\frac{w\pi}{2\sigma}}\right)$, where $c_{12} = \frac{2\sigma}{\pi e} e^{\frac{c\pi}{2\sigma}}$, satisfies the conditions of Lemma 2 because $g_2 \in L^2\left(\partial D_\sigma\right)$. Then $g_2 \in E^2[D_\sigma]$, hence condition (5) is valid for $c = c_{12}$.

Theorem 2 is proved. □

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Received 7.12.2009