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COARSE STRUCTURES AND FUZZY METRICS

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It is known that every fuzzy metric on a set generates a uniform structure on this set. The aim of this note is to show that every fuzzy metric on a set generates a coarse structure on this set. In the case of fuzzy non-Archimedean metric space, the obtained coarse space turns out to be asymptotically zero-dimensional in the sense of Gromov.

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Известно, что каждая метрика на множестве порождает равномерную структуру на этом множестве. Цель настоящей заметки — показать, что каждая нечеткая метрика на множестве порождает грубую структуру на этом множестве. Показано, что в случае нечеткого неархимедового метрического пространства полученное грубое пространство является асимптотически нульмерным в смысле Громова.

1. Introduction. The notion of fuzzy metric space is tightly related with that of probabilistic metric space introduced by Menger [1] (see also [2]). In probabilistic metric spaces, the distance is defined on distribution functions.

Let $\mathcal{D}^+$ be the set of all probability distribution functions (i.e. nondecreasing, left continuous mappings $F\colon \mathbb{R} \to [0,1]$ such that $F(0) = 0$ and $\sup F(x) = 1$).

An ordered pair (S,d) is said to be a probabilistic metric space if S is a nonempty set and $d\colon S\times S \to \mathcal{D}^+$ be a map that satisfies the following conditions (in the following, $d(p,q)$ is denoted by $d_{p,q}$ for every $(p,q) \in S\times S$:

(1) $d_{u,v}(x) = 1$ for all $x > 0$ and $u = v$;

(2) $d_{u,v}(x) = d_{v,u}(x)$;

(3) $d_{u,v}(x) = 1$ and $d_{v,w}(y) = 1$ imply $d_{u,w}(x+y) = 1$.

We obtain the definition of a Menger space, if we replace condition (3) by the following one: (3') $d_{u,v}(x) * d_{v,w}(y) \le d_{u,w}(x+y)$ (here $*$ is a continuous t-norm; see the definition below). In turn, the notion of fuzzy metric space is obtained from that of Menger space by means of the exponential law: informally, a fuzzy metric is a function $(u,v,x) \mapsto d_{u,v}(x)$.

The notion of a fuzzy metric space is therefore a generalization of that of metric space. It is known that any metric on a set generates both uniform structure on this set and coarse structure ([4]; see the definition below). Recall that the notion of coarse structure is, in some sense, dual to the notion of uniform structure and is intended to formalize the macroscopic properties of spaces.

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It is proved in [3] that every fuzzy metric on a set generates a uniform structure on this set. In this note we show that every fuzzy metric on a set generates also a coarse structure on this set.

M. Gromov [5] defined the notion of asymptotic dimension of metric spaces. This notion can be easily extended over the coarse spaces. We prove that, in the case of non-Archimedean fuzzy metric, the obtained coarse space is asymptotically zero-dimensional.

2. Preliminaries.

2.1 Fuzzy metric spaces. We start with the definition of fuzzy metric spaces (see, e.g., [6]). A continuous t-norm is a continuous map $(x, y) \mapsto x * y \colon [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following properties:

1. $(x * y) * z = x * (y * z)$;
2. $x * y = y * x$;
3. $x * 1 = x$;
4. if $x \leq x'$ and $y \leq y'$, then $x * y \leq x' * y'$.

In other words, a continuous t-norm is a continuous Abelian monoid with unit 1 and with the monotonic operation. The following are examples of continuous t-norms:

1. $x * y = \min\{x, y\}$;
2. $x * y = \max\{0, x + y - 1\}$.

Definition 1. A *fuzzy metric space* is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X \times (0, \infty)$ (i.e. M is a map from $X \times X \times (0, \infty)$ to $[0, 1]$) satisfying the following properties:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, s) * M(y, z, t) \leq M(x, z, s + t)$;
(v) the function $M(x, y, -) \colon (0, \infty) \to [0, 1]$ is continuous.

We obtain the notion of a fuzzy pseudometric space if we replace condition (ii) from the above definition by the following condition:

(ii') $M(x, x, t) = 1$.

We obtain the notion of non-Archimedean fuzzy metric space if we replace condition (iv) in Definition 1 by the following condition:

$$\text{(iv')}\ M(x, y, s) * M(y, z, t) \leq M(x, z, \max\{t, s\}),$$

or, equivalently,

$$M(x, y, t) * M(y, z, t) \leq M(x, z, t)$$

(see, e.g., [7]).

In a fuzzy metric space $(X, M, *)$, we say that the set

$$B_M(x, r, t) = \{y \in X \mid M(x, y, t) > 1 - r\},\ x \in X,\ r \in (0, 1),\ t \in (0, \infty),$$

is the *open ball* of radius $r > 0$ centered at x for t. It is proved in [6] that the family of all open balls is a base of a topology on X; this topology is denoted by τ_M.

