УДК 517.574

K. G. Malyutin

INVERSE FORMULAE FOR FOUIRIER COEFFICIENTS OF DELTA-SUBHARMONIC FUNCTIONS IN A HALF-PLANE

K. G. Malyutin. *Inverse formulae for Fourier coefficients of delta-subharmonic functions in a half-plane*, Mat. Stud. **32** (2009), 210–215.

Applying the Fourier series method we obtain inverse relations for the Fourier coefficients of delta-subharmonic function in the upper complex half-plane. These inverse relations as well as direct ones can be considered as generalizations of well-known Carleman's formula.

К. Г. Малютин. *Обратные формулы для коэффициентов Фурье дельта-субгармонических функций в полуплоскости* // Мат. Студiї. – 2009. – Т.32, №2. – C.210–215.

Методом рядов Фурье получены обратные формулы для коэффициентов Фурье дельта-субгармонических функций в верхней комплексной полуплоскости. Эти формулы, как и прямые формулы, можно рассматривать в качестве обобщений известной формулы Карлемана.

Introduction. In the 60s several American authors (Rubel [1], Rubel and Taylor [2], Miles [3], Shea and others) started to use the Fourier series method for the study of the properties of entire and meromorphic functions. This method is efficient in the solution of several general problems of the theory of meromorphic functions and establishes its connections with Fourier series theory. One advantage of this method is its suitability for the investigation of functions of fairly irregular growth at infinity and functions of infinite order. In the 80s important results in this direction were obtained by Kondratyuk [4], [5], [6], who generalized the Levin-Pflüger theory of entire functions of completely regular growth to meromorphic functions of arbitrary γ-type. Inverse relations for the Fourier coefficients of meromorphic functions have played an important role in his researches.

At the beginning of this century some results of Rubel, Taylor and Miles were extended to delta-subharmonic functions in the upper half-plane ([7]). These researches were continued in works [8], [9], [10]. In present paper we establish inverse formulae for the Fourier coefficients of delta-subharmonic function in the upper half-plane. These formulae as well as direct ones can be considered as generalizations of well-known Carleman's formula.

Let $J\delta$ be the class of just δ-subharmonic functions, (we present the definitions of this class below) in the upper half-plane. We will formulate a basic result.

Let $v \in J\delta$, $v(0) = 0$. Then

$$\Lambda_k(r) = \Lambda_k(r, v) \overset{def}{=} \int_{r_0}^{r} \frac{\widetilde{\lambda}_k(t)}{t}\, dt = \frac{\pi}{2} c_k(r) - \frac{\pi k^2}{2} \int_{r_0}^{r} \frac{dt}{t} \int_{r_0}^{t} \frac{c_k(\tau)}{\tau}\, d\tau -$$

$$-\big(k\alpha_k r_0^k + \frac{2\lambda_k(r_0)}{\pi r_0^k}\big) \ln r - \alpha_k r_0^k + \big(k\alpha_k r_0^k + \frac{2\lambda_k(r_0)}{\pi r_0^k}\big) \ln r_0\, ,$$

2000 *Mathematics Subject Classification:* 31A05, 31A10.

for all $k \in \mathbb{N}$ and all $r > r_0 > 0$, where $c_k(r)$ are the Fourier coefficients of v.

1. Classes of delta-subharmonic functions. Let $\mathbb{C}$ be the complex plane. By $C(a, r)$ and $B(a, r)$ denote respectively open and closed disks of radius r centered at a point a.

We mean a delta-subharmonic function in a domain D as difference of two subharmonic in this domain functions: $v(z) = v_1(z) - v_2(z)$, where v_i, $i = 1, 2$, are functions subharmonic in a domain D.

We will use terminology from [11] and [12]. Let $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ and $\Omega_+ = \Omega \cap \mathbb{C}_+$ for a set Ω. A subharmonic function v in $\mathbb{C}_+$ is said to be just subharmonic if

$$\limsup_{z \to t} v(z) \le 0$$

for each $t \in \mathbb{R}$. By JS denote the class of just subharmonic functions in $\mathbb{C}_+$. Let SK be the class of subharmonic functions in $\mathbb{C}_+$ possessing a positive harmonic majorant in each bounded subdomain of $\mathbb{C}_+$. The properties of functions in SK see [11, 12].

