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INDEPENDENT SETS AND PARTITIONS OF GRAPHS

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Let $\Gamma(V,E)$ be a connected graph with the set of vertices V and the set of edges E, d be the path metric on V, $B(v,r)=\{x\in V\colon d(v,x)\le r\}$ be the ball of radius r with the center $v\in V$. A subset $S\subseteq V$ is of finite index if there exists r such that $\bigcup_{v\in S}B(v,r)=V$, and the minimal r satisfying this condition is called the index of S. The paper consists of two parts. In the first part we use the independent subsets in $\Gamma(V,E)$ to partition V in the subsets of controlled finite indices. In the second part we propose an approach to the balanced partitions of infinite graphs in the subsets of finite index.

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Пусть $\Gamma(V,E)$ — связный граф с множеством вершин V и множеством ребер E, d — естественная метрика на V, $B(v,r)=\{x\in V\colon d(v,x)\le r\}$ — шар радиуса r с центром в вершине $v\in V$. Подмножество $S\subseteq V$ имеет конечный индекс, если существует число r такое, что $\bigcup_{v\in S}B(v,r)=V$, а минимальное число r, удовлетворяющее этому условию, называется индексом S. Статья состоит из двух частей. В первой части независимые подмножества $\Gamma(V,E)$ применяются для разбиения V на подмножества контролируемых конечных индексов. Во второй части предлагается подход к уравновешенным разбиениям бесконечных графов на подмножества конечных индексов.

1. Introduction. A *graph* $\Gamma=\Gamma(V,E)$ consists of a set V of *vertices* and a set E of *edges*, where each edge is an unordered pair $\{a,b\}$ of distinct vertices (thus we do not allow loops). All graphs under consideration are supposed to be connected. For any $a,b\in V$, we denote by $d(a,b)$ the length of a shortest path between a,b. Given any $a\in V$, $A\subseteq V$ and a non-negative integer m, we put

$$B(a,m)=\{x\in V\colon d(a,x)\le m\}, B(A,m)=\textstyle\bigcup_{a\in A}B(a,m).$$

A subset $A\subseteq V$ is of finite index if there exists m such that $B(A,m)=V$. In this case we define the *index* of A by $\operatorname{ind}A=\min\{m\colon B(A,m)=V\}$. The following two theorems are from [7] (see also [3, Theorems 2.1 and 2.3])

Theorem 1. *Let $\Gamma(V,E)$ be a graph, r be a natural number such that $|V|\ge r$. Then there exists a partition $V=V_1\cup V_2\cup\ldots\cup V_r$ such that $\operatorname{ind}V_i\le r-1$ for each $i\in\{1,\ldots,r\}$.*

Let X be a finite set, $|X|=n$, r be a natural number , $1\le r\le n$, $n=rs+t$, $0\le t<r$. A partition $X=X_1\cup X_2\cup\ldots\cup X_r$ is called *balanced* if

$$|X_1|=|X_2|=\ldots=|X_t|=s+1,\ |X_{t+1}|=|X_{t+2}|=\ldots=|X_r|=s.$$

In the case $r|n$, we have $|X_1|=|X_2|=\ldots=|X_r|$.

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Theorem 2. *Let $\Gamma(V, E)$ be a finite graph, r be a natural number such that $|V| \geq r$. Then there exists a balanced partition $V = V_1 \cup V_2 \cup ... \cup V_r$ such that* ind $V_i \leq r$ *for each* $i \in \{1, ..., r\}$.

A graph $\Gamma(V, E)$ is *k-regular* if $|B(x, 1)| = k + 1$ for every $x \in V$. A k-regular graph $\Gamma(V, E)$ is called *kaleidoscopical* if there exists a coloring $\chi \colon V \longrightarrow \{0, 1, ..., k\}$ such that each ball $B(x, 1)$ has no distinct monochrome points. The coloring χ determines the partition $V = \chi^{-1}(0) \cup \chi^{-1}(1) \cup ... \cup \chi^{-1}(k)$ and ind $\chi^{-1}(i) = 1$ for each $i \in \{0, 1, ..., k\}$. If $\Gamma(V, E)$ is finite, then $k+1||V|$ and this partition is balanced. For kaleidoscopical graphs see [3, Chapter 6], [4], [5], [6], for some graphs closed to kaleidoscopical see [1], [9], [10].

