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CAPACITY FUNCTOR ON THE CATEGORY OF ULTRAMETRIC SPACES

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The set of all capacities with compact support on an ultrametric space can be endowed with a natural ultrametric. We show that in that way we obtain a functor in the category of ultrametric spaces and nonexpanding maps. This functor contains the inclusion hyperspace functor as a subfunctor.

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Множество всех емкостей с компактным носителем на ультраметрическом пространстве может быть наделено естественной ультраметрикой. Показано, что таким образом получен функтор в категории ультраметрических пространств и нерастягивающих отображений. Этот функтор содержит функтор гиперпространств включения в качестве подфунктора.

1. Introduction. The notion of capacity was first introduced by Choquet [1]. The capacities have been widely used in decision making and related areas, e.g., pattern recognition (see [3]). The set of upper-semicontinuous capacities on a compact Hausdorff space bears a natural topology [11]. Recently, M. Zarichnyi and O. Nykyforchyn [10] considered the capacity functor in the category **Comp** of compact Hausdorff spaces and established some its properties. In particular, they proved that this functor is open (i.e. preserves the class of open maps) and is the functorial part of a monad on the category **Comp**.

Since the capacities are generalizations of probability measures, it is natural to ask whether results known for the measures have their counterparts also for capacities. In this note we consider the question of ultrametrization of the set of capacities on the ultrametric spaces. Recall that a metric d on a set X is said to be an *ultrametric* if the following strong triangle inequality holds:

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}$$

for all $x, y, z \in X$. The natural ultrametric on the set of probability measures in defined in [9] and investigated, in particular, in [4]. The obtained ultrametric space of probability measures has applications to different problems of programming language semantics.

In [5], a natural ultrametric was defined on the set of idempotent measures of ultrametric space. The aim of this note is to define analogously an ultrametric on the set of capacities with compact supports on an ultrametric space. We show that this ultrametric is functorial in the category of ultrametric spaces and nonexpanding maps. There is, however, a substantial difference between the case of capacities and that of probability (idempotent) measures

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caused by the fact that the functor of capacities does not preserve the preimages, i.e. is not a normal functor in the sense of E. Shchepin. The obtained capacity functor in the category of ultrametric spaces is not a functor with continuous supports.

Finally, we demonstrate that a construction used to define monads in the category of ultrametric spaces and non-expanding maps generated by functors of idempotent and probability measures does not work for the case of the capacity functor.

2. Preliminaries.

2.1. Ultrametric spaces. By $O_r(A)$ we denote the r-neighborhood of a set A in a metric space. We write $O_r(x)$ if $A = \{x\}$.

In [9], an ultrametric is defined on the set of probability measures with compact supports defined on an ultrametric space.

Recall that a map $f\colon X \to Y$, where (X, d) and $(Y, \varrho$, are metric spaces, is called *nonexpanding* if $\varrho(f(x), f(y)) \le d(x, y)$, for every $x, y \in X$.

Let (X, d) be an ultrametric space. By $\exp X$ we denote the set of all nonempty compact subsets in X endowed with the Hausdorff metric $d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}$. For a continuous map $f\colon X \to Y$ the map $\exp f\colon \exp X \to \exp Y$ is defined as $\exp f(A) = f(A)$. It is well-known that $\exp f$ is a nonexpanding map if so is f. We denote by $s_X\colon X \to \exp X$ the singleton map, $s_X(x) = \{x\}$.

By $\mathrm{CL}(X)$ we denote the family of nonempty closed subsets of X. For any $r > 0$, we define $\mathcal{O}_r = \mathcal{O}_r^X = \{U \subset X \mid U$ is a union of balls of radius $r\}$.

Let (X, d) be an ultrametric space and $r > 0$. Let $\mathcal{F}_r = \{f\colon X \to \mathbb{R} \mid f^{-1}(f(x)) \in \mathcal{O}_r$ for any $x \in X\}$.