A fuzzy metric space is called *proper* if every open ball in it is precompact.

2.2 Coarse structures.

Definition 2. Let X be a set. A coarse structure on X is a *family* $\mathcal{E}$ *of entourages of the diagonal in* $X \times X$ that satisfies the following properties:

1. for every $U \in \mathcal{E}$ there is $V \in \mathcal{E}$ such that $U^{-1} \subset V$;
2. if $U, V \in \mathcal{E}$, then there exists $W \in \mathcal{E}$ such that $U \cup V \subset W$;
3. for every $U, V \in \mathcal{E}$ there exists $W \in \mathcal{E}$ such that $UV \subset W$.

The pair $(X, \mathcal{E})$, where $\mathcal{E}$ is a coarse structure on X, is called a *coarse space.*

A coarse space $(X, \mathcal{E})$ is called *unital* if the diagonal is contained in an entourage.

A coarse space $(X, \mathcal{E})$ is called *connected* if every point of $X \times X$ is contained in an entourage.

Given $x \in X$ and $U \in \mathcal{E}$, let $U(x) = \{y \in X \mid (x, y) \in U\}$.

Let X be a topological space. Let $\mathcal{E}$ be a coarse structure on X. We say that $(X, \mathcal{E})$ is a *topological coarse space* (see [8]) if the following conditions are satisfied:

1. every $U \in \mathcal{E}$ is an open subset of $X \times X$;
2. every set of the form $U(x)$, where $x \in X$ and $U \in \mathcal{E}$, is precompact.

Two coarse spaces $(X, \mathcal{E}_i$, $i = 1, 2$, are said to be equivalent if, every $U \in \mathcal{E}_1$ is contained in some $V \in \mathcal{E}_2$ and also every $U \in \mathcal{E}_2$ is contained in some $V \in \mathcal{E}_1$.

Every metric d on a set X generates the *bounded* coarse structure: a set $A \subset X \times X$ is an entourage if $\sup(d|A) = \sup\{d(x, y) \mid (x, y) \in A\}$ is finite.

One can define the notion of the asymptotic dimension for the coarse spaces [9]. In the sequel, we are interested only the notion of asymptotic dimension zero.

Definition 3. We say that the asymptotic dimension of a coarse space $(X, \mathcal{E})$ is *zero (written* $\operatorname{asdim}(X, \mathcal{E}) = 0$*)* if, for every $U \in \mathcal{E}$ there exists a U-discrete uniformly bounded cover of X.

Here, *U-discreteness* of a family $\mathcal{A}$ of subsets of X means that every set of the form $U(x)$ intersects at most one of the elements of $\mathcal{A}$. A family $\mathcal{A}$ is called *uniformly bounded* if there is $V \in \mathcal{E}$ such that $\mathcal{A}$ is inscribed into $\{V(x) \mid x \in X\}$.

3. Coarse structure generated by a fuzzy metric. Let $(X, M, *)$ be a fuzzy metric space. Let $x_0 = \inf\{x \in [0, 1] \mid x * x > 0\}$. Since $1 * 1 = 1$, we see that $x_0 \in [0, 1)$. There exists a decreasing sequence $(x_i)_{i=0}^{\infty}$ such that: (1) $x_i > x_0$, for every i, (2) $x_{i+1} \leq x_i * x_i$, for every i, and (3) $\lim_{i\to\infty} x_i = x_0$. We let $U_i = \{(x, y) \in X \times X \mid M(x, y, i) > x_i\}$.

Theorem 1. *The family* $\mathcal{E} = \{U_i \mid i \in \mathbb{N}\}$ *is a unital connected coarse structure on the set* X*. The coarse space* $(X, \mathcal{E})$ *is equivalent to a bounded coarse space generated by some metric on* X*.*

Proof. The symmetry of M implies condition (1) from the definition of the coarse structure.

Condition (2) follows from the obvious fact that the family $\mathcal{E}$ is ordered by inclusion.

Let $U, V \in \mathcal{E}$. Without loss of generality, one may assume that $U = V = U_i$, for some i. Then, given $(x, z) \in U_iU_i$, one can find $y \in X$ such that $(x, y), (y, z) \in U_i$. Then

$$M(x, z, 2i) \geq M(x, y, i) * M(y, z, i) > x_i * x_i \geq x_{i+1} \geq x_{2i},$$

whence $(x, z) \in U_{2i}$. This proves condition (3).

If $(x, y) \in X \times X$, then $M(x, y, t) > 0$, for every t. There exist $i, j \in \mathbb{N}$ such that $M(x, y, j) > x_i$. Let $k = \max\{i, j\}$, then

$$M(x, y, k) \geq M(x, y, j) > x_i \geq x_k,$$

i.e. $(x, y) \in U_k$. This proves the connectedness of $\mathcal{E}$.