By [3, Theorem 3.4], for every infinite graph $\Gamma(V, E)$, the set V can be partitioned in countably many subsets of finite index. This is one of key points in the proof of Theorem 3.12 from [3] stating that every infinite group can be partitioned in countably many large subsets.

For general combinatorial motivation to study the partitions of graphs in subsets of finite index see the Preface to [3].

The paper consists of two parts. In the first part we use the independent subsets in a graph $\Gamma(V, E)$ to partition V in the subsets of controlled finite indices. In Theorem 3 and 7 we deal with the finite partitions, the remaining theorems concern the infinite partitions. In the second part we prove a refinement of Theorem 2 for $r = 2$ and propose an approach to the balanced partitions of infinite graphs.

2. Partitions based on independent sets. Given a graph $\Gamma(V, E)$, a subset $S \subseteq V$ is called *independent* if $d(u, v) > 1$ for all $u, v \in S$. The existence of a maximal (by inclusion) independent set in a finite graph is evident, to construct such a subset in an infinite graph one may use the Axiom of Choice, say, in the form of Zorn Lemma.

Lemma 1. *Let Y be a maximal independent set in a graph $\Gamma(V, E)$ with $|V| > 1$. Then* ind $Y =$ ind $(V \setminus Y) = 1$.

Proof. It follows directly from the definition of a maximal independent set that each unit ball in Γ contains at least one point from Y and at least one point from $V \setminus Y$. □

Lemma 2. *Let Y be a maximal independent set in a graph $\Gamma(V, E)$. Then, for any $u, v \in Y$, there exist $y_1, ..., y_n \in Y$ such that $y_1 = u$, $y_n = v$ and $d(y_i, y_{i+1}) \leq 3$ for every $i \in \{1, 2, ..., n-1\}$.*

Proof. We define an equivalence $\sim$ on Y by the rule: $u \sim v$ if and only if there exist $y_1, ..., y_n \in Y$ such that $y_1 = u$, $y_n = v$ and $d(y_i, y_{i+1}) \leq 3$ for every $i \in \{1, 2, ..., n-1\}$. We show that there is only one $\sim$-class. Suppose the contrary. We define the distance between two $\sim$-classes A, B as the minimal distance between representatives $a \in A$, $b \in B$ of these classes. Then we chose two distinct classes A_0, B_0 with the minimal distance p between them. Clearly, $p > 3$. We pick $a_0 \in A_0$, $b_0 \in B_0$ so that $d(a_0, b_0) = p$. Let $x_0 = a_0$, $x_1, x_2, ..., x_p = b_0$ be the shortest path between a_0 and b_0. Since Y is a maximal independent set in Γ, $B(x_2, 1) \cap Y \neq \varnothing$. For $y \in B(x_2, 1) \cap Y$, we have $d(a_0, y) \leq 3$, so $y \in Y_0$. Then $d(y, b_0) \leq p - 1$, contradicting the choice of a_0, b_0. □

Lemma 3. *For any graph $\Gamma(V, E)$, there exists a maximal independent set Y such that*

(∗) *for any $u, v \in Y$, there exist $y_1, ..., y_n \in Y$ such that $y_1 = u$, $y_n = v$ and $d(y_i, y_{i+1}) \leq 2$ for every $i \in \{1, 2, ..., n-1\}$.*

Proof. Using Zorn Lemma, we choose a maximal by inclusion independent set Y satisfying $(*)$. Suppose that Y is not a maximal independent set in Γ. Then the set $Z = \{v \in V\colon B(v,1) \cap Y = \varnothing\}$ is non-empty. We choose $z \in Z$ at the minimal distance p to Y, and $y \in Y$ such that $d(y,z) = p$. Let $v_0, v_1, ..., v_p$ be the shortest path from $z = v_0$ to $y = v_p$. By the definition of Z, $v_1 \notin Y$. By the choice of z, y, we have $v_1 \notin Z$. Hence, $B(v_1, 1) \cap Y \neq \varnothing$, so we can join $z \in Y$ so that $Y \cup \{z\}$ is independent and satisfies $(*)$, contradicting the choice of Y. □

Remark 1. By Turan's theorem [8], for every finite graph $\Gamma(V,E)$, there exists an independent set $S \subset V$ such that $|S| \geq \sum_{v\in V} |B(v,1)|^{-1}$. It would be interesting to find an analogue of this theorem for independent sets satisfying $(*)$ of Lemma 3.