2.2. Capacities with compact supports. Let X be a metric space. A *capacity* on X is a function $c\colon \mathrm{CL}(X) \cup \{\varnothing\} \to I$ satisfying the properties (below F, G are closed subsets in X):

1. $c(\varnothing) = 0$, $c(X) = 1$;
2. if $F \subset G$, then $c(F) \le c(G)$ (monotonicity);
3. if $c(F) < a$, then there exists an open subset $U \supset F$ such that for every $G \subset U$ we have $c(G) < a$ (upper semicontinuity).
4. there is $F \in \exp X$ such that $c(F) = 1$.

   We say that a capacity c is *convex* if:

5. for every F, G we have $c(F \cup G) + c(F \cap G) \ge c(F) + c(G)$.

   Similarly, c is called *concave* if:

6. for every F, G we have $c(F \cup G) + c(F \cap G) \le c(F) + c(G)$.

A capacity c on a metric space X is said to be of *compact support* if there exists $A \in \exp X$ such that $c(B) = c(B \cap A)$, for every $B \in \exp X$. The minimal (with respect to the inclusion) set A with the property defined above is called the *support* of c and is denoted by $\mathrm{supp}(c)$. We denote by $M(X)$ the set of capacities on X with compact supports. Note that every capacity with compact support is completely determined by its values on the family of compact subsets.

For every $x \in X$, denote by $\eta_X(x)$ the capacity defined as follows: $\eta_X(x)(A) = 1$ whenever $x \in A$ and $\eta_X(x)(A) = 0$ otherwise. Obviously $\mathrm{supp}(\eta_X(x)) = \{x\}$.

Given a continuous map $f\colon X \to Y$ of metric spaces, we define a map $M(f)\colon M(X) \to M(Y)$ by the condition $M(f)(c)(A) = c(\mathrm{supp}(c) \cap f^{-1}(A))$, for every $A \in \exp Y$. One can easily see that the capacity $M(f)(c)$ is well-defined by this condition.

Given a capacity $c \in M(X)$ and an open set (respectively closed) $U \subset X$, we define $c(U) = \sup\{c(A) \mid A \in \exp X,\ A \subset U\}$. It is easy to see that, for any two open (respectively closed) sets U, V with $U \subset V$, we have $c(U) \leq c(V)$. Actually, the capacities could be extended also on the class of Borel subsets but we do not need this extension for our exposition.

Let X be a compact Hausdorff space. By $C(X)$ we denote, as usual, the Banach space of continuous functions on X. Given $\varphi \in C(X)$ and $t \in \mathbb{R}$, we let $(\varphi \geq t) = \{x \in X \mid \varphi(x) \geq t\}$. If $c \in M(X)$, then the *Choquet integral* of φ with respect to c is defined as follows:

$$\int_X \varphi dc = \int_0^\infty c(\varphi \geq 0)dt + \int_{-\infty}^0 (c(\varphi \geq 0) - 1)dt.$$

3. Ultrametric on the set of capacities. There are different ways to define a metric on the set of capacities on a metric space (see [10]). Here we assume that (X, d) is an ultrametric space. Given $c_1, c_2 \in M(X)$, we let

$$\hat{d}(c_1, c_2) = \inf\left\{r > 0 \mid \int_X \varphi dc_1 = \int_X \varphi dc_2 \text{ for every } \varphi \in \mathcal{F}_r\right\}.$$

Theorem 1. *The function $\hat{d}$ is an ultrametric on the set $M(X)$.*

Proof. We first show that the function $\hat{d}$ is well defined. Let $r > \mathrm{diam}(\mathrm{supp}(c_1) \cup \mathrm{supp}(c_2))$. Then any $\varphi \in \mathcal{F}_r$ is constant on the set $\mathrm{supp}(c_1) \cup \mathrm{supp}(c_2)$, whence $\int_X \varphi dc_1 = \int_X \varphi dc_2$. Therefore, $\hat{d}(c_1, c_2) \leq r$.

By the definition, $\hat{d}(c_1, c_2) \geq 0$, for every $c_1, c_2 \in M(X)$. Obviously, $\hat{d}(c, c) = 0$, for every $c \in M(X)$.

