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UNIQUENESS OF MONADS FOR THE CAPACITY FUNCTOR AND ITS SUBFUNCTORS

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It is proved that the capacity functor in the category of compacta and two its subfunctors, namely the functors of cap-capacities (also called necessity measures) and cup-capacities (also known as sup-measures or possibility measures) are functorial parts of unique monads that admit a special kind of automorphisms.

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Доказано, что функтор ёмкостей в категории компактов и два его подфунктора, а именно функтор cap-ёмкостей (называемых также мерами необходимости) и cup-ёмкостей (известных также как меры возможности или sup-меры) являются функториальными частями единственных монад, допускающих особый вид автоморфизмов.

1. Introduction. Capacities were introduced by Choquet [2] as a natural generalization of measures. They found numerous applications, e.g. in decision making theory in conditions of uncertainty [8]. The upper semicontinuous capacities were defined and studied in [10], and a Kolmogorov-type theorem on existence of a capacity on a product of compacta with specified marginal capacities was also proved. Algebraic and topological properties of capacities on compact Hausdorff spaced were investigated in [9]. In particular, the capacity functor in the category of compacta was defined. A remarkable fact is that this functor is a functorial part of a monad that is also described in [9]. In [3] two dual submonads of the capacity monad in the category of compacta are constructed, and it is proved that each capacity on a compact Hausdorff space can be obtained (in a certain sense) out of capacities of the two described classes by the multiplication of the capacity monad.

The capacity functor belongs to the class of weakly normal functors that contains also important and well studied functors in the category of compacta such as the hyperspace functor, the probability measure functor, the inclusion hyperspace functor, the superextension functor etc. These functors also are functorial parts of monads, and it was shown that the hyperspace monad $\mathbb{H} = (\exp, s, u)$ [13], the superextension monad $\mathbb{L} = (\lambda, \eta_L, \mu_L)$ [11] and the probability measure monad $\mathbb{P} = (P, \eta_P, \mu_P)$ [14] are unique for the respective functors $\exp$, λ and P. The last result in this line [6] states that the inclusion hyperspace monad [12] is unique for the inclusion hyperspace functor G. This functor is closely connected to the

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capacity functor, therefore in this paper we use the latter result to obtain a partial answer to the problems of uniqueness of monads for the capacity functor and two its subfunctors.

2. Terminology, notations and basic facts. A *compactum* is a compact Hausdorff topological space. We regard the unit segment $I = [0;1]$ as a subspace of the real line with the natural topology. We write $A \underset{\mathrm{cl}}{\subset} B$ or $A \underset{\mathrm{op}}{\subset} B$ if A is a closed or resp. an open subset of a space B.

See [4] for the definitions of category, functor, natural transformation, monad, morphism of monads. We denote by $\mathcal{C}\mathrm{omp}$ the *category of compacta* that consists of all compacta and their continuous mappings, and $\mathbf{1}_{\mathcal{C}\mathrm{omp}}$ is the identity functor in $\mathcal{C}\mathrm{omp}$. If there is a natural transformation of one functor in $\mathcal{C}\mathrm{omp}$ to another with all components being topological embeddings, then the first functor is called a *subfunctor* of the latter [15]. Similarly, an *embedding of monads* in $\mathcal{C}\mathrm{omp}$ is a morphism of monads with all components being topological embeddings. If there exists an embedding of one monad in $\mathcal{C}\mathrm{omp}$ into another one, then the former monad is called a *submonad* of the latter. For a topological space X, we denote by $\exp X$ the set of all nonempty closed subsets of X. The *Vietoris topology* [15] on $\exp X$ is determined by a base that consists of all sets of the form

$$\langle U_1, U_2, \dots, U_n\rangle = \{F \in \exp X \mid F \subset U_1 \cup U_2 \cup \dots \cup U_n, F \cap U_i \neq \varnothing, i = 1, 2, \dots, n\},$$