Given a graph $\Gamma(V,E)$, we put $\operatorname{diam}\Gamma(V,E) = \sup\{d(u,v)\colon u, v \in V\}$. If $\operatorname{diam}\Gamma(V,E) = \aleph_0$, we put $\operatorname{rad}\Gamma = \aleph_0$. If $\operatorname{diam}\Gamma < \aleph_0$ and $v \in V$, we put

$$r(v) = \max\{d(v,x)\colon x \in V\},\ \operatorname{rad}\Gamma = \min\{r(v)\colon v \in V\}.$$

Theorem 3. *Let $\Gamma(V,E)$ be a graph, r be a natural number. If $\operatorname{rad}\Gamma(V,E) > 2^{r-1} - 1$, then V can be partitioned $V = V_1 \cup V_2, ..., V_{r+1}$ so that $\operatorname{ind}\ V_1 = 1$, $\operatorname{ind}\ V_2 \leq 3$, ... , $\operatorname{ind}\ V_i \leq 2^i - 1$, ..., $\operatorname{ind}\ V_r \leq 2^r - 1$, $\operatorname{ind}\ V_{r+1} \leq 2^r - 1$.*

Proof. We choose a maximal independent set Y_1 satisfying Lemma 3, and put $V_1 = V \setminus Y_1$. If $r = 1$, put $V_2 = Y_1$ and apply Lemma 1.

Suppose that $r > 1$ and construct the auxiliary graph $\Gamma_1(Y_1, E_1)$, where $\{u, v\} \in E_1$ if and only if $d_\Gamma(u, v) = 2$. Since Y_1 satisfies Lemma 3, Γ_1 is connected. Since $\operatorname{rad}\Gamma > 2^{r-1} - 1$ and $r > 1$, we have $|Y_1| > 1$. Apply Lemma 3 to Γ_1 and choose corresponding independent set $Y_2 \subset Y_1$. Put $V_2 = Y_1 \setminus Y_2$. Every ball of radius 2 in Γ centered in Y_2 contains at least one point from Y_2, and every ball of radius 2 in Γ centered in Y_2 contains at least one point from Y_2. Moreover, every unit ball in Γ with the center in Y_2 has at least one point from Y_1. Since $Y_1 = V_2 \cup Y_2$, then

$$\operatorname{ind}\ Y_2 \leq 3,\ \operatorname{ind}\ Y_2 \leq 3.$$

If $r = 2$, we put $V_3 = Y_2$.

Suppose that $r > 2$ and construct the auxiliary graph $\Gamma_2(Y_2, E_2)$, where $\{u, v\} \in E_2$ if and only if $d_\Gamma(u, v) = 4$. Since Y_2 satisfies Lemma 3 for Γ_2, we see that Γ_2 is connected. Since $\operatorname{rad}\Gamma > 2^{r-1} - 1 \geq 3$, we see that $|Y_2| > 1$. We apply Lemma 3 to Γ_2 and choose corresponding independent subset $Y_3 \subset Y_2$. Put $V_3 = Y_2 \setminus Y_3$. Every ball of radius 4 in Γ centered in Y_3 has at least one point from V_3 and vice versa. Every ball of radius $1 + 2$ centered in $V_1 \cup V_2$ has at least one point from Y_2. Since $Y_2 = Y_3 \cup V_3$, we have

$$\operatorname{ind}\ V_3 \leq 1 + 2 + 2^2, \operatorname{ind}\ Y_3 \leq 1 + 2 + 2^2.$$

If $r = 3$, we put $V_4 = Y_3$.

At last, suppose that we have constructed the subsets $V_1, V_2, ..., V_i$, $i < r + 1$, and the subset Y_i such that the corresponding graph $\Gamma_i(Y_i, E_i)$ is connected and $\operatorname{ind}\ Y_i \leq 2^i - 1$. Applying above arguments, we partition Y_i in two subsets V_{i+1} and Y_{i+1}. □

Theorem 4. *For every infinite graph $\Gamma(V,E)$, there exists a partition $V = V_1 \cup V_2 \cup ...$ such that each subset V_i is of finite index.*

Proof. If $\operatorname{rad}\Gamma = \aleph_0$, we apply the construction from the proof of Theorem 3 to get the pairwise disjoint subsets $V_1, V_2, ...$ such that $\operatorname{ind}\ V_i \leq 2^i - 1$. If $\operatorname{rad}\Gamma < \aleph_0$, then every singleton in V is of finite index. □

Remark 2. For an alternative proof of Theorem 4 see [2].