We define the class $J\delta$ of δ-subharmonic functions as follows

$$J\delta = JS - JS.$$

For a fixed measure λ let

$$d\lambda_m(\zeta) = \frac{\sin m\varphi}{\sin\varphi}\tau^{m-1}\, d\lambda(\zeta)\,\left(\zeta = \tau e^{i\varphi}\right), \lambda_m(r) = \lambda_m\left(\overline{C(0,r)}\right),$$

where $\frac{\sin m\varphi}{\sin\varphi}$ is defined for $\varphi = 0, \pi$ by continuity.

The next relation is Carleman's formula in Grishin's notation

$$\frac{1}{r^k}\int_0^\pi v(re^{i\varphi})\sin k\varphi\, d\varphi = \int_{r_0}^r \frac{\lambda_k(t)}{t^{2k+1}}\, dt + \frac{1}{r_0^k}\int_0^\pi v(r_0e^{i\varphi})\sin k\varphi\, d\varphi\,, \tag{1}$$

in particular, for $k = 1$ we have

$$\frac{1}{r}\int_0^\pi v(re^{i\varphi})\sin\varphi\, d\varphi = \int_{r_0}^r \frac{\lambda(t)}{t^3}\, dt + \frac{1}{r_0}\int_0^\pi v(r_0e^{i\varphi})\sin\varphi\, d\varphi\,. \tag{2}$$

for all $r > r_0$.

Let $D_+(R_1, R_2) = \overline{C_+(0, R_2)\setminus C_+(0, R_1)}$, $R_1 < R_2$. Functions $v \in J\delta$ have the following representation in the half-annulus $z \in D_+(R_1, R_2)$

$$v(z) = -\frac{1}{2\pi}\iint_{D_+(R_1,R_2)} K(z,\zeta)\, d\lambda(\zeta) + \frac{R_2}{2\pi}\int_0^\pi \frac{\partial G(z, R_2e^{i\varphi})}{\partial n} v\left(R_2e^{i\varphi}\right)\, d\varphi +$$

$$+\frac{R_1}{2\pi}\int_0^\pi \frac{\partial G(z, R_1e^{i\varphi})}{\partial n} v(R_1e^{i\varphi})\, d\varphi\,, \tag{3}$$

and in the half-disk $z \in C_+(0, R)$

$$v(z) = -\frac{1}{2\pi}\iint_{\overline{C_+(0,R)}} K(z,\zeta)\, d\lambda(\zeta) + \frac{R}{2\pi}\int_0^\pi \frac{\partial G(z, Re^{i\varphi})}{\partial n} v(Re^{i\varphi})\, d\varphi\,,$$

where $G(z, \zeta)$ is the Green's function of the half-annulus (the half-disk), $\dfrac{\partial G}{\partial n}$ is its derivative in the inward normal direction, and the function $K(z, \zeta) = G(z, \zeta)/\operatorname{Im}\zeta$, $\zeta \in D_+(R_1, R_2)$ (and $z \in C_+(0, R)$), is extended by continuity to the points on the real axis with $R_1 \le |t| \le R_2$.

2. Fourier coefficients of delta-subharmonic functions. Following [13] denote by $c_k(r,v)$ the Fourier coefficients of delta-subharmonic in a disk $C(0,R)$ function $v(z)$,

$$c_k(r,v)=\frac{1}{2\pi}\int\limits_0^{2\pi} e^{-ik\theta}v(re^{I\theta})\,d\theta,\quad k\in\mathbb{Z},\quad r<R\,.$$

Let μ be a measure. We use the following notations

$$\mu(r)=\mu(B(0,r)),\quad N(r)=\int_0^r\frac{\mu(t)}{t}\,dt\,,$$

$$d\mu_k(\zeta)=\frac{d\mu(\zeta)}{\zeta^k},\quad \mu_k(r)=\mu_k(B(0,r))\,.$$

The following lemma gives expressions for the Fourier coefficients $c_k(r,v)$, which slightly differ from another ones got in [13] as well as from the Fourier coefficients of meromorphic functions [14, 15].