with all U_i being open in X. The space $\exp X$ is a compactum for a compactum X, and the assignment exp extends to a functor in $\mathcal{C}\mathrm{omp}$ if for a continuous mapping of compacta $f\colon X \to Y$ the mapping $\exp f\colon \exp X \to \exp Y$ is defined as follows :$\exp f(F) = \{f(x) \mid x \in F\}$. We call a subset $\mathcal{A} \in \exp X$ an *inclusion hyperspace* [15] if $\mathcal{A}$ is closed in $\exp X$ and $A, B \in \exp X$, $B \supset A \in \mathcal{A}$ implies $B \in \mathcal{A}$. The set GX of all inclusion hyperspaces in X is closed in $\exp^2 X = \exp(\exp X)$, thus is a compactum. The assignment G is a functor in $\mathcal{C}\mathrm{omp}$ if for a continuous mapping of compacta $f\colon X \to Y$ the mapping $Gf\colon GX \to GY$ is defined as follows :$Gf(\mathcal{A}) = \{B \in \exp Y \mid B \supset f(A)$ for some $A \in \mathcal{A}\}$.

We follow a terminology of [9] and call a function $c\colon \exp X \cup \{\varnothing\} \to I$ a *capacity* on a compactum X if the three following properties hold for all closed subsets F, G in X :

1. $c(\varnothing) = 0$, $c(X) = 1$;
2. if $F \subset G$, then $c(F) \leqslant c(G)$ (monotonicity);
3. if $c(F) < a$, then there exists an open set $U \supset F$ such that for any $G \subset U$ we have $c(G) < a$ (upper semicontinuity).

We extend a capacity c to all open subsets in X by the formula :

$$c(U) = \sup\{c(F) \mid F \underset{\mathrm{cl}}{\subset} X, F \subset U\}.$$

It is proved in [9] that the set MX of all capacities on a compactum X is a compactum as well, if a topology on MX is determined by a subbase that consists of all sets of the form

$$O_-(F, a) = \{c \in MX \mid c(F) < a\},$$

where $F \underset{\mathrm{cl}}{\subset} X$, $a \in \mathbb{R}$, and $O_+(U, a) = \{c \in MX \mid c(U) > a\} = \{c \in MX \mid$ there exists a compactum $F \subset U, c(F) > a\}$, where $U \underset{\mathrm{op}}{\subset} X$, $a \in \mathbb{R}$.

The assignment M extends to the *capacity functor* M in the category of compacta, if the map $Mf\colon MX \to MY$ for a continuous map of compacta $f\colon X \to Y$ is defined by the formula

$$Mf(c)(F) = c(f^{-1}(F)),$$

where $c \in MX$, $F \underset{\mathrm{cl}}{\subset} Y$. This functor is the functorial part of the *capacity monad* $\mathbb{M} = (M, \eta, \mu)$ that was described in [9]. Its unit and multiplication are defined by the formulae

$$\eta X(x)(F) = \begin{cases} 1, & \text{if } x \in F, \\ 0, & \text{if } x \notin F, \end{cases}$$

$$\mu X(\mathcal{C})(F) = \sup\{\ \alpha \in I \mid \mathcal{C}(\{c \in MX \mid c(F) \geqslant \alpha\}) \geqslant \alpha\},$$

where $x \in X$, $\mathcal{C} \in M^2X$, $F \underset{\mathrm{cl}}{\subset} X$.

If c is a capacity on X, then it is proved in [9] that the function $\varkappa X(c)$, that is defined on $\exp X \cup \{\varnothing\}$ by the formula $\varkappa X(c)(F) = 1 - c(X \setminus F)$, is a capacity as well. It is called the *dual capacity* (or *conjugate capacity*) to c. The mapping $\varkappa X\colon MX \to MX$ is a homeomorphism and an involution, and the collection of $\varkappa X$ for all compacta X is an isomorphism of the capacity monad into itself.

We call a capacity $c \in MX$ a $\cap$-*capacity* (cap-capacity [3], also called *necessity measure*), if for each $\alpha \in I$ there is a set $A_{\alpha,c} \underset{\mathrm{cl}}{\subset} X$ such that for any set $F \underset{\mathrm{cl}}{\subset} X$ the inequality $c(F) \geq \alpha$ holds if and only if $F \supset A_{\alpha,c}$.