Remark 3. Let κ be an infinite cardinal, $\kappa > \aleph_0$, $\Gamma(V, E)$ be a graph such that $|V| = \kappa$. In view of Theorem 4, it is naturally to ask whether V can be partitioned in κ subsets of finite index. To give the negative answer, we fix an arbitrary graph $\Gamma'(V', E')$ with $|V'| = \kappa$. Then we fix an arbitrary vertex $x \in V$, take a countable set $Y = \{y_n : n \in \omega\}$ such that $Y \cap V = \varnothing$, and put

$$V = V' \cup Y, E = E' \cup \{x_0, y_0\} \cup \{\{y_n, y_{n+1}\} : n \in \omega\}.$$

Thus, we have defined a graph $\Gamma(V, E)$ with $|V| = \kappa$. We note that every subset X of V of finite index contains some infinite subset of Y. Hence, V cannot be partitioned in uncountably many subsets of finite index.

In [7] Theorem 1 was proved in the following stronger form: given a graph $\Gamma(V, E)$ and a natural number r such that $|V| \geq r$, there exists an r-coloring $\chi : V \longrightarrow \{1, 2, ..., r\}$ such that $|\chi(B(v, k))| \geq k + 1$ for all $v \in V$, $k \in \{0, 1, ..., r - 1\}$.

Theorem 5. *For every infinite graph $\Gamma(V, E)$, there exists a coloring $\chi : V \longrightarrow \omega$ such that $|\chi(B(v, k))| \geq k + 1$ for all $v \in V$, $k \in \omega$.*

Proof. If every ball $B(v, 1)$, $v \in V$ is finite, then V is countable and we can take an arbitrary bijection $\chi : V \longrightarrow \omega$.

Suppose that some vertex $z \in V$ is the center of the infinite ball $B(z, 1)$. For each $i \in \omega$, we denote

$$S_{i+1} = B(z, i + 1) \setminus B(z, i),$$

and put $N_1 = \{1, 3, 5, ...\}$. We put $\chi(z) = 0$ and extend χ onto S_1 so that $\chi(S_1) = N_1$. For each $x \in S_i$, $i \geq 2$, we put $\chi(x) = 2(i - 1)$. Thus, we have defined the coloring $\chi : V \longrightarrow \omega$. If $x \in S_1 \cup \{z\}$, then $|\chi(B(x, 1))| \geq 1$ and $\chi(B(x, 2))$ is infinite. If $x \in S_i$, $i \geq 2$, then $|\chi(B(x, k))| \geq k$ for every $k \leq i$, and $\chi(B(x, k))$ is infinite for each $k > i$. □

Given a graph $\Gamma(V, E)$ and a natural number m, a subset $Y \subset V$ is called *m-independent* if $d(u, v) > m$ for all $u, v \in Y$.

Theorem 6. *Let $\Gamma(V, E)$ be a graph, m be a natural number, κ be a cardinal. If $|(B(v, m))| \geq \kappa$ for every $v \in V$, then V can be partitioned in κ subsets of index $\leq 3m$.*

Proof. Let Y be a maximal m-independent subset of V. Then any two balls of radius m centered in distinct point of Y are disjoint. In each ball we take κ points and color them in κ distinct colors. Extend this coloring to κ-coloring of V arbitrarily. Since Y is a maximal m-independent subset, for every $v \in V$, there exists $y \in V$ such that $B(v, m) \cap B(y, m) \neq \varnothing$. Thus, $d(y, v) \leq 2m$ and $B(y, m)$ contains the points of all colors. It follows that $B(v, 3m)$ has the points of all colors, so this coloring determines desired partition of V. □

Theorem 7. *Let $\Gamma(V, E)$ be a graph, $m_1, m_2, ..., m_n$ be natural numbers. Suppose that $|B(v, m_i)| \geq 2^i$ for all $v \in V$, $i \in \{1, 2, ..., n\}$. Then there exists a partition $V = V_1 \cup V_2 \cup ... \cup V_n$ such that*

$$\text{ind } V_i \leq 2(m_i + 2m_{i-1} + 2^2 m_{i-2} + ... + 2^{i-1}_{m_1})$$

for every $i \in \{1, 2, ..., n\}$.

