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RADICAL FILTERS OF NOETHERIAN RINGS AND SOME EXAMPLES

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Radical filters of noetherian rings are considered.

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Рассматриваются радикальные фильтры нетеровых колец.

We recall some necessary definitions. We consider only associative rings with $1 \neq 0$. All modules are unitary. Let R be a ring.

The category of right R-modules will be denoted by $\mathrm{Mod}-R$. A subfunctor of an identity functor $1_{\mathrm{Mod}-R}\colon \mathrm{Mod}-R \longrightarrow \mathrm{Mod}-R$ is called a preradical of $\mathrm{Mod}-R$.

A preradical t is said to be a radical in case for every right R-module M $t(M/t(M)) = 0$. A preradical t is said to be idempotent in case for every right R-module M $t(t(M)) = t(M)$. A preradical t is said to be hereditary in case for every right R-module M and for every its submodule N $t(N) = t(M) \cap N$.

It is obvious that every hereditary preradical is idempotent. Let t be a preradical of $\mathrm{Mod}-R$. Put $T(t) = \{M \in \mathrm{Mod}-R \mid t(M) = M\}$, $F(t) = \{M \in \mathrm{Mod}-R \mid t(M) = 0\}$. Let $L_r(R)$ be the set of all right ideals of the ring R. Let $\mathrm{Spec}_r(R)$ be the set of all maximal right ideals of R. If N is a submodule of a module M we shall write $N \leq M$.

Definition 1. A right ideal A is *similar to a right ideal* B if $R/A \cong R/B$. In this case we shall write $A \sim B$.

Definition 2. A set $G \subseteq L_r(R)$ is said to be *similarly-closed* in case $\forall A \in G\ \forall B \in L_r(R)\colon A \sim B \Rightarrow B \in G$.

Let $[G] = \{P \mid \exists A \in G\colon P \cong R/A\}$ for $G \subseteq \mathrm{Spec}_r(R)$. Let R be a ring and $J \subseteq R$, $a \in R$. Put $(J : a) = \{r \in R \mid ar \in J\}$.

Definition 3 ([1,2]). A set $E \subseteq L_r(R)$ is called a *radical filter* if the following conditions are fulfilled:

G1. $I \in E,\ I \subseteq J,\ J \in L_r(R) \Rightarrow J \in E$.

G2. $I \in E,\ a \in R \Rightarrow (I : a) \in E$.

G3. $I \in E,\ J \subseteq I,\ J \in L_r(R),\ \forall a \in I\colon (J : a) \in E \Rightarrow J \in E$.

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Definition 4 ([1,2]). A set $E \subseteq L_r(R)$ satisfying Gl and G2 is called a *preradical filter* if the following condition is fulfilled:

G4. $I \in E,\ J \in E \Rightarrow I \cap J \in E$.

It is well known that every radical filter is a preradical filter. Let

$$E_G = \{I \mid \exists n \in \mathbb{N} \cup \{0\}\ \exists A_0, A_1, \dots, A_n \in L_r(R)\colon I = A_0 \subseteq A_1 \subseteq \dots \subseteq A_n = R \wedge \\ \wedge\ \forall i \in \{1, 2, \dots, n\}\colon A_i/A_{i-1} \in [G]\}.$$

Theorem 1. *Let R be a right noetherian ring. If for every right ideal I of R the module R/I has a composition series if and only if I is non-zero, then for every radical filter $E \neq L_r(R)$ there exists a similarly-closed set $G \subseteq \operatorname{Spec}_r(R)$ such that $E = E_G$.*

Proof. Let $E \neq L_r(R)$ be a radical filter. Put $G = \operatorname{Spec}_r(R) \cap E$. We shall prove that the set G is similarly-closed. Let t be the hereditary radical corresponding to the radical filter E ([1]). Let $A \in G$, $B \in L_r(R)$, $A \sim B$.

Then $R/A \cong R/B$ and $R/A \in T(t)$. Since the class $T(t)$ is closed under epimorphic images, $R/B \in T(t)$. Thus $B \in G$. Therefore $G \subseteq \operatorname{Spec}_r(R)$ is a similarly-closed set.

Let $P \in [G]$. Then there exists $A \in G$ such that $P \cong R/A$. Since $A \in G \subseteq E$, $R/A \in T(t)$. Since the class $T(t)$ is closed under epimorphic images and $P \cong R/A$, $P \in T(t)$. Therefore $[G] \subseteq T(t)$.

