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**TOPOLOGICAL SEMIGROUPS OF MATRIX UNITS AND
COUNTABLY COMPACT BRANDT λ^0 -EXTENSIONS
OF TOPOLOGICAL SEMIGROUPS**

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We show that a topological semigroup of finite partial bijections \mathcal{S}_λ^n of an infinite set with a compact subsemigroup of idempotents is absolutely H -closed and any countably compact topological semigroup does not contain \mathcal{S}_λ^n as a subsemigroup. We give sufficient conditions onto a topological semigroup \mathcal{S}_λ^1 to be non- H -closed. Also we describe the structure of countably compact Brandt λ^0 -extensions of topological monoids and study the category of countably compact Brandt λ^0 -extensions of topological monoids with zero.

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В работе показано, что топологическая полугруппа конечных частичных биекций \mathcal{S}_λ^n бесконечного множества с компактной подполугруппой идемпотентов является абсолютно H -замкнутой полугруппой и произвольная счётно компактная топологическая полугруппа не содержит \mathcal{S}_λ^n как подполугруппу. Мы даём достаточные условия на топологическую полугруппу \mathcal{S}_λ^1 , чтобы последняя не была H -замкнутой. В работе также описана структура счётно компактных λ^0 -расширений Брандта топологических моноидов с нулём и изучается категория счётно компактных λ^0 -расширений Брандта топологических моноидов с нулём.

1. Introduction and preliminaries. In this paper all spaces are Hausdorff. Further we follow the terminology of [9, 10, 12, 14]. By ω we denote the first infinite cardinal. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\text{cl}_Y(A)$ we denote the topological closure of A in Y .

An algebraic semigroup S is called *inverse* if for any element x in S there exists the unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called *inverse to $x \in S$* . If S is an inverse semigroup, then the function $\text{inv}: S \rightarrow S$ which assigns to every element x of S an inverse element x^{-1} is called *inversion*.

If S is a semigroup, then by $E(S)$ we denote the band (the subset of idempotents) of S , and by S^1 [S^0] we denote the semigroup S with the adjoined unit [zero] (see [10]). Also if a semigroup S has zero 0_S , then for any $A \subseteq S$ we denote $A^* = A \setminus \{0_S\}$. For an inverse semigroup S we define the maps $\varphi: S \rightarrow E(S)$ and $\psi: S \rightarrow E(S)$ by the formulae $\varphi(x) = x \cdot x^{-1}$ and $\psi(x) = x^{-1} \cdot x$.

If E is a semilattice, then the semilattice operation on E determines the partial order \leq on E :

$$e \leq f \quad \text{if and only if} \quad ef = fe = e.$$

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This order is called *natural*. For every element e of a semilattice E we denote $\uparrow e = \{f \in E: e \leq f\}$ and $\downarrow e = \{f \in E: f \leq e\}$. An element e of a partially ordered set X is called *minimal* if $f \leq e$ implies $f = e$ for $f \in X$. An idempotent e of a semigroup S without zero (with zero) is called *primitive* if e is a minimal element in $E(S)$ (in $(E(S))^*$).

A *topological (inverse) semigroup* is a topological space together with a continuous multiplication (and an inversion, respectively).

A topological space X is called *countably compact* if any countable open cover of X contains a finite subcover [12]. A topological space X is called *pseudocompact (discretely pseudocompact)* if every locally finite (discrete) family of non-open subsets of X is finite [12]. A Tychonoff topological space X is *pseudocompact* if and only if each continuous real-valued function on X is bounded (see [12, Theorem 3.10.22]). Obviously that every countably compact space is pseudocompact and every pseudocompact space is discretely pseudocompact. Also we observe that every quasi-regular discretely pseudocompact space is pseudocompact. We recall that the Stone-Ćech compactification of a Tychonoff space X is a compact Hausdorff space βX containing X as a dense subspace so that each continuous map $f: X \rightarrow Y$ to a compact Hausdorff space Y extends to a continuous map $\bar{f}: \beta X \rightarrow Y$ [12].