Proof. We fix an arbitrary maximal $2m_1$-independent set V_1. Clearly, ind $V_1 \le 2m_1$. Put $U_1 = V \setminus V_1$. We choose an arbitrary $j \in \{2, 3, ..., n\}$, $v \in U_1$ and show that $|B(v, m_1 + m_j) \cap U_1| \ge 2^{j-1}$. By the assumption, $|B(v, m_j)| \ge 2^j$. If $|B(v, m_j) \cap U_1| \ge 2^{j-1}$, the statement is evident. Suppose that $|B(v, m_j) \cap V_1| > 2^{j-1}$. Since $B(u, m_1) \cap U_1 \ne \varnothing$ for each $u \in V_1$, we have $|B(v, m_1 + m_j) \cap U_1| \ge 2^{j-1}$. Clearly, ind $V_1 \le m_1$.

Assume that, for some $k < n$, we have constructed the subsets $V_1, V_2, ..., V_k$ of V such that, for the subsets $U_i = V \setminus (V_1 \cup V_2 \cup ... \cup V_i)$, $i \in \{1, 2, ..., k\}$, the following conditions hold

(1) ind $V_i \le 2(m_i + 2m_{i-1} + ... + 2^{i-1}m_1)$; (2) ind $U_i \le 2(m_i + 2m_{i-1} + ... + 2^{i-1}m_1)$;

(3) for all $j \in \{k+1, k+2, ..., n\}$ and $v \in U_k$, $B(v, m_j + m_k + 2m_{k-1} + ... + 2^{k-1}m_1) \cap U_k| \ge$ $\ge 2^{j-k}$.

We put $s = m_{k+1} + m_k + 2m_{k-1} + ... + 2^{k-1}m_1$. By (3), $|B(v, s) \cap U_k| \ge 2$ for every $v \in U_k$. We choose a maximal by inclusion $2s$-independent subset V_{k+1} of U_k. Since $U_k \subseteq B(V_k, 2s)$ and ind $U_k \le 2(m_k + 2m_{k-1} + ... + 2^{k-1}m_1)$, we have

$$\text{ind } U_{k+1} \le 2s + 2(m_k + 2m_{k-1} + ... + 2^{k-1}m_1) = 2(m_{k+1} + 2m_k + ... + 2^k m_1).$$

Thus, (1) holds for V_k. Put $U_{k+1} = V \setminus (V_1 \cup ... \cup V_{k+1})$. Since $|B(v, s) \cap U_{k+1}| \ge 1$ for each $v \in U_{k+1}$, we have $U_k \subseteq B(U_{k+1}, s)$. Hence,

$$\text{ind } U_{k+1} \le s + m_k + 2m_{k-1} + ... + 2m_1 = m_{k+1} + 2m_k + ... + 2^k m_1,$$

and (2) holds for U_{k+1}. To verify (3) for U_{k+1}, we fix $j \in \{k+2, k+3, ..., n\}$ and $v \in U_{k+1}$. If $|B(v, m_j + m_k + 2m_{k-1} + ... + 2^{k-1}m_1) \cap U_{k+1}| \ge 2^{j-k-1}$ the statement is evident. Otherwise, $|B(v, m_j + m_k + 2m_{k-1} + ... + 2^{k-1}m_1) \cap U_{k+1}| \ge 2^{j-k-1}$. On the other hand $B(u, s) \cap U_{k+1} \ne \varnothing$ for each $u \in V_{k+1}$. Since V_{k+1} is $2s$-independent, (3) holds for V_{k+1}. □

Theorem 8. *Let $\Gamma(V, E)$ be an infinite graph, $m_1, m_2, ...$ be a sequence of natural numbers. Suppose that $|B(v, m_i)| \ge 2^i$ for all i and $v \in V$. Then V can be partitioned $V = V_1 \cup V_2 \cup ...$ so that* ind $V_i \le 2(m_i + 2m_{i-1} + 2^2 m_{i-2} + ... + 2^{i-1}m_1)$ *for every natural number i.*

Proof. Apply arguments proving Theorem 7. □

Lemma 4. *Let $\Gamma(V, E)$ be an infinite graph, $\kappa_1, \kappa_2, ...$ be an increasing sequence of infinite cardinals, $\kappa = \sup\{\kappa_1, \kappa_2, ...\}$. Suppose that, for every natural number n, there exists a natural number m_n such that $|B(v, m_n)| \ge \kappa_n$ for each $v \in V$. Then V can be partitioned in κ subsets of finite index.*

Proof. We put $V_1 = V$ and choose a maximal $2m_1$-independent subset X_1 of V_1. For each $v \in X_1$, in the ball $B(v, m_1)$ we fix two disjoint subsets $V_1(v)$, $Z_1(v)$ of cardinality κ_1. For every $v \in X_1$, we pick one point from $Y_1(v)$ and get a subset of index $\le 3m_1$. Thus, $Y_1 = \bigcup_{v \in X_1} Y_1(v)$ can be partitioned in κ_1 subsets of finite index.