Suppose now that $\hat{d}(c_1, c_2) = 0$. Then, for every $\varphi \in \mathcal{F} = \cup\{\mathcal{F}_r \mid r > 0\}$, we have $\int_X \varphi dc_1 = \int_X \varphi dc_2$. We let $\mathcal{F}' = \{\varphi|(\mathrm{supp}(c_1) \cup \mathrm{supp}(c_2)) \mid \varphi \in \mathcal{F}\}$.

Note that $\mathcal{F}'$ is a subalgebra of $C(\mathrm{supp}(c_1) \cup \mathrm{supp}(c_2))$ which contains all the constants and separates the points. By the Stone-Weierstrass theorem (see, e.g., [2]), the set $\mathcal{F}'$ is dense in $C(\mathrm{supp}(c_1) \cup \mathrm{supp}(c_2))$, whence $c_1 = c_2$. Evidently, the function $\hat{d}$ is symmetric.

We are going to prove the strong triangle inequality for $\hat{d}$. Let $c_1, c_2, c_3 \in M(X)$ and suppose that $\hat{d}(c_1, c_2) < r$, $\hat{d}(c_2, c_3) < s$. Then, for every $t > \sup\{r, s\}$, and every $\varphi \in \mathcal{F}_t$, we have

$$\int_X \varphi dc_1 = \int_X \varphi dc_2 = \int_X \varphi dc_3,$$

whence $\hat{d}(c_1, c_3) < t$. Passing to the limit we obtain the desired inequality. □

Lemma 1. *Let (X, d) be an ultrametric space, $c_1, c_2 \in M(X)$. Then*

$$\hat{d}(c_1, c_2) = \inf\{r > 0 \mid c_1(U) = c_2(U),\ \textit{for all } U \in \mathcal{O}_r\}. \tag{1}$$

Proof. We denote the right-hand side of 1 by $\varrho(c_1, c_2)$. Suppose that $r > \varrho(c_1, c_2)$ and $\varphi \in \mathcal{F}_r$. Without loss of generality, one may assume that $\varphi \geq 0$ and, since the set $\mathrm{supp}(c_1) \cup \mathrm{supp}(c_2)$ is compact, that the set $\varphi(X) = \{x_1, \ldots, x_n, x_{n+1}\}$ is finite. Moreover, we assume that $x_1 \geq \cdots \geq x_n \geq x_{n+1} = 0$.

Then

$$\int_X \varphi dc_1 = \sum_{i=1}^{n}(x_i - x_{i+1})c_1(\varphi \geq x_i) = \sum_{i=1}^{n}(x_i - x_{i+1})c_2(\varphi \geq x_i) = \int_X \varphi dc_2,$$

because $(\varphi \geq x_i) \in \mathcal{O}_r$, for every i. Thus $\hat{d}(c_1, c_2) \leq \varrho(c_1, c_2)$.

The proof of the opposite inequality is left to the reader. □

In the sequel, we use formula (1) as an equivalent definition of the ultrametric $\hat{d}$. The following metric $\tilde{d}$ on $M(X)$, which is a counterpart of the Prohorov metric on the set of probability measures, is introduced in [10]. Let $c_1, c_2 \in M(X)$, we define

$$\tilde{d}(c_1, c_2) = \inf\{\varepsilon > 0 \mid c_1(F) \leq c_2(O_\varepsilon(F)) + \varepsilon,\ c_2(F) \leq c_1(O_\varepsilon(F)) + \varepsilon,\ F \in \exp X\}.$$

Proposition 1. *The identity map* $1_X\colon (X, \hat{d}) \to (X, \tilde{d})$ *is nonexpanding.*