Lemma 1. *Let v be a delta-subharmonic in a disk $C(0,R)$ function, $v(0)=0$. Let μ be the Riesz measure of v and*

$$v(z)=\sum_{k=1}^{\infty}\frac{r^k}{2}\left(\alpha_ke^{ik\theta}+\overline{\alpha}_ke^{-ik\theta}\right)$$

in some neighborhood of the origin $z=0$. Then for all $0<r<R_0$

$$c_0(r,v)=N(r,-v)-N(r,v);\tag{4}$$

$$c_k(r,v)=\frac{r^k}{2}\alpha_k+\frac{1}{2k}\iint\limits_{|\zeta|\le r}\frac{r^{2k}-\tau^{2k}}{r^k}\,d\mu_k(\zeta),\quad z=re^{i\theta},\zeta=\tau e^{i\varphi},\tag{5}$$

where $k\ge1$, and $c_k=\overline{c}_{-k}$ for all $k\le-1$.

The proof word for word repeats the proof of lemma 2.1 from [15]. Further denote

$$d\widetilde{\mu}_k(\tau e^{i\varphi})=e^{-ik\varphi}\,d\mu(\tau e^{i\varphi}),\ \widetilde{\mu}_k(r)=\widetilde{\mu}_k(B(0,r))\quad k\in\mathbb{Z}\,.$$

Then (5) can be written in a following way

$$c_k(r,v)=\frac{r^k}{2}\alpha_k+\frac{1}{2k}\int\limits_0^r\left[\left(\frac{r}{t}\right)^k-\left(\frac{t}{r}\right)^k\right]\,d\widetilde{\mu}_k(t)\,,\tag{6}$$

for all $k\ge1$ and $c_k=\overline{c}_{-k}$ for all $k\le-1$.

The Fourier coefficients of a function $v\in J\delta$ are defined as usual [7]

$$c_k(r,v)=\frac{2}{\pi}\int\limits_0^{\pi} v(re^{i\theta})\sin k\theta\,d\theta,\quad k\in\mathbb{N}\,.$$

By (1), we obtain the following expressions for the Fourier coefficients for $r>r_0$:

$$c_k(r,v)=\alpha_kr^k+\frac{2r^k}{\pi}\int\limits_{r_0}^{r}\frac{\lambda_k(t)}{t^{2k+1}}\,dt,\quad k\in\mathbb{N}\,,\tag{7}$$

where $\alpha_k=r_0^{-k}c_k(r_0,v)$.

Integrating by parts in (7), we have

$$c_k(r,v)=\alpha_k r^k+\frac{r^k}{\pi k r_0^{2k}}\iint_{\overline{C_+(0,r_0)}}\frac{\sin k\varphi}{\operatorname{Im}\zeta}\tau^k\,d\lambda(\zeta)+$$
$$\frac{r^k}{\pi k}\iint_{D_+(r_0,r)}\frac{\sin k\varphi}{\tau^k\operatorname{Im}\zeta}\,d\lambda(\zeta)-\frac{1}{r^k\pi k}\iint_{\overline{C_+(0,r)}}\frac{\sin k\varphi}{\operatorname{Im}\zeta}\tau^k\,d\lambda(\zeta)\,, \tag{8}$$

where $\zeta=\tau e^{i\varphi}$.

Proposition 1. *The Fourier coefficients $c_k(r,v)$ of a function v in the class $J\delta$ are continuous functions of r.*

This follows immediately from the continuity of the functions in the right-hand side of (8).