We put $V_2 = V_1 \setminus Y_1$. Since V_2 contains the subset $Z = \bigcup_{v \in X_1} Z_1(v)$, then ind $V_2 \le 3m_1$. We show that, for every natural number n, $n > 1$, there exists a natural number m'_n such that, for every $v \in V_2$,

$$|B(v, m'_n) \cap V_2| \ge \kappa_n.$$

To this end, we fix $v \in V_2$ and put $m'_n = m_n + 2m_1$. By the assumption, $|B(v, m_n)| \ge \kappa_n$. If $B(v, m_n)$ contains κ_n vertices from V_2, the statement is true because $m_n < m'_n$. Otherwise, there exists a subset $S \subseteq B(v, m_n) \cap Y_1$ of cardinality κ_n. For each $x \in X_1$, we fix some bijection $f_x \colon Y_1(x) \longrightarrow Z_1(x)$ and denote by f the bijection between Y_1 and Z_1 such that $f|_{Y_1(x)} \equiv f(x)$ for each $x \in X_1$. Since $d(y, f(y)) \le 2m_1$ for every $y \in Y_1$, we have $f(S) \subseteq B(v, m_n + 2m_1)$, and it suffices to note that $f(S) \subseteq V_2$.

On the second step we choose $2m_2'$-independent subset X_2 of V_2 maximal by inclusion. For each $v \in X_2$, we fix two disjoint subsets $Y_2(v)$, $Z_2(v)$ such that $Y_2(v) \subseteq B(v, m_2' \cap V_2)$, $Z_2(v) \subseteq B(v, m_2' \cap V_2)$, and $Y_2(v) = Z_2(v) = \kappa_2$. We pick one element from every subset $Y_2(v)$ and get a subset of index $3m_2' + 3m_1$. Thus, $Y_2 = \bigcup_{v \in X_2} Y_2(v)$ can be partitioned in κ_1 subsets of finite index. Put $V_3 = V_2 \setminus Y_2$. Then V_3 is of finite index and, for every natural number $n, n > 2$, there exists a natural number n_n'' such that each ball $B(v, m_n'')$, $v \in V_2$ contains κ_n vertices from V_3.

After ω step we get κ pairwise disjoint subsets of finite index. □

Theorem 9. *Let $\Gamma(V,E)$ be an infinite graph, κ be a limit cardinal, $|V| = \kappa$. Suppose that, for each $\kappa' < \kappa$, there exists a natural number m such that $|B(v,m)| \geq \kappa'$ for every $v \in V$. Then V can be partitioned in κ subsets of finite index.*

Proof. For every natural number m, we put

$$\kappa_m = \min\{|B(x,m)| \colon x \in V\}, \; s = \sup\{\kappa_1, \kappa_1, ...\}.$$

We show that $\kappa = s$. Since $\kappa_m \leq \kappa$ for each m, we see that $s \leq \kappa$. Assume that $s < \kappa$. Since κ is a limit cardinal, there exists κ' such that $s < \kappa' < \kappa$. By the assumption of theorem, there exists a natural number m_0 such that $|B(v, m_0)| \geq \kappa'$ for each $v \in V$. Hence, $\kappa' \leq \kappa_{m_0}$ and $\kappa_{m_0} > s$, contradicting the choice of s. We consider two cases.

Case 1. There exists a natural number n such that $\kappa_n = \kappa$. By Theorem 6, V can be partitioned in κ subsets of index $3n$.