Let $R \neq I \in E$. Since $E \neq L_r(R)$, I is non-zero. Since the module R/I has a composition series, there exist $A_0, A_1, \dots, A_n \in L_r(R)$ such that

$$I = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = R \wedge \forall i \in \{1, 2, \dots, n\}\colon A_i/A_{i-1} \text{ is simple.}$$

Since $I \subseteq A_{i-1}$ for every $i \in \{1, 2, \dots, n\}$, taking into consideration Gl, $A_{i-1} \in E$ for every $i \in \{1, 2, \dots, n\}$. Hence $R/A_{i-1} \in T(t)$ for every $i \in \{1, 2, \dots, n\}$. But $A_i/A_{i-1} \subseteq R/A_{i-1}$ for every $i \in \{1, 2, \dots, n\}$. Since the class $T(t)$ is closed under submodules, $A_i/A_{i-1} \in T(t)$ for every $i \in \{1, 2, \dots, n\}$. Since A_i/A_{i-1} is simple, $A_i/A_{i-1} \cong R/M_i$ for some maximal right ideal M_i of R for every $i \in \{1, 2, \dots, n\}$. Since the class $T(t)$ is closed under epimorphic images and $R/M_i \cong A_i/A_{i-1} \in T(t)$ for every $i \in \{1, 2, \dots, n\}$, $R/M_i \in T(t)$ for every $i \in \{1, 2, \dots, n\}$. Therefore $M_i \in E$ for every $i \in \{1, 2, \dots, n\}$. Since M_i is a maximal right ideal of R for every $i \in \{1, 2, \dots, n\}$, $M_i \in G$ for every $i \in \{1, 2, \dots, n\}$. Since $M_i \in G$ and $R/M_i \cong A_i/A_{i-1}$ for every $i \in \{1, 2, \dots, n\}$, $A_i/A_{i-1} \in [G]$ for every $i \in \{1, 2, \dots, n\}$. We obtain that $I \in E_G$. Hence $E \subseteq E_G$.

Now let $R \neq I \in E_G$. It follows from this that

$\exists A_0, A_1, \dots, A_n \in L_r(R)\colon I = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = R \wedge \forall i \in \{1, 2, \dots, n\}\colon A_i/A_{i-1} \in [G]$.

It is clear that $A_n = R \in E$.

Let $A_i \in E$ for some $i \in \{1, 2, \dots, n\}$. Since $A_i/A_{i-1} \in [G]$, $R/M_i \cong A_i/A_{i-1}$ for some $M_i \in G$. But $R/M_i \in T(t)$. Since the class $T(t)$ is closed under epimorphic images and $A_i/A_{i-1} \cong R/M_i \in T(t)$, $A_i/A_{i-1} \in T(t)$. Since $A_i \in E$,

$$(R/A_{i-1})/(A_i/A_{i-1}) \cong R/A_i \in T(t).$$

Since the class $T(t)$ is closed under epimorphic images, $(R/A_{i-1})/(A_i/A_{i-1}) \in T(t)$. Hence

$$A_i/A_{i-1} \in T(t) \wedge (R/A_{i-1})/(A_i/A_{i-1}) \in T(t) \wedge A_i/A_{i-1} \subseteq R/A_{i-1}.$$

But the class $T(t)$ is closed under extensions. Hence $R/A_{i-1} \in T(t)$. Therefore $A_{i-1} \in E$. It means that $A_0 \in E$. Hence $I \in E$. Therefore $E_G \subseteq E$. Now we have that $E = E_G$. □

Definition 5. An element a is *similar to an element* b if $aR \sim bR$.

A set $\mathcal{P} \subseteq R$ is said to be similarly-closed in case $\forall p \in \mathcal{P}\ \forall r \in R$: $r \sim p \Rightarrow r \in \mathcal{P}$. Let $\mathcal{E}_{\mathcal{P}} := \{I \in L_r(R) \mid (\exists n \in \mathbb{N}\ \exists a_1, a_2, \ldots, a_n \in \mathcal{P} : I = a_1 a_2 \ldots a_n R) \vee I = R\}$. We shall say that R is a domain if $\forall a, b \in R \setminus \{0\}$: $ab \in R \setminus \{0\}$.

A ring R is said to be a principal ideal domain in case it is a domain such that every its right ideal is a right principal ideal and every its left ideal is a left principal ideal.

Let R be a principal ideal domain. An element $p \in R$ is said to be an atom in case $p \neq 0 \wedge p \notin U(R) \wedge (\forall a, b \in R : (p = ab \Rightarrow a \in U(R) \vee b \in U(R)))$.

The set of all atoms of R will be denoted by Ω_R.