Denote now

$$d\widetilde{\lambda}_k(\tau e^{i\varphi})=\frac{\sin k\varphi}{\operatorname{Im}\zeta}\,d\lambda(\tau e^{i\varphi}),\ \widetilde{\lambda}_k(r)=\widetilde{\lambda}_k(B(0,r)),\ k\in\mathbb{N}\,.$$

Taking into account these notations the relation (8) can be written as follows

$$c_k(r,v)=\alpha_k r^k+\frac{1}{\pi k}\left[\left(\frac{r}{r_0^2}\right)^k-\frac{1}{r^k}\right]\lambda_k(r_0)+$$
$$\frac{1}{\pi k}\int_{r_0}^{r}\left[\left(\frac{r}{t}\right)^k-\left(\frac{t}{r}\right)^k\right]\,d\widetilde{\lambda}_k(t)\,. \tag{9}$$

3. Inverse formulae for Fourier coefficients. We now prove inverse relations for the Fourier coefficients of delta-subharmonic in the complex plane $\mathbb{C}$ functions. This formulae as well as direct ones (5), (6) can be considered as generalizations of the Jensen formula. Respective formulae for meromorphic functions were obtained by Kondratyuk ([4, 5]).

Lemma 2. *Let v be a delta-subharmonic function in the complex plane $\mathbb{C}$, $v(0)=0$, and let μ be Riesz measure of v. Then*

$$M_k(r)\stackrel{def}{=}\int_0^r\frac{\widetilde{\mu}_k(t)}{t}dt=c_k(r)-k^2\int_0^r\frac{dt}{t}\int_0^t\frac{c_k(\tau)}{\tau}\,d\tau$$

for all $k\in\mathbb{Z}$ and $r>0$.

The proof word for word repeats the proof of lemma 8.6 from [15].

Let $v\in J\delta$ and let λ be the complete measure of function v. We set

$$\Lambda_k(r)=\Lambda_k(r,v)\stackrel{def}{=}\int_{r_0}^{r}\frac{\widetilde{\lambda}_k(t)}{t}\,dt,\quad k\in\mathbb{N}\,.$$

where $r>r_0$.

Theorem 1. *Let $v\in J\delta$, $v(0)=0$. Then*

$$\Lambda_k(r)=\frac{\pi}{2}c_k(r)-\frac{\pi k^2}{2}\int_{r_0}^{r}\frac{dt}{t}\int_{r_0}^{t}\frac{c_k(\tau)}{\tau}\,d\tau-$$
$$\left(k\alpha_k r_0^k+\frac{2\lambda_k(r_0)}{\pi r_0^k}\right)\ln r-\alpha_k r_0^k+\left(k\alpha_k r_0^k+\frac{2\lambda_k(r_0)}{\pi r_0^k}\right)\ln r_0\,.$$

for all $k\in\mathbb{N}$ and all $r>r_0>0$.

◁ Integrating twice by parts in (9) we obtain

$$c_k(r) := c_k(r,v) = \alpha_k r^k + \frac{1}{\pi k}\left[\left(\frac{r}{r_0^2}\right)^k - \frac{1}{r^k}\right]\lambda_k(r_0) +$$

$$+\frac{2}{\pi}\Lambda_k(r) + \frac{k}{\pi}\int\limits_{r_0}^{r}\left[\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k\right]\frac{\Lambda_k(t)}{t}\,dt\,, \tag{10}$$

where $k \in \mathbb{N}$.

Denote

$$\Phi_k(r) \overset{def}{=} \frac{k}{\pi}\int\limits_{r_0}^{r}\left[\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k\right]\frac{\Lambda_k(t)}{t}\,dt\,.$$

Note that the function $\Phi_k(r)$ has continuous second derivative. Thus

$$\Phi_k'(r) = \frac{k^2}{\pi r}\int\limits_{r_0}^{r}\left[\left(\frac{r}{t}\right)^k + \left(\frac{t}{r}\right)^k\right]\frac{\Lambda_k(t)}{t}\,dt\,,$$

$$\Phi_k''(r) = \frac{k^3}{\pi r^2}\int\limits_{r_0}^{r}\left[\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k\right]\frac{\Lambda_k(t)}{t}\,dt -$$

$$-\frac{k^2}{\pi r^2}\int\limits_{r_0}^{r}\left[\left(\frac{r}{t}\right)^k + \left(\frac{t}{r}\right)^k\right]\frac{\Lambda_k(t)}{t}\,dt + \frac{2k^2}{\pi r^2}\Lambda_k(r)\,,$$

$$\Phi_k'(r_0) = 0, \quad \Phi_k''(r_0) = 0\,. \tag{11}$$

Function $\Phi_k(r)$ satisfies differential equation

$$r^2\Phi_k''(r) + r\Phi_k'(r) = k^2 c_k(r) - k^2\alpha_k r^k - \frac{k}{\pi}\left[\left(\frac{r}{r_0^2}\right)^k - \frac{1}{r^k}\right]\lambda_k(r_0)\,,$$

with initial conditions (11).