We denote the set of all $\cap$-capacities on a compactum X by $M_\cap X$. It is proved in [3] that $M_\cap X$ is closed in MX and therefore is a compactum with the induced topology. Moreover, the assignment $M_\cap$ extends to a subfunctor of the functor M in $\mathcal{C}$omp, that is the functorial part of a submonad $\mathbb{M}_\cap = (M_\cap, \eta_\cap, \mu_\cap)$ of the capacity monad $\mathbb{M} = (M, \eta, \mu)$.

We call a capacity $c \in MX$ a $\cup$-*capacity* (cup-capacity [3], also called *possibility measure* or *sup measure*, see [1]), if for each number $\alpha \in I$ there is a set $A^{\alpha,c} \underset{\mathrm{cl}}{\subset} X$ such that for each set $F \underset{\mathrm{cl}}{\subset} X$ the inequality $c(F) \geq \alpha$ holds if and only if $F \cap A^{\alpha,c} \neq \varnothing$. The set of all $\cup$-capacities on a compactum X is denoted by $M_\cup X$. It is proved in [3] that $M_\cup X$ is closed in MX as well and the assignment $M_\cup$ extends to a subfunctor of the functor M in $\mathcal{C}$omp, that is the functorial part of a submonad $\mathbb{M}_\cup = (M_\cup, \eta_\cup, \mu_\cup)$ of the capacity monad $\mathbb{M} = (M, \eta, \mu)$. See also [3] for the proof that a capacity c on a compactum X is a $\cap$-capacity if and only if the dual capacity $c' = \varkappa X(c)$ is a $\cup$-capacity. The restrictions of the map $\varkappa X$ to the subspaces $M_\cup X$ and $M_\cap X$ are components of mutually inverse natural isomorphisms of the functors $M_\cup \to M_\cap$ and $M_\cap \to M_\cup$.

Let $h\colon I \to I$ be an increasing bijection. Then the formula $s_h X(c)(F) = h(c(F))$, $c \in MX$, $F \in \exp X \cup \{\varnothing\}$, defines a homeomorphism $s_h X\colon MX \to MX$. Its restrictions $s_h^\cap X$ and $s_h^\cup X$ provide homeomorphisms $M_\cap X \to M_\cap X$ and $M_\cup X \to M_\cup X$. It is proved in [9] that the collection s_h of all $s_h X$ is an isomorphism of the monad $\mathbb{M}$ onto itself. It is straightforward to check that the collection $s_h^\cap$ of all $s_h^\cap X$ is an isomorphism of the monad $\mathbb{M}_\cap$ onto itself, as well as the collection $s_h^\cup$ of all $s_h^\cup X$ is an isomorphism of the monad $\mathbb{M}_\cup$ onto itself. For the "practical" interpretation of stability of the mentioned monads under s_h see [9].

3. Uniqueness of monads. The three following theorems are main results of this paper.

Theorem 1. *The monad $\mathbb{M} = (M, \eta, \mu)$ is a unique monad in $\mathcal{C}$omp such for any increasing bijection $h\colon I \to I$ the collection s_h is a morphism of this monad to itself.*

Theorem 2. *The monad $\mathbb{M}_\cap = (M_\cap, \eta_\cap, \mu_\cap)$ is a unique monad in $\mathcal{C}$omp such for any increasing bijection $h\colon I \to I$ the collection $s_h^\cap$ is a morphism of this monad to itself.*

Theorem 3. *The monad $\mathbb{M}_\cup = (M_\cup, \eta_\cup, \mu_\cup)$ is a unique monad in $\mathcal{C}$omp such for any increasing bijection $h\colon I \to I$ the collection $s_h^\cup$ is a morphism of this monad to itself.*

Proof of Theorem 1. Sinse the capacity functor M preserves singletons and continuous injections of compacta, the defined above η is a unique natural transformation from the identity functor to M [15]. Therefore assume that there is a natural transformation $\mu'\colon M^2 \to M$ such that $\mathbb{M}' = (M, \eta, \mu')$ is a monad, and the described above natural transformations $s_h\colon M \to M$ (with $h\colon I \to I$ being increasing bijections) are morphisms of monads $\mathbb{M}' \to \mathbb{M}'$.

Let K be a finite subset of $(0,1]$ that contains 1. Denote by $M_K X$ the set of all capacities on a compactum X with values in $K \cup \{0\}$. It is easy to see that $M_K X$ is closed in MX, and for any continuous mapping of compacta $f\colon X \to Y$ we have $Mf(M_K X) \subset M_K Y$. Thus M_K extends to a subfunctor of the capacity functor M and the respective natural transformation e_K consists of all embeddings $e_K X\colon M_K X \hookrightarrow MX$.