Case 2. For every natural number m, $\kappa_m < \kappa$. Apply Lemma 4. □

Remark 4. Is Theorem 9 true for an infinite non-limit cardinal κ? To give the negative answer, let $\kappa = s^+$. We take the complete graph $\Gamma(V_1, E_1)$ with $|V_1| = \kappa$, fix some vertex $x \in V$ and denote by $\Gamma(V_2, E_2)$ an s-regular tree with the root y. We suppose that $V_1 \cap V_2 = \varnothing$. Then we define the graph $\Gamma(V,E)$ putting

$$V = V_1 \cup V_2, E = E_1 \cup E_2 \cup \{x, y\}.$$

Each unit ball in $\Gamma(V,E)$ has at least s point. Every subset of finite index in V meets V_2. Since $|V_2| = s$, V cannot be partitioned in $> s$ subsets of finite index.

Theorem 10. *Let $\Gamma(V,E)$ be a graph, κ be an infinite cardinal such that $|B(v,1)| = \kappa$ for each $v \in V$. Then V can be partitioned in κ subsets of index 1.*

Proof. We note that $|V| = \kappa$ and construct a subset $X \subseteq V$, $|X| = \kappa$ and κ-coloring χ of X such that each unit ball in Γ has vertices of all colors. Let $\{B_\alpha \colon \alpha < \kappa\}$ be an enumeration of the set of unit balls in Γ. We fix a vertex $x(0,0)$ in B_0 and put $X_0 = \{x(0,0)\}$, $\chi(x(0,0)) = 0$. Assume that, for some cardinal $\gamma < \kappa$, we have constructed the subsets $X_{\gamma'} = \{x(\alpha,\beta) \colon \alpha, \beta < \gamma'\}$, $\gamma' < \gamma$, and the colorings of $X_{\gamma'}$, $\gamma' < \gamma$ such that every ball B_λ, $\lambda < \gamma'$, contains the points of each color from $[0, \gamma']$.

Now we construct X_γ. Since each unit ball in Γ is of cardinality κ and we have colored $< \kappa$ vertices, we can choose distinct non-colored vertices $x(\alpha, \gamma) \in B_\alpha$, $\alpha \leq \gamma$ and $x(\gamma, \beta)$, $\beta < \gamma$. Put $\chi(x(\alpha,\gamma)) = \gamma$, $\alpha \leq \gamma$ and $\chi(x(\gamma,\beta)) = \beta$, $\beta < \gamma$. Thus, each ball B_λ, $\lambda \leq \gamma$ has the point of all γ colors. After κ steps we get a desired set $X = \bigcup_{\gamma < \kappa} X_\gamma$. □

3. Balanced partitions. We begin with the following refinement of Theorem 2 for $r = 2$.

Theorem 11. *For every finite graph $\Gamma(V,E)$, $|V| = n$, $n \geq 2$, there exists a balanced coloring $V = V_1 \cup V_2$ such that* $\operatorname{ind} V_1 = 1$, $\operatorname{ind} V_2 = 2$.

Proof. Passing to a spanning tree, we may suppose that Γ itself is a tree. We define a balanced coloring of V in black and white such that each unit ball has a black point and each ball of radius 2 has a white point. We proceed inductively by n. For $n=2$, the statement is evident. Suppose that we have defined corresponding coloring for all trees with $< n$ vertices.

Let d be a distance from the root x of Γ to the remotest vertex x_d of Γ. If $d=1$ we color x in black and the remaining vertices in black and white to get a balanced coloring. Suppose that $d>1$. Let $x=x_0, x_1, ..., x_{d-2}, x_{d-1}, x_d$ be the path from x to x_d. We delete the edge $\{x_{d-2}, x_{d-1}\}$. After that Γ splits in two trees: Γ_1 is the tree with root x, Γ_2 is the tree with the root x_{d-1}. Applying the inductive assumption, we color Γ_1 so that each unit ball has a black point, each ball of radius 2 has a white point and this coloring is balanced. Then we color x_{d-2} in black, x_{d-1} in white and extend this coloring to a balanced coloring of V. □

Remark 5. In connection with Theorem 11 we define two games between two players. In both games the game field is a finite graph, at each step the first player color a vertex in black and the second player in white until all vertices are colored. In the first game the second player win if the set of white vertices is of index ≤ 2. In the second game the first player win if the set of black points is of index ≤ 1. For every finite graph, in the first game the second player has a winning strategy, while in the second game the first player has a winning strategy only for some special graphs.

Let $\Gamma(V,E)$ be a countable graph. We say that a sequence $(A_n)_{n\in\omega}$ of finite subsets of V is *covering* if

(i) $V=\cup_{n\in\omega}A_n$; (ii) $A_n\subseteq A_m$ for all $n\leq m$;

(iii) every induced subgraph $\Gamma[A_n]$ is connected.