Proof. Suppose that $c_1, c_2 \in M(X)$ and $\hat{d}(c_1, c_2) = a$. Let $F \in \exp X$. For any $\varepsilon > 0$, there exists a finite disjoint cover $\mathcal{U}$ of F by closed balls of radius $a + \varepsilon$ and we have

$$c_1(F) \leq c_1(\cup\mathcal{U}) = c_2(\cup\mathcal{U}) < c_2(\cup\mathcal{U}) + a + \varepsilon, c_2(F) \leq c_2(\cup\mathcal{U}) = c_1(\cup\mathcal{U}) < c_1(\cup\mathcal{U}) + a + \varepsilon,$$

whence $\tilde{d}(c_1, c_2) \leq a + \varepsilon$. Because of arbitrariness of ε, we are done. □

For compact ultrametric X, it follows from Proposition 1 that the topology generated by the metric $\hat{d}$ on $M(X)$ is stronger that the weak topology considered in [11].

Proposition 2. *The set* $M_\omega(X)$ *of the points with finite support is dense in* $M(X)$.

Proof. Let $c \in M(X)$ and $r > 0$. Choose a finite disjoint cover $\{O_r(x_i) \mid i = 1, \dots, k\}$ of the set $\mathrm{supp}(c)$ by closed balls. Define the capacity c' by the condition $c'(A) = c(\bigcup\{O_r(x_i) \mid x_i \in A\})$, for every $A \in \exp X$. One can easily see that $c' \in M(X)$ is well-defined and $\mathrm{supp}(c') \subset \{x_1, x_2, \dots, x_k\}$.

We are going to prove that $\hat{d}(c, c') \leq r$. Let $\varphi \in \mathcal{F}_r$. Without loss of generality, one may assume that the set $\varphi(X)$ is finite. We may also assume that

$$\varphi(X) = \{y_1, \dots, y_k, y_{k+1}\},\ \varphi(y_1) \geq \dots \geq \varphi(y_k) \geq \varphi(y_{k+1}) = 0.$$

Then

$$\int_X \varphi dc = \sum_{i=1}^{k}(y_i - x_{y+1})c(\varphi \geq y_i) = \sum_{i=1}^{k}(y_i - y_{i+1})c'(\varphi \geq y_i) = \int_X \varphi dc',$$

by the definition of c'.

□

It is easy to see that one can similarly prove the following generalization of this proposition.

Proposition 3. *Let* Y *be a dense subset in* X. *The set* $\{c \in M_\omega(X) \mid \mathrm{supp}(c) \subset Y$ *is dense in* $M(X)$.

Proposition 4. *The map* $\eta_X\colon X \to M(X)$ *is an isometric embedding.*

Proof. Suppose that $d(x,y) \le c$, for some $c \ge 0$. Then, for every $\varepsilon > 0$ and every $f \in \mathcal{F}_{c+\varepsilon}$, we have $f(x) = f(y)$, whence

$$\int_X f d\eta_X(x) = f(x) = f(y) = \int_X f d\eta_X(y),$$

and we conclude that $\hat{d}(\eta_X(x), \eta_X(y)) \le c$.

Conversely, if $c - \varepsilon > 0$, for some $\varepsilon > 0$, then there exists $f \in \mathcal{F}_{c-\varepsilon}$ such that $f(x) = 0$, $f(y) = 1$. Then

$$\int_X f d\eta_X(x) = f(x) = 0 \ne 1 = f(y) = \int_X f d\eta_X(y),$$

whence $\hat{d}(\eta_X(x), \eta_X(y)) \ge c - \varepsilon$. Letting $\varepsilon \to 0$, we are done. □

Proposition 5. *Let $f\colon X \to Y$ be a nonexpanding map of ultrametric spaces. Then the map $M(f)\colon M(X) \to M(Y)$ is also nonexpanding.*

Proof. We denote the metrics in X and Y by d and ϱ respectively. Suppose that $c_1, c_2 \in M(X)$ and $\hat{d}(c_1, c_2) = a$. Let $b > a$ and $U \in \mathcal{O}_b^Y$. Then $f^{-1}(U) \in \mathcal{O}_b^X$, because f is nonexpanding. We have

$$M(f)(c_1)(U) = c_1(f^{-1}(U)) = c_2(f^{-1}(U)) = M(f)(c_2)(U).$$

Therefore, $\hat{\varrho}(M(f)(c_1), M(f)(c_2)) \le b$. Since $b > a$ is arbitrary, we are done. □

Unlikely to the case of probability measures (see, e.g., [4]), the map $\mathrm{supp}\colon M(X) \to \exp X$ is not nonexpanding, as the following example demonstrates.