Resolving this differential equation, we have

$$\Phi_k(r) = k^2\int_{r_0}^{r}\frac{dt}{t}\int_{r_0} t\frac{c_k(\tau)}{\tau}\,d\tau - \alpha_k r^k - \frac{1}{\pi k}\left[\left(\frac{r}{r_0^2}\right)^k - \frac{1}{r^k}\right]\lambda_k(r_0) +$$

$$+\left(k\alpha_k r_0^k + \frac{2\lambda_k(r_0)}{\pi r_0^k}\right)\ln r + \alpha_k r_0^k - \left(k\alpha_k r_0^k + \frac{2\lambda_k(r_0)}{\pi r_0^k}\right)\ln r_0\,.$$

Together with (10) this completes the proof. ▷

REFERENCES

1. Rubel L.A. *A Fourier series method for entire functions*, Duke Math. J. **30**(1963), 437-442.
2. Rubel L.A., Taylor B.A. *Fourier series method for meromorphic and entire functions*, Bull. Soc. Math. France. **96**(1968), 53-96.
3. Miles J.B. *Quotient representations of meromorphic functions*, J. d′ Analyse Math. **25**(1972), 371-388.
4. Kondratyuk A.A. *The Fourier series method for entire and meromorphic functions of completely regular growth. I*, Mat. Sb. **106(148)** (1978), 386-408; English transl. in Math. USSR-Sb. **35** (1979).

5. Kondratyuk A.A. *The Fourier series method for entire and meromorphic functions of completely regular growth. II*, Mat. Sb. **113(155)** (1980), 118-132; English transl. in Math. USSR-Sb. **41** (1982).
6. Kondratyuk A.A. *The Fourier series method for entire and meromorphic functions of completely regular growth. III*, Mat. Sb. **120(162)** (1983), 331-343; English transl. in Math. USSR-Sb. **48** (1984).
7. Malyutin K.G. *Fourier series and δ-subharmonic functions of finite γ-type in a half-plane*, Mat. Sb. **192(6)** (2001), 51-70; English transl. in Sb.: Math. **192(6)** (2001).
8. Malyutin K.G., Nazim Sadik. *Delta-subharmonic functions of completely regular growth in the half-plane*, Russian Acad. Sci. Docl. Math. **380(3)** (2001), 1-3.
9. Malyutin K.G., Nazim Sadik. *Presentation of subharmonic functions in a half-plane*, Mat. Sb. **198(12)** (2007), 47-62; English transl. in Sb.: Math. **198(12)** (2007).
10. Malyutin K.G., Malyutina T.I. *Fourier series and delta-subharmonic functions of zero-type in a half-plane*, Matematychni Studii. **30(2)** (2008), 133 - 138.
11. Grishin A.F. *Continuity and asymptotical continuity of subharmonic functions*, Mathematical Physics, Analysis and Geometry. **1(2)** (1994), 193-215.
12. Fedorov M.A., A.F. Grishin A.F. *Some Questions of the Nevanlinna Theory for the Complex Half-Plane*, Mathematical Physics, Analysis and Geometry. **1(3)** (1998), 223-271.
13. Malyutin K.G. *Fourier series and delta-subharmonic functions*, Proceedings of IPMM NAN Ukraine. **3** (1998), 146-157.
14. Rubel L.A. *Entire and meromorphic functions*. Springer, New York, Berlin, Heidelberg 1995.
15. Kondratyuk A.A. *The Fourier series and meromorphic functions*. "Vyshcha shkola Lviv 1988.

State University of Sumy,
malyutinkg@yahoo.com

Received 4.06.2009
Revised 16.11.2009