Corollary 1 ([5]). *Let R be a principal ideal domain. Then for every radical filter $E \neq L_r(R)$ there exists a similarly-closed set $\mathcal{P} \subseteq \Omega_R$ such that $E = \mathcal{E}_{\mathcal{P}}$.*

Proof. Let $E \neq L_r(R)$ be a radical filter. Put $G = \mathrm{Spec}_r(R) \cap E$. By the proof of Theorem 1, $E = E_G$. It is clear that $\forall M \in G\ \exists a \in \Omega_R$: $M = aR$.

Put $\mathcal{P} = \{a \in R \mid aR \in G\}$. It is clear that $P \subseteq \Omega_R$. G is similarly-closed. Hence $\mathcal{P}$ is similarly-closed. Let $I = a_1 a_2 \ldots a_n R \in E_G$ for some $\{a_1, a_2, \ldots, a_n\} \subseteq \Omega_R$ (see Theorem 5, Chapter 3 [3]).

Put $V_s := a_1 a_2 \ldots a_{n-s} R$, $s \in \{0, 1, \ldots, n-1\}$, $V_n := R$. Then we have that $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = R \wedge V_i / V_{i-1} \cong R / a_{n-i+1} R$, $i \in \{1, 2, \ldots, n\}$. Let t be the hereditary radical corresponding to the radical filter E. Since $V_0 = I \in E$ and $V_0 \subseteq V_{i-1}$ ($i \in \{1, 2, \ldots, n\}$), by Gl, $V_{i-1} \in E$ ($i \in \{1, 2, \ldots, n\}$). It follows from this that $R/V_{i-1} \in T(t)$. Since the class $T(t)$ is closed under submodules and $V_i/V_{i-1} \subseteq R/V_{i-1} \in T(t)$, $i \in \{1, 2, \ldots, n\}$, $V_i/V_{i-1} \in T(t)$, $i \in \{1, 2, \ldots, n\}$. But the class $T(t)$ is closed under epimorphic images. Now taking into consideration that $V_i/V_{i-1} \cong R/a_{n-i+1}R$, $i \in \{1, 2, \ldots, n\}$, we obtain $R/a_{n-i+1}R \in T(t)$, $i \in \{1, 2, \ldots, n\}$. Therefore $a_{n-i+1}R \in E$, $i \in \{1, 2, \ldots, n\}$. Since $a_{n-i+1} \in \Omega_R$, $a_{n-i+1}R \in G$. Therefore $a_{n-i+1} \in \mathcal{P}$, $i \in \{1, 2, \ldots, n\}$. Hence $\{a_1, a_2, \ldots, a_n\} \subseteq \mathcal{P}$. Therefore $I = a_1 a_2 \ldots a_n R \in \mathcal{E}_{\mathcal{P}}$. It follows from this that $E_G \subseteq \mathcal{E}_{\mathcal{P}}$.

Let $I = a_1 a_2 \ldots a_n R \in \mathcal{E}_{\mathcal{P}}$ for some $\{a_1, a_2, \ldots, a_n\} \subseteq \mathcal{P}$. Then

$$V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = R \wedge V_i/V_{i-1} \cong R/a_{n-i+1}R,\ i \in \{1, 2, \ldots, n\}.$$

It is clear that $R/a_{n-i+1}R \in [G]$, $i \in \{1, 2, \ldots, n\}$. Hence $V_i/V_{i-1} \in [G]$, $i \in \{1, 2, \ldots, n\}$. Therefore $I = a_1 a_2 \ldots a_n R \in E_G$. Hence $\mathcal{E}_{\mathcal{P}} \subseteq E_G$.

It follows from this that $E = E_G = \mathcal{E}_{\mathcal{P}}$. □

Corollary 2. *Let R be a Dedekind domain. Then for every radical filter $E \neq L_r(R)$ there exists a similarly-closed set $G \subseteq \mathrm{Spec}(R)$ such that $E = T_G$, where*

$$T_G := \{I \in L_r(R) \mid (\exists n \in \mathbb{N}\ \exists I_1, I_2, \ldots, I_n \in G\colon I = I_1 I_2 \ldots I_n) \vee I = R\}.$$