Let S be a semigroup with zero and λ be cardinal ≥ 1 . On the set $B_\lambda(S) = \lambda \times S \times \lambda \cup \{0\}$ we define the semigroup operation as follows

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$. If $S = S^1$ is a semigroup with unity then the semigroup $B_\lambda(S)$ is called the *Brandt λ -extension of the monoid S* [19]. Obviously, $\mathcal{J} = \{0\} \cup \{(\alpha, \mathcal{O}, \beta): \mathcal{O} \text{ is the zero of } S\}$ is an ideal of $B_\lambda(S)$. We put $B_\lambda^0(S) = B_\lambda(S)/\mathcal{J}$ and we shall call $B_\lambda^0(S)$ the *Brandt λ^0 -extension of the monoid S with zero* [22]. Further, if $A \subseteq S$ then we shall denote $A_{\alpha,\beta} = \{(\alpha, s, \beta): s \in A\}$ if A does not contain zero, and $A_{\alpha,\beta} = \{(\alpha, s, \beta): s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in \lambda$. If \mathcal{I} is a trivial semigroup (i.e. \mathcal{I} contains only one element), then by \mathcal{I}^0 we denote the semigroup \mathcal{I} with the adjoined zero. Obviously, for any $\lambda \geq 2$ the Brandt λ^0 -extension of the semigroup \mathcal{I}^0 is isomorphic to the semigroup of $\lambda \times \lambda$ -matrix units and any Brandt λ^0 -extension of a monoid with zero contains the semigroup of $\lambda \times \lambda$ -matrix units. Further by B_λ we shall denote the semigroup of $\lambda \times \lambda$ -matrix units and by $B_\lambda^0(1)$ the subsemigroup of $\lambda \times \lambda$ -matrix units of the Brandt λ^0 -extension of a monoid S with zero.

Let \mathcal{S}_λ denote the set of all partial one-to-one transformations of a set X of cardinality λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if } x \in \text{dom}(\alpha\beta) = \{y \in \text{dom } \alpha: y\alpha \in \text{dom } \beta\}, \quad \text{for } \alpha, \beta \in \mathcal{S}_\lambda.$$

The semigroup \mathcal{S}_λ is called the *symmetric inverse semigroup* over the set X (see [10]). The symmetric inverse semigroup was introduced by Wagner [39] and it plays a major role in the theory of semigroups.

We denote $\mathcal{S}_\lambda^n = \{\alpha \in \mathcal{S}_\lambda: \text{rank } \alpha \leq n\}$, for $n = 1, 2, 3, \dots$. Obviously, \mathcal{S}_λ^n ($n = 1, 2, 3, \dots$) is an inverse semigroup, \mathcal{S}_λ^n is an ideal of \mathcal{S}_λ for each $n = 1, 2, 3, \dots$. Further, we shall call the semigroup \mathcal{S}_λ^n the *symmetric inverse semigroup of finite transformations of the rank n* . The elements of the semigroup \mathcal{S}_λ^n are called *finite one-to-one transformations (partial bijections)* of the set X . By

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

we denote a partial one-to-one transformation which maps x_1 onto y_1 , x_2 onto y_2 , \dots , and x_n onto y_n , and by 0 the empty transformation. Obviously, in such case we have $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$ ($i, j = 1, 2, 3, \dots, n$). We observe that the symmetric inverse semigroup \mathcal{S}_λ^1 of finite transformations of the rank 1 is isomorphic to the semigroup of matrix units B_λ .

A semigroup S is called *congruence-free* if it has only two congruences: identical and universal [34]. Obviously, a semigroup S is congruence-free if and only if every homomorphism h of S into an arbitrary semigroup T is an isomorphism “into” or is an annihilating homomorphism (i. e. there exists $c \in T$ such that $h(a) = c$ for all $a \in S$).

Let \mathcal{S} be a class of topological semigroups.