Let $h\colon I \to I$ be an increasing bijection such that $h(k) = k$ for all $k \in K$, and let $\mathbf{C}$ be a capacity that belongs to $M_K(M_K X) \subset M(MX)$. It is easy to verify that $s_h X \circ e_K X = e_K X$ for any compactum X. Thus

$$\begin{aligned} s_h MX \circ M s_h X(\mathbf{C}) &= s_h MX \circ M s_h X \circ M e_K X \circ e_K M_K X(\mathbf{C}) = \\ = s_h MX \circ M(s_h X \circ e_K X) \circ e_K M_K X(\mathbf{C}) &= s_h MX \circ M e_K X \circ e_K M_K X(\mathbf{C}) = \\ = M e_K X \circ s_h M_K X \circ e_K M_K X(\mathbf{C}) &= M e_K X \circ e_K M_K X(\mathbf{C}) = \mathbf{C}. \end{aligned}$$

This equality and the definition of morphism of monads imply that

$$s_h X \circ \mu' X(\mathbf{C}) X = \mu' X \circ s_h MX \circ M s_h X(\mathbf{C}) = \mu' X(\mathbf{C}).$$

Obviously all capacities on X that are invariant under all mappings $s_h X$ for increasing bijections $h\colon I \to I$ that preserve elements of K, have values only in K, i.e. belong to $M_K X$. Thus we obtain $\mu' X(M_K(M_K X)) \subset M_K X$, and the restriction $\mu'_K X\colon M_K^2 X \to M_K X$ of $\mu' X$ is well defined and makes the diagram

$$\begin{array}{ccccc} M_K^2 X & \xrightarrow{M_K e_K X} & M_K M X & \xrightarrow{e_K MX} & M^2 X \\ \downarrow{\scriptstyle \mu'_K X} & & & & \downarrow{\scriptstyle \mu' X} \\ M_K X & & \xrightarrow[e_K X]{} & & MX \end{array}$$

commute. This implies that the collection μ'_K of all $\mu'_K X$ is a natural transformation $M_K^2 \to M_K$. Observe also that $\eta X(X) \subset M_K X$, thus we can define a natural transformation $\eta_K\colon \mathbf{1}_{\mathcal{C}\mathrm{omp}} \to M_K$ by putting $\eta_K X(x) = \eta X(x)$. Thus we obtain a submonad (M_K, η_K, μ'_K) of the monad (M, η, μ').

Now let $K = \{1\}$, then $M_K X$ is the set of capacities that have only values 0 and 1. There is an isomorphism of functors $i\colon G \to M_{\{1\}}$. Its components are defined as follows :

$$iX(\mathcal{A})(F) = \begin{cases} 1, F \in \mathcal{A}, \\ 0, F \notin \mathcal{A}. \end{cases}$$

Thus we can transport the monad structure along the natural isomorphism i to G. It is proved in [6] that there is a unique monad with the functorial part G, therefore $M_{\{1\}}$ is also the functorial part of a unique monad in $\mathcal{C}$omp. This implies that $\mu' X$ and μX coincide on $M_{\{1\}}(M_{\{1\}}X)$.

Now let again K be a finite subset of $(0,1]$ that contains 1. We put all its elements in increasing order : $k_1 < k_2 < \dots < k_n = 1$. It is convenient to denote $k_0 = 0$. Choose $k \in K$. Let $\widetilde{k} = k_{l-1}$ if $k = k_l$ and define a mapping $h_k \colon I \to I$ as follows:

$$h_k(t) = \begin{cases} 0 & \text{for } t \leqslant \widetilde{k}, \\ \frac{t-\widetilde{k}}{k-\widetilde{k}} & \text{for } \widetilde{k} < t < k, \\ 1 & \text{for } t \geqslant k. \end{cases}$$