Proof. Let $E \neq L_r(R)$ be a radical filter. Put $G = \mathrm{Spec}_r(R) \cap E$. By the proof of Theorem 1, $E = E_G$ and G is similarly-closed. Let $I = I_1 I_2 \ldots I_n \in E_G$ for some $\{I_1, I_2, \ldots, I_n\} \subseteq \mathrm{Spec}_r(R)$ (see [6]). Since E is a radical filter and $I = I_1 I_2 \ldots I_n \in E \wedge \forall i \in \{1, 2, \ldots, n\}$: $I \subseteq I_i$, by Gl, $\forall i \in \{1, 2, \ldots, n\} : I_i \in E$. Therefore $\{I_1, I_2, \ldots, I_n\} \subseteq G$. Hence $I \in T_G$. It follows from this that $E \subseteq T_G$. Let $I = I_1 I_2 \ldots I_n \in T_G$ for some $I_1, I_2, \ldots, I_n \in G$. Since radical filters are closed under products and $I_1, I_2, \ldots, I_n \in E$, $I \in E$. Hence $T_G \subseteq E$. Therefore $E = T_G$. □

Recall that a right ideal I of a ring R is said to be essential in case for every non-zero right ideal L of R $I \cap L \neq 0$ (Chapter 5 [4]).

Let R be a ring. It is well known that the set of all essential right ideals is a preradical filter of R. This preradical filter is not always a radical filter.

Recall that a right R-module M is semiartinian if every its non-zero factor-module has the non-zero socle. A ring R is called right semiartinian in case R_R is semiartinian (see [2, p.54]).

Theorem 2. *Let R be a right noetherian ring. If for every right ideal I of R the module R/I has a composition series if and only if I is non-zero, then the set of all non-zero right ideals of R is a radical filter of R.*

Proof. Let E be the radical filter corresponding to the hereditary torsion theory generated by the class of all simple right R-modules. Since for every non-zero right ideal I of the ring R the module R/I has a composition series, for every non-zero right ideal I of the ring R the module R/I is artinian. Since every artinian module is semiartinian (see [7, p.182]), for every non-zero right ideal I of the ring R the module R/I is semiartinian. Therefore the radical filter E contains all non-zero right ideals. If $0 \in E$ then R is right semiartinian. But R is right noetherian. By Proposition 2.1 ([7, p.183]), R is right artinian. Therefore $R/0 \cong R$ has a composition series. And we have a contradiction. Therefore $0 \notin E$. □

Corollary 3. *Let R be a right noetherian ring. If for every right ideal I of R the module R/I has a composition series if and only if I is non-zero, then every non-zero right ideal of R is essential.*

Proof. By Theorem 4, the set of all non-zero right ideals of R is a radical filter of R. Now apply G4. □

Corollary 4. *If R is a principal ideal domain then every non-zero right ideal of R is essential.*

Proof. Apply Corollary 5. □

Remark 1. Let R be a right semiartinian ring. Then the preradical filter D consisting of all essential right ideals of R is a radical filter if and only if the ideal $\mathrm{soc}(R_R)$ is idempotent.

Let D be a radical filter.

By Proposition 9.7 ([8]), $\forall I \in D\colon \mathrm{soc}(R_R) \subseteq I$. It is obvious that the ideal $\mathrm{soc}(R_R)$ is an essential right ideal of R. It means that $\mathrm{soc}(R_R) \in D$. Since D is a radical filter, by Lemma 6.7 ([2, p.39]), $(\mathrm{soc}(R_R))^2 \in D$. But $\mathrm{soc}(R_R)$ is the smallest right ideal of D. Therefore $(\mathrm{soc}(R_R))^2 = \mathrm{soc}(R_R)$.

Let $(\mathrm{soc}(R_R))^2 = \mathrm{soc}(R_R)$. Since $\forall I \in D : \mathrm{soc}(R_R) \subseteq I$ and $\mathrm{soc}(R_R) \in D$, $D = \{I \in L_r(R) \mid \mathrm{soc}(R_R) \subseteq I\}$. Since $(\mathrm{soc}(R_R))^2 = \mathrm{soc}(R_R)$, D is a radical filter (see [2, p.59]).

Example 1. Let $R = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$. It is clear that R is a right semiartinian ring. Then $\mathrm{soc}(R_R) = \begin{pmatrix} 0 & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$. It is clear that $(\mathrm{soc}(R_R))^2 = \mathrm{soc}(R_R)$. Therefore the preradical filter D consisting of all essential right ideals of R is a radical filter.

Example 2. Let $R = \mathbb{Z}/4\mathbb{Z}$. It is obvious that R is a right semiartinian ring. Then $\mathrm{soc}(R_R) = 2\mathbb{Z}/4\mathbb{Z}$. It is clear that $(\mathrm{soc}(R_R))^2 = 0 \neq \mathrm{soc}(R_R)$. Therefore the preradical filter D consisting of all essential right ideals of R is not a radical filter.

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