Definition 1 ([19, 36]). A semigroup $S \in \mathcal{S}$ is called *H-closed in \mathcal{S}* , if S is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains S as a subsemigroup. If \mathcal{S} coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*.

Definition 2 ([20, 37]). A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed in the class \mathcal{S}* if any continuous homomorphic image of S into $T \in \mathcal{S}$ is *H-closed in \mathcal{S}* . If \mathcal{S} coincides with the class of all topological semigroups, then the semigroup S is called *absolutely H-closed*.

A semigroup S is called *algebraically closed in \mathcal{S}* if S with any semigroup topology τ such that $(S, \tau) \in \mathcal{S}$ is *H-closed in \mathcal{S}* [19]. If \mathcal{S} coincides with the class of all topological semigroups, then the semigroup S is called *algebraically closed*. A semigroup S is called *algebraically h-closed in \mathcal{S}* if S with discrete topology \mathfrak{d} is absolutely *H-closed in \mathcal{S}* and $(S, \mathfrak{d}) \in \mathcal{S}$. If \mathcal{S} coincides with the class of all topological semigroups, then the semigroup S is called *algebraically h-closed*.

Absolutely *H-closed* semigroups and algebraically *h-closed* semigroups were introduced by Stepp in [37]. There they were called *absolutely maximal* and *algebraic maximal*, respectively.

Many topologists have studied topological properties of topological spaces of partial continuous maps $\mathcal{PC}(X, Y)$ from a topological space X into a topological space Y with various topologies such as the Vietoris topology, generalized compact-open topology, graph topology, τ -topology, and others (see [1, 8, 11, 13, 26, 27, 28, 29]). Since the set of all partial continuous self-transformations $\mathcal{PC}\mathcal{T}(X)$ of the space X with the operation composition is a semigroup, many semigroup theorists have considered the semigroup of continuous transformations (see surveys [30] and [17]), or the semigroup of partial homeomorphisms of an arbitrary topological space (see [2, 3, 4, 5, 16, 31, 35, 40]). Beřida [7], Orlov [32, 33], and Subbiah [38] have considered semigroup and inverse semigroup topologies of semigroups of partial homeomorphisms of some classes of topological spaces. In this context the results of our paper yield some notable results about the topological behavior of the finite rank symmetric inverse semigroups setting inside larger function space semigroups, or larger semigroups in general. For example, under reasonably general conditions, the inverse semigroup of partial finite bijections \mathcal{S}_λ^n of rank $\leq n$ is a closed subsemigroup of a topological semigroup which contains \mathcal{S}_λ^n as a subsemigroup.

Gutik and Pavlyk in [21] consider the partial case of the semigroup \mathcal{S}_λ^n : an infinite topological semigroup of $\lambda \times \lambda$ -matrix units B_λ . There they showed that an infinite topological semigroup of $\lambda \times \lambda$ -matrix units B_λ does not embed into a compact topological semigroup, B_λ is algebraically *h-closed* in the class of topological inverse semigroups, described the Bohr

compactification of B_λ and minimal semigroup and minimal semigroup inverse topologies on B_λ .

Gutik, Lawson and Repovš in [18] introduced the notion of a semigroup with a tight ideal series and investigated their closures in semitopological semigroups, particularly inverse semigroups with continuous inversion. As a corollary they showed that the symmetric inverse semigroup of finite transformations \mathcal{S}_λ^n of infinite cardinal λ is algebraically closed in the class of (semi)topological inverse semigroups with continuous inversion. They also derived related results about the nonexistence of (partial) compactifications of classes of considered semigroups.

In [23] Gutik and Reiter showed that the topological inverse semigroup \mathcal{S}_λ^n is algebraically h -closed in the class of topological inverse semigroups. Also they proved that a topological semigroup S with countably compact square $S \times S$ does not contain the semigroup \mathcal{S}_λ^n for infinite cardinal λ and showed that the Bohr compactification of an infinite topological semigroup \mathcal{S}_λ^n is the trivial semigroup.