Evidently, $\mathcal{E}$ is unital.

That $(X, \mathcal{E})$ is equivalent to the bounded coarse structure generated by some metric on X follows from the fact that $\mathcal{E}$ is countable (see [10]). □

Remark 1. The coarse structure defined at the beginning of this section does not depend on the choice of the sequence (x_i) in the following sense. Let also $(x'_i)_{i=0}^{\infty}$ such that: (1) $x'_i > x_0$, for every i, (2) $x'_{i+1} \leq x_i * x_i$, for every i, and (3) $\lim_{i\to\infty} x'_i = x_0$. We let $U'_i = \{(x, y) \in X \times X \mid M(x, y, i) > x'_i\}$. Then the coarse spaces $(X, \mathcal{E})$ and $(X, \mathcal{E}' = \{U'_i \mid i \in \mathbb{N}\})$ are equivalent.

Indeed, consider $U_i \in \mathcal{E}$, $U_i = \{(a, b) \in X \times X \mid M(a, b, i) > x_i\}$. Then there exists $j \geq i$ such that $x_i \geq x'_j$ and we have

$$M(a, b, j) \geq M(a, b, i) > x_i \geq x'_j,$$

whence $(a, b) \in U'_j$, i.e. $U_i \subset U'_j$.

Theorem 2. *Let $(X, M, *)$ be a proper fuzzy metric space. Then $(X, \tau_M, \mathcal{E})$ is a coarse topological space.*

Proof. Given $a \in X$ and $i \in \mathbb{N}$, we see that the set

$$U_i(a) = \{b \in X \mid M(a, b, i) > x_i\} = B(a, 1 - x_i, i)$$

is precompact.

We are going to prove that every U_i is open in $X \times X$. Let $(a, b) \in U_i$. Since $M(a, b, i) > x_i$, from the continuity of $M(a, b, -)$ it follows that there exists $\eta \in (0, i)$ for which $M(a, b, i - \eta) > x_i$.

There exist $\alpha, \beta \in (0, 1)$ such that

$$B(a, \alpha, \eta/2) \subset B(b, 1 - x_i, i),\ \ B(b, \beta, \eta/2) \subset B(a, 1 - x_i, i).$$

Since 1 is a unit for $*$ and $*$ is continuous, one may assume that α and β are small enough so that $(1 - \alpha) * M(a, b, i - \eta) * (1 - \beta) > x_i$.

Let $a' \in B(a, \alpha, \eta/2)$, $b' \in B(b, \beta, \eta/2)$. Then

$$M(a', b', i) \geq M(a', a, \eta/2) * M(a, b, i - \eta) * M(b, b', \eta/2) \geq$$
$$\geq (1 - \alpha) * M(a, b, i - \eta) * (1 - \beta) > x_i$$

and therefore $(a', b') \in U_i$. We conclude that $B(a, \alpha, \eta/2) \times B(b, \beta, \eta/2) \subset U_i$ and thus U_i is open. □

Theorem 3. *Let $(X, M, \min)$ be a non-Archimedean fuzzy metric space. Then* $\operatorname{asdim}(X, \mathcal{E}) = 0$, *where $\mathcal{E}$ denotes the coarse structure constructed above.*

Proof. Consider $U_i \in \mathcal{E}$. We are going to show that the cover $\mathcal{U} = \{U_i(x) \mid x \in X\}$ is U_i-discrete. To this end, let $c \in U_i(a) \cap U_i(b)$, then $c \in B(a, 1 - x_i, i) \cap B(b, 1 - x_i, i)$ (see the proof of Theorem 2) and we obtain

$$M(a, b, i) \geq \min\{M(a, c, i), M(c, b, i)\} > \min\{x_i, x_i\} = x_i,$$

whence $b \in U_i(a)$. This proves the U_i-discreteness of $\mathcal{U}$.

Clearly, $\mathcal{U}$ is uniformly bounded. Thus, $\operatorname{asdim}(X, \mathcal{E}) = 0$. □

4. Remarks and open questions. I. Protasov [11] introduced the ball structures as follows. A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a ball of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. We see that every fuzzy metric $(M, *)$ on a set X generates a ball structure $\mathcal{B} = (X, P, B)$, where $P = (0, 1] \times (0, \infty)$ and, for every $x \in X$ and $(r, t) \in (0, 1] \times (0, \infty)$, the set $B(x, r, t)$ is defined as above.

The notion of fuzzy metric space in the sense of Kramosil and Michalek [12] differs from the above one by the condition

(v') the function $M(x, y, -)\colon (0, \infty) \to [0, 1]$ is left continuous.

It is natural to ask whether the fuzzy metric in this sense also generates a coarse structure.

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