For every subset $B\subseteq V$ we define the sequence $(p_n(B))_{n\in\omega}$ by the rule $p_n(B)=\frac{|B\cap A_n|}{A_n}$.

If this sequence is convergent and $p(B)=\lim_{n\to\infty}p_n(B)$, we say that $p(B)$ is the *density of* B *with respect to the covering sequence* $(A_n)_{n\in\omega}$. We say that a partition $V=V_1\cup V_2\cup ...\cup V_r$ is balanced with respect to the covering sequence $(A_n)_n$ if $p(V_1)=p(V_2)=...=p(V_r)=\frac{1}{r}$.

Recall that a graph Γ is *locally finite* if every unit ball in Γ is finite.

Theorem 12. *For every locally finite countable graph* $\Gamma(V,E)$ *and every natural number* r, *there exists a partition* $V=V_1\cup ...\cup V_r$ *balanced with respect to some covering sequence* $(A_n)_{n\in\omega}$ *and such that* ind $V_i\leq r$ *for each* $i\in\{1,...,r\}$.

To prove Theorem 12, we need the following auxiliary lemma.

Lemma 5. *Let* $\Gamma(V,E)$ *be a countable locally finite graph,* r *be a natural number. Then there exists a partition* $V=\bigcup_{n\in\omega}U_n$ *in finite subsets such that, for all* $n\in\omega$,

(i) $|U_n|\geq r$; (ii) *induced subgraph* $\Gamma[U_n]$ *is connected;*

(iii) induced subgraph $\Gamma[U_0\cup U_1\cup ...\cup U_n]$ *is connected.*

Proof. We may suppose that Γ is a tree with the root $x\in V$. For each $\kappa\in\omega$, the κ*-level* of Γ is the set of all vertices at distance κ from. Since Γ is locally finite, each κ-level is finite. We color the set of vertices V in yellow and green by the following rule. A vertex $v\in V$ is colored in green if and only if the set of all infinite paths from x going through v is infinite. Clearly, the set of green vertices is infinite, but the set of yellow vertices could be empty.

We arrange the set of green vertices as follows: x is the first vertex in the list, then we enumerate the green vertices on the 1-level, then on the 2-level and so on.

We insert to U_0 the root x and all yellow vertices on the ways from x to yellow vertices going through yellow vertices on the 1-level. In particular, all yellow on the 1-level are inserted in U_0. If we got $\geq r$ vertices, U_0 is determined. Otherwise, we add to U_0 the first green vertices on the 1-level and repeat this procedure for the tree with the root y, and so on. To construct U_1, we take the first unlisted green vertices and repeat above procedure. After ω steps we get the partition $V = \bigcup_{n\in\omega} U_n$. We omit a routine verification of $(i), (ii), (iii)$ for $(U_n)_{n\in\omega}$. □

Proof of Theorem 12. We use the partition $V = \bigcup_{n\in\omega} U_n$ from Lemma 5 and put $A_n = U_0 \cup U_1 \cup ... \cup U_n$. Evidently, $(A_n)_{n\in\omega}$ is a covering sequence.

By Theorem 2, each graph $\Gamma[U_n]$ admits a balanced r-coloring of index $\leq r$. Applying [7, Lemma 5], we get a sequence $(\chi_n)_{n\in\omega}$ such that χ_n is a balanced r-colorings of $\Gamma[A_n]$ of index $\leq r$ and χ_{n+1} is an extension of χ_n. Thus, $\chi = \bigcup_{n\in\omega} \chi_n$ determines desired r-coloring of $\Gamma(V, E)$. □

Let $\Gamma(V, E)$ be a countable locally finite graph with the pointed vertex v. The most natural covering sequence in Γ is the sequence of balls $(B(v, n))_{n\in\omega}$. We do not know whether Theorem 12 is true for this covering sequence. We conclude the paper with the following partial result in this direction.

Theorem 13. *Let $\Gamma(V, E)$ be a countable locally finite graph, $x_0 \in V$, r be a natural number. Suppose that $|B(v, 1)| \geq r$ for each $v \in V$. Then there exists a partition $V = V_1 \cup V_2 \cup ... \cup V_r$ of index ≤ 3 balanced with respect to the sequence of balls $(B(x_0, n))_{n\in\omega}$.*

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