Gutik and Repovš in [24] study algebraic properties of Brandt λ^0 -extensions of monoids with zero and non-trivial homomorphisms between Brandt λ^0 -extensions of monoids with zero. Also they described a category whose objects are ingredients of the construction of Brandt λ^0 -extensions of monoids with zeros. There they introduce finite, compact topological Brandt λ^0 -extensions of topological semigroups and countably compact topological Brandt λ^0 -extensions of topological inverse semigroups in the class of topological inverse semigroups and establish the structure of such extensions and non-trivial continuous homomorphisms between such topological Brandt λ^0 -extensions of topological monoids with zero. They also describe a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt λ^0 -extensions of topological monoids with zeros.

In this paper we show that a topological semigroup of finite partial bijections \mathcal{S}_λ^n of an infinite set with a compact subsemigroup of idempotents is absolutely H -closed. We prove that any countably compact topological semigroup and any Tychonoff topological semigroup with pseudocompact square do not contain \mathcal{S}_λ^n as a subsemigroup. Moreover every continuous homomorphism from topological semigroup \mathcal{S}_λ^n into a countably compact topological semigroup or Tychonoff topological semigroup with pseudocompact square is annihilating. We give sufficient conditions onto a topological semigroup \mathcal{S}_λ^1 to be non- H -closed and show that the topological inverse semigroup \mathcal{S}_λ^1 is absolutely H -closed if and only if the band $E(\mathcal{S}_\lambda^1)$ is compact. Also we describe the structure of countably compact Brandt λ^0 -extensions of topological monoids and establish the category of countably compact Brandt λ^0 -extensions of topological monoids with zero.

2. On the closure and embedding of the semigroup of matrix units.

Lemma 1. *Let E be a topological semilattice with zero 0 such that every non-zero idempotent of E is primitive. Then every non-zero element of E is an isolated point in E . Moreover for the infinite topological semilattice E the following conditions are equivalent:*

- (i) E is compact;
- (ii) E is countably compact;
- (iii) E is pseudocompact;
- (iv) E is discretely pseudocompact;
- (v) E is homeomorphic to the one-point Alexandroff compactification of the discrete space X of cardinality $|E|$ with zero 0 as the remainder.

Proof. Let $x \in E^*$. Since E is a Hausdorff topological semilattice, for any open neighbourhood $U(x) \not\ni 0$ of the point x there exists an open neighbourhood $V(x)$ of x such that $V(x) \cdot V(x) \subseteq U(x)$. If x is not an isolated point of E then $V(x) \cdot V(x) \ni 0$ which contradicts to the choice of the neighbourhood $U(x)$. This implies the first assertion of the lemma.

We observe that the implications $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are trivial.

To show the implication $(iv) \Rightarrow (i)$ suppose that the semilattice E is discretely pseudo-compact and E satisfies the assertion of the lemma. Suppose to the contrary that E is not compact. Let $\mathcal{C} = \{U_s : s \in \mathcal{J}\}$ be any open cover of E such that \mathcal{C} does not contain a finite subcover. Let $U_{s_0} \in \mathcal{C}$ such that $0 \in U_{s_0}$. We denote $A = E \setminus U_{s_0}$. Since the topological semilattice E is non-compact, the set A is infinite. Put $\mathcal{W} = \{A\} \cup \{\{x\} : x \in A\}$. Then \mathcal{W} is an infinite discrete family of open non-empty subsets of E . This contradicts to discrete compactness of A . The obtained contradiction implies that E is a compact semilattice.