We define a mapping $s_{h_k}X \colon MX \to MX$ for an a compactum X by the same formula $s_{h_k}X(c)(F) = h_k(c(F))$ as before. Obviously h_k is not injective, but it is the limit of increasing bijections $I \to I$ w.r.t. compact-open topology, e.g. we can take functions $h_{l,\lambda}(t) = \lambda h_k(t) + (1-\lambda)t$ for $\lambda \in (0,1)$, $\lambda \to 1$. From this and from continuity [15, 9] of the functor M we can deduce by [5] that for $\lambda \to 1$ the mappings $s_{h_{l,\lambda}}X$, $s_{h_{l,\lambda}}MX$ and $Ms_{h_{l,\lambda}}X$ converge respectively to $s_{h_l}X$, $s_{h_l}MX$ and $Ms_{h_l}X$ w.r.t. compact-open topology. As composition of continuous mappings of compacta is continuous w.r.t. compact-open topology, we obtain that the following diagrams are commutative:

$$\begin{array}{ccccc} M^2X & \xrightarrow{s_{h_k}MX} & M^2X & \xrightarrow{Ms_{h_k}X} & M^2X \\ {\scriptstyle \mu X}\downarrow & & & & \downarrow{\scriptstyle \mu X} \\ MX & & \xrightarrow[s_{h_k}X]{} & & MX \end{array} \qquad \begin{array}{ccccc} M^2X & \xrightarrow{s_{h_k}MX} & M^2X & \xrightarrow{Ms_{h_k}X} & M^2X \\ {\scriptstyle \mu' X}\downarrow & & & & \downarrow{\scriptstyle \mu' X} \\ MX & & \xrightarrow[s_{h_k}X]{} & & MX \end{array}$$

for any compactum X.

Observe that $s_{h_k}X(M_KX) \subset M_{\{1\}}X$ for any compactum X, therefore if $\mathbf{C} \in M_K(M_KX)$, then $s_{h_k}MX \circ Ms_{h_k}X(\mathbf{C}) \in M_{\{1\}}(M_{\{1\}}X)$. Denote $c = \mu X(\mathbf{C})$, $c' = \mu' X(\mathbf{C})$, then $s_{h_k}X(c) = \mu X \circ s_{h_k}MX \circ Ms_{h_k}X(\mathbf{C}) = \mu' X \circ s_{h_k}MX \circ Ms_{h_k}X(\mathbf{C}) = s_{h_k}X(c')$ for all $k \in K$. But

$$s_{h_k}X(c)(F) = \begin{cases} 1, c(F) \geqslant k, \\ 0, c(F) < k. \end{cases}$$

Thus $c(F) = \max\{k \in K \mid s_{h_k}X(c)(F) = 1\} = \max\{k \in K \mid s_{h_k}X(c')(F) = 1\} = c'(F)$ (here we assume $\max\varnothing = 0$).

Thus μX and $\mu' X$ coincide on all subsets of the form $M_K(M_KX) \subset M^2X$. Since their union is dense in M^2X, we obtain $\mu X = \mu' X$, thus $\mathbb{M} = \mathbb{M}'$ and the monad in question is unique. □

As the functors $M_\cap$ and $M_\cup$ are isomorphic through a restriction of $\varkappa$ and $\varkappa \circ s_h = s_{\widetilde{h}} \circ \varkappa$ for $\widetilde{h}(t) = 1 - h(1-t)$, one can easily see that Theorem 2 and Theorem 3 are equivalent. Thus it is sufficient to prove, e.g., only Theorem 2. Its proof is obtained *mutatis mutandis* from the proof of Theorem 2, mostly by replacing M_K with $M_K^\cap$. The only notable distinction is that the functor $M_{\{1\}}^\cap$ is isomorphic to exp instead of G. Components of the natural isomorphism $i^\cap \colon \exp \to M_{\{1\}}^\cap$ are of the form :

$$i^\cap X(A)(F) = \begin{cases} 1, F \supset A, \\ 0, F \not\supset A. \end{cases}$$

It is well known [15] that the hyperspace functor is the functorial part of a unique monad im $\mathcal{C}$omp. Thus Theorem 2 (and therefore Theorem 3) are proved.

4. Final remarks. It is still unknown to the author whether the requirement of stability of the mentioned monads under natural transformations of the form s_h is essential. The problem of uniqueness of monads for the functors of capacities with values in compact Hausdorff Lawson lattices [7] is also open.

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