Example 1. Let $X = \{x, y, z\}$ and the ultrametric d on X is defined by: $d(x, y) = 1$, $d(x, z) = d(y, z) = 2$. Let $c_1, c_2 \in M(X)$ be defined as follows: $c_1(\{x, y\}) = c_1(X) = 1$, $c_1(A) = 0$ for the other subsets A of X; $c_2(\{x, z\}) = 1/2$, $c_2(\{x, y\}) = c_2(X) = 1$, $c_2(A) = 0$ for the other subsets A of X.

Then $\mathrm{supp}(c_1) = \{x, y\}$, $\mathrm{supp}(c_2) = X$ and therefore $d_H(\mathrm{supp}(c_1), \mathrm{supp}(c_2)) = 2$. At the same time, note that $\mathcal{U}_1 = \{\{x, y\}, \{z\}, X\}$ and $c_1(A) = c_2(A)$ for every $A \in \mathcal{U}_1$. Thus, $\hat{d}(c_1, c_2) = 1$.

By $G(X)$ we denote the subset of $M(X)$ consisting of capacities taking only two values, 0 and 1. Given $c \in G(X)$, one can easily see that the family $c^{-1}(1)$ is an inclusion hyperspace of X (see [8]).

Proposition 6. *The set $G(X)$ is closed in $M(X)$.*

Proof. Let $c \in M(X) \setminus G(X)$. Then there exists $F \in \exp X$ such that $0 < c(F) < 1$. By the definition, there exists a neighborhood U of F such that $c(F') < 1$, for every $F' \in \exp X$, $F' \subset U$. Then there exists $r > 0$ and a finite disjoint cover $\mathcal{U}$ of F by closed r-balls such that $\cup\mathcal{U} \subset U$. Let φ denote the characteristic function of the set $\cup\mathcal{U}$.

Now, suppose that $c' \in M(X)$ and $\hat{d}(c, c') < r$. Then $\int_X \varphi dc = \int_X \varphi dc_1 < 1$, whence $c' \in M(X) \setminus G(X)$. This demonstrates that $M(X) \setminus G(X)$ is an open subset in $M(X)$. □

Remark 1. One can similarly prove that, for any closed subset $K \subset [0, 1]$ with $\{0, 1\} \subset K$, the set $M_K(X) = \{c \in M(X) \mid c(A) \in K \text{ for any } A \in \exp X\}$ is closed in $M(X)$.

By **UMET** we denote the category of ultrametric spaces and nonexpanding maps. It follows from Theorem 1 and Proposition 5 that M is a covariant functor from the category **UMET** to itself.

Given $\varphi \in C(X)$, define $\bar{\varphi}\colon M(X) \to \mathbb{R}$ as follows: $\bar{\varphi}(c) = \int_X \varphi dc$, $c \in M(X)$.

Lemma 2. *If $\varphi \in \mathcal{F}_r$, for some $r > 0$, then $\bar{\varphi} \in \mathcal{F}_r(M(X))$.*

Proof. If $\hat{d}(c_1, c_2) < r$, for some $c_1, c_2 \in M(X)$, then

$$\bar{\varphi}(c_1) = \int_X \varphi dc_1 = \int_X \varphi dc_2 = \bar{\varphi}(c_2),$$

whence we conclude that $\bar{\varphi}$ is constant on all the balls of radius r. Thus, $\bar{\varphi} \in \mathcal{F}_r(M(X))$. $\square$

Further, we proceed similarly as in the case of probability measures (see [4]). Given $C \in M(M(X))$, define $\mu(C)\colon C(X) \to \mathbb{R}$ as follows: $\mu(C)(\varphi) = C(\bar{\varphi}))$, $\varphi \in C(X)$. The following example shows that, unlikely to the case of probability measures (see [4]), in general, $\mu(C) \notin M(X)$.