Simple verifications show that if the semilattice E is homeomorphic to the one-point Alexandroff compactification of the discrete space X of cardinality $|E|$ with zero 0 as the remainder the semilattice operation is continuous. This implies the implications $(v) \Rightarrow (i)$. Also the first assertion of the lemma implies the implications $(i) \Rightarrow (v)$. \square

Lemma 2. *Let T be a topological semigroup which contains the infinite semigroup of matrix units B_λ as a dense subsemigroup. Then the following conditions hold:*

- (i) *the zero 0 of B_λ is the zero of T ;*
- (ii) *if $T \setminus B_\lambda \neq \emptyset$ then $x^2 = 0$ for all $x \in T \setminus B_\lambda$; and*
- (iii) *$E(T) = E(B_\lambda)$.*

Proof. (i) The set $\{x \in T : x \cdot 0 = 0 \cdot x = 0\}$ is closed and contains the dense subset B_λ , so coincides with T .

(ii) Let $T \setminus B_\lambda \neq \emptyset$. Suppose to the contrary that there exists $x \in T \setminus B_\lambda$ such that $x^2 = y \neq 0$. Then for any open neighbourhoods $U(y)$ and $U(0)$ of y and 0 such that $U(y) \cap U(0) = \emptyset$ there exists an open neighbourhood $V(x)$ such that $V(x) \cdot V(x) \subseteq U(y)$ and $V(x) \cap U(0) = \emptyset$. Since the closure of semilattice in a topological semigroup is subsemilattice (see [20, Corollary 19]) Theorem 9 of [37] implies that the band $E(B_\lambda)$ is a closed subsemigroup of T . Hence without loss of generality we can assume that $V(x) \subseteq T \setminus E(B_\lambda)$. Since the neighbourhood $V(x)$ of the point x contains infinitely many point from $B_\lambda \setminus E(B_\lambda)$, we have that $0 \in V(x) \cdot V(x)$. This contradicts to the assumption that $U(y) \cap U(0) = \emptyset$. Therefore $x^2 = 0$.

(iii) The statement follows from statement (ii). \square

Theorem 1. *A topological semigroup of matrix units B_λ with a compact band $E(B_\lambda)$ is an H -closed topological semigroup.*

Proof. Since the statement theorem is trivial in case when the set $E(B_\lambda)$ is finite, we consider the case when the band $E(B_\lambda)$ is infinite.

Suppose to the contrary that there exists a topological semigroup T which contains B_λ as a non-closed subsemigroup. Without loss of generality we can assume that B_λ is a dense subsemigroup of T and $T \setminus B_\lambda \neq \emptyset$. Let $x \in T \setminus B_\lambda$. By Lemma 2 we have that zero 0 of the semigroup B_λ is zero in the topological semigroup T and $x^2 = 0$.

Since $0 \cdot x = x \cdot 0 = 0$ for any open neighbourhoods $U(x)$ and $U(0)$ in T of x and 0 , respectively, such that $U(x) \cap U(0) = \emptyset$, there exist open neighbourhoods $V(x)$ and $V(0)$ in

T of x and 0 , respectively, such that

$$V(0) \cdot V(x) \subseteq U(0), \quad V(x) \cdot V(0) \subseteq U(0), \quad V(x) \subseteq U(x) \quad \text{and} \quad V(0) \subseteq U(0).$$

Since by Lemma 1 any non-zero idempotent of B_λ is an isolated point in $E(B_\lambda)$, compactness of $E(B_\lambda)$ implies that the set $E(B_\lambda) \setminus V(0)$ is finite and $V(x) \cap E(B_\lambda) = \emptyset$. Since the neighbourhood $V(x)$ contains infinitely many element of the semigroup B_λ and the set $E(B_\lambda) \setminus V(0)$ is finite, there exists $(\alpha, \beta) \in V(x)$ such that either $(\alpha, \alpha) \in V(0)$ or $(\beta, \beta) \in V(0)$. Therefore, we have that at least one of the following conditions holds:

$$(V(x) \cdot V(0)) \cap V(x) \neq \emptyset \quad \text{and} \quad (V(0) \cdot V(x)) \cap V(x) \neq \emptyset.$$

This contradicts to the assumption that $U(x) \cap U(0) = \emptyset$. The obtained contradiction implies the statement of the theorem. \square

Lemma 1 and Theorem 1 imply the following:

Corollary 1. *A topological semigroup of matrix units B_λ with a discretely pseudocompact (pseudocompact, countably compact) band $E(B_\lambda)$ is an H -closed topological semigroup.*

By Theorem 1 [15] the semigroup of matrix units B_λ is congruence free and hence any homomorphic image of B_λ is either the semigroup of matrix units or the trivial semigroup. Since a continuous image of a compact space is a compact space (see [12, Thorem 3.1.10]), Theorem 1 implies the following:

Theorem 2. *A topological semigroup of matrix units B_λ with a compact band $E(B_\lambda)$ is an absolutely H -closed topological semigroup.*

Lemma 1 and Theorem 2 imply the following:

Corollary 2. *A topological semigroup of matrix units B_λ with a discretely pseudocompact (pseudocompact, countably compact) band $E(B_\lambda)$ is an absolutely H -closed topological semigroup.*

Question 1. *Assume that a topological semigroup B_λ is absolutely H -closed. Is the idempotent band $E(B_\lambda)$ compact?*

The following theorem shows that the converse statement to Theorem 1 is true when B_λ is a topological inverse semigroup.

Theorem 3. *If B_λ is an H -closed topological inverse semigroup, then the band $E(B_\lambda)$ is compact.*

Proof. Suppose the contrary: the band $E(B_\lambda)$ is a non-compact subset in (B_λ, τ) . By [21, Lemma 4] any non-zero element of the semigroup B_λ is an isolated point in (B_λ, τ) and hence there exists an open neighbourhood $U(0)$ of zero 0 in (B_λ, τ) such that $A = E(B_\lambda) \setminus (E(B_\lambda) \cap U(0))$ is an infinite subset of $E(B_\lambda)$. Without loss of generality we can assume that A is countable. Next we enumerate the set A by positive integers:

$$A = \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (\alpha_3, \alpha_3), \dots\}.$$

Then A is a closed subset of $E(B_\lambda)$ and hence the continuity of the inversion in (B_λ, τ) implies that $I_A = \varphi^{-1}(E(B_\lambda) \setminus A) \cup \psi^{-1}(E(B_\lambda) \setminus A)$ is an open subset of the topological space (B_λ, τ) .

Let $x \notin B_\lambda$. Put $S = B_\lambda \cup \{x\}$. We extend the semigroup operation from B_λ onto S as follows:

$$x \cdot x = y \cdot x = x \cdot y = 0, \quad \text{for all } y \in B_\lambda.$$

Simple verifications show that such defined operation is associative.

Put $A_n = \{(\alpha_{2k-1}, \alpha_{2k}) : k = n, n+1, n+2, \dots\}$ for any positive integer n . We determine a topology τ^* on S as follows:

- (i) for every $y \in B_\lambda$ the bases of topologies τ and τ^* at y coincide; and
- (ii) $\mathcal{B}(x) = \{U_n(x) = \{x\} \cup A_n : n = 1, 2, 3, \dots\}$ is the base of the topology τ^* at x .

For any open neighbourhood $V(0)$ of zero 0 such that $V(0) \subseteq U(0)$ we have

$$V(0) \cdot U_n(0) = U_n(0) \cdot V(0) = U_n(0) \cdot U_n(0) = \{0\} \subseteq V(0).$$

We observe that the definition of the set A_n implies that for any non-zero element (α, β) of the semigroup B_λ there exists the smallest positive integer $i_{(\alpha, \beta)}$ such that $(\alpha, \beta) \cdot A_{i_{(\alpha, \beta)}} = A_{i_{(\alpha, \beta)}} \cdot (\alpha, \beta) = \{0\}$. Then we have

$$(\alpha, \beta) \cdot U_{i_{(\alpha, \beta)}}(0) = U_{i_{(\alpha, \beta)}}(0) \cdot (\alpha, \beta) = \{0\} \subseteq V(0).$$

Therefore (S, τ^*) is a topological semigroup which contains (B_λ, τ) as a dense subsemigroup. The obtained contradiction implies that $E(B_\lambda)$ is a compact subset of (B_λ, τ) . \square

Since the semigroup of matrix units is congruence-free, Theorems 1 and 3 imply the following theorem:

Theorem 4. *A topological inverse semigroup B_λ is H -closed if and only if the band $E(B_\lambda)$ is compact.*

Gutik and Pavlyk in [21] show that an infinite semigroup of matrix units with the discrete topology is not H -closed. The following proposition gives the sufficient conditions on an infinite topological semigroup of matrix units to be non- H -closed.