Let $X = \{a_i \mid i \in \{0\} \cup \mathbb{N}\} \cup \{b_i \mid i \in \mathbb{N}\}$. (We suppose that all the a_is and b_is in X are distinct.) We define a metric $d\colon X \times X \to \mathbb{R}$ as follows:

1. $d(b_i, x) = 1$, for all $x \in X \setminus \{b_i\}$;
2. $d(a_0, a_i) = 2^{-i}$, $i \in \mathbb{N}$;
3. $d(a_i, a_j) = 2^{-\min\{i,j\}}$, $i, j \in \mathbb{N}$, $i \neq j$.

One can easily see that the obtained metric is an ultrametric on X.

For every $i \in \mathbb{N}$, define $c_i \in M(X)$ as follows: $c_i(A) = 1$, if and only if $|A \cap \{a_0, a_i, b_i\}| \geq 2$, otherwise $c_i(A) = 0$. Then the sequence $(c_i)_{i=1}^{\infty}$ in $(M(X), \hat{d})$ converges to $c_0 = \eta_X(a_0)$.

Define $C \in M(M(X))$ as follows:

$$\int_{M(X)} \psi dC = \max\left\{\psi(c_i) \mid i \in \{0\} \cup \mathbb{N}\right\}.$$

In order to prove that $\mu(C) \notin M(X)$, we are going to demonstrate that the functional $\mu(C)\colon C(X) \to \mathbb{R}$ is not of compact support. The latter means that there is no compact subset $K \subset X$ with the following property: $\mu(C)(\varphi) = \mu(C)(\varphi')$, whenever $\varphi|K = \varphi'|K$.

Suppose the contrary, i.e. that there is such a K. Let $i \in \mathbb{N}$. Define $\varphi_i\colon X \to \mathbb{R}$ as follows: $\varphi_i(b_i) = 2$, $\varphi_i(a_i) = 1$, and $\varphi_i(x) = 0$ whenever $x \notin \{a_i, b_i\}$. Then $\bar{\varphi}_i(c_j) = 0$, for all $j \neq i$. Therefore,

$$\mu(C)(\varphi_i) = \bar{\varphi}_i(c_i) = \int_X \varphi_i dc_i = 2 \cdot c_i(\varphi_i \geq 2) + 1 \cdot c_i(\varphi_i \geq 1) = 2 + 1 = 3.$$

Now, define $\varphi_i'\colon X \to \mathbb{R}$ as follows: $\varphi_i'(b_i) = 3$, $\varphi_i'(x) = \varphi_i(x)$ otherwise. Then $\mu(C)(\varphi_i') = 4 \neq 3 = \mu(C)(\varphi_i)$ and therefore $b_i \in K$. We conclude that $\{b_i \in \mathbb{N}\} \subset K$, which contradicts to the compactness of K.

4. Remarks and open questions. The following question is open.

Question 1. *Are the subspaces of convex and concave capacities closed in the space of all capacities?*

It is proved in [4] and [5] that the spaces of probability and idempotent measures of an ultrametric space are complete.

Question 2. *Is the space $M(X)$ complete if so is X?*

One can endow the set $M(X)$, for an ultrametric space (X,d), with the metric $\bar{d}$ defined as follows: $\bar{d}(c_1,c_2) = \max\{\hat{d}(c_1,c_2), d_H(\mathrm{supp}(c_1),\mathrm{supp}(c_2))\}$. One can easily see that the metric $\bar{d}$ is functorial on the category **UMET**. Let us denote the obtained functor by $\overline{M}$.

Question 3. *Does the functor $\overline{M}$ determine a monad (see [6] for the necessary definitions) on the category* **UMET**?

Note that the monad of capacities in the category of compact Hausdorff spaces and continuous maps is defined in [10].

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