Proposition 1. *Let τ be a semigroup topology on an infinite semigroup of matrix units B_λ . If there exists an open neighbourhood $U(0)$ of zero in (B_λ, τ) such that $(\varphi^{-1}(A) \cup \psi^{-1}(A)) \cap U(0) = \emptyset$ for some infinite subset A of $E(B_\lambda)$, then (B_λ, τ) is not an H -closed topological semigroup.*

Proof. We observe that without loss of generality we can assume that the set A is countable.

Let $x \notin B_\lambda$. Put $S = B_\lambda \cup \{x\}$. We extend the semigroup operation from B_λ onto S as follows:

$$x \cdot x = x \cdot a = a \cdot x = 0 \quad \text{for all } a \in B_\lambda.$$

Simple verifications show that such defined binary operation is associative.

Further we enumerate the elements of the set A by the positive integers, i. e. $A = \{(\alpha_i, \alpha_i) : i = 1, 2, 3, \dots\}$. Let $A_n = \{(\alpha_{2k-1}, \alpha_{2k}) : k \geq n\}$ for each positive integer n . A topology τ_0 on S is defined as follows:

- a) bases of topologies τ and τ_0 coincide at any point $a \in B_\lambda$;
- b) $\mathcal{B}(x) = \{U_n(0) = \{x\} \cup A_n : n \text{ is an positive integer}\}$ is a base of the topology τ_0 at the point $x \in S$.

Such defined topology τ_0 on S implies that it is complete to show that the semigroup operation on (S, τ_0) is continuous in the following cases:

$$1) x \cdot x = 0, \quad 2) x \cdot 0 = 0, \quad 3) 0 \cdot x = 0, \quad 4) x \cdot a = 0, \quad 5) a \cdot x = 0, \quad \text{for } a \in B_\lambda \setminus \{0\}.$$

Then

$$U_n(x) \cdot U_n(x) = U_n(x) \cdot V(0) = V(0) \cdot U_n(x) = \{0\} \subseteq V(0),$$

for any $U_n(x) \in \mathcal{B}(x)$ and any open neighbourhood $V(0)$ of zero in S such that $V(0) \subseteq U(0)$. For every $a \in B_\lambda \setminus \{0\}$ there exists a positive integer j such that

$$(\varphi^{-1}(a \cdot a^{-1}) \cup \psi^{-1}(a \cdot a^{-1}) \cup \varphi^{-1}(a^{-1} \cdot a) \cup \psi^{-1}(a^{-1} \cdot a)) \cap A_j = \emptyset.$$

Then we have

$$\{a\} \cdot U_j(x) = U_j(x) \cdot \{x\} = \{0\} \subseteq V(0),$$

for any open neighbourhood $V(0)$ of zero in S such that $V(0) \subseteq U(0)$.

Obviously that (B_λ, τ) is a dense subsemigroup of the topological semigroup (S, τ_0) . \square

Theorem 5. *An infinite semigroup of matrix units does not embed into a countably compact topological semigroup.*

Proof. Suppose to the contrary that there exists a countably compact topological semigroup S which contains an infinite semigroup of matrix units B_λ for some infinite cardinal λ . Since a closed subset of a countably compact space is countably compact (see [12, Theorem 3.10.4]), without loss of generality we can assume that B_λ is a dense subsemigroup of S . Then by Lemma 2 (iii), we have that $E(S) = E(B_\lambda)$. Theorem 1.5 [9, Vol. 1] and Theorem 3.10.4 of [12] implies that $E(B_\lambda)$ is a countably compact band of S . By Lemma 1, $E(B_\lambda)$ is compact and hence by Theorem 1, B_λ is a closed subgroup of S . Therefore by Theorem 3.10.4 [12], B_λ is a countably compact topological semigroup. A contradiction to the fact that on an infinite semigroup of matrix units there does not exist a countably compact semigroup topology (see [21, Theorem 6]). The obtained contradiction implies the assertion of the theorem. \square

Since the semigroup of matrix units is congruence-free, Theorem 5 implies the following:

Theorem 6. *Any continuous homomorphism of an infinite semigroup of matrix units into a countably compact topological semigroup is annihilating.*

Since the semigroup of matrix units B_λ is isomorphic to the semigroup \mathcal{S}_λ^1 and \mathcal{S}_λ^1 is a subsemigroup of \mathcal{S}_λ^n for all cardinals $\lambda \geq 1$, Theorem 6 implies the following:

Corollary 3. *Let λ be an infinite cardinal and n be a positive integer. Then there exists no countably compact topological semigroup S which contains \mathcal{S}_λ^n .*

Theorem 7. *Let λ be an infinite cardinal and n be a positive integer. Then every continuous homomorphism of the topological semigroup \mathcal{S}_λ^n into a countably compact topological semigroup is annihilating.*

Proof. We shall prove the assertion of the theorem by induction. By Theorem 6 every continuous homomorphism of the topological semigroup \mathcal{S}_λ^1 into a countably compact topological semigroup S is annihilating. We suppose that the assertion of the theorem holds for $n = 1, 2, \dots, k-1$ and we shall prove that it is true for $n = k$.

Obviously it is sufficient to show that the statement of the theorem holds for the discrete semigroup \mathcal{S}_λ^k . Let $h: \mathcal{S}_\lambda^k \rightarrow S$ be arbitrary homomorphism from \mathcal{S}_λ^k with the discrete topology into a countably compact topological semigroup S . Then by Theorem 6 the restriction $h|_{\mathcal{S}_\lambda^1}: \mathcal{S}_\lambda^1 \rightarrow S$ of homomorphism h onto the subsemigroup \mathcal{S}_λ^1 of \mathcal{S}_λ^k is an annihilating homomorphism. Let $(\mathcal{S}_\lambda^1)h|_{\mathcal{S}_\lambda^1} = (\mathcal{S}_\lambda^1)h = e$, where $e \in E(S)$. We fix any $\alpha \in \mathcal{S}_\lambda^k$ with $\text{ran}(\alpha) = i \geq 2$. Let $\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_i \\ y_1 & y_2 & \cdots & y_i \end{pmatrix}$ (where $x_1, x_2, \dots, x_i, y_1, y_2, \dots, y_i \in X$ for some set X of cardinality λ). We fix $y_1 \in X$ and define subsemigroup T_{y_1} of \mathcal{S}_λ^k as follows:

$$T_{y_1} = \left\{ \beta \in \mathcal{S}_\lambda^k : \begin{pmatrix} y_1 \\ y_1 \end{pmatrix} \cdot \beta = \beta \cdot \begin{pmatrix} y_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1 \end{pmatrix} \right\}.$$

Then the semigroup T_{y_1} is isomorphic to the semigroup $\mathcal{S}_\lambda^{k-1}$, the element $\begin{pmatrix} y_1 \\ y_1 \end{pmatrix}$ is zero of T_{y_1} and hence by induction assumption we have $\left(\begin{pmatrix} y_1 \\ y_1 \end{pmatrix} \right)h = (\beta)h$ for all $\beta \in T_{y_1}$.

Since $\begin{pmatrix} y_1 \\ y_1 \end{pmatrix} \in \mathcal{S}_\lambda^1$, we have that $(\beta)h = (0)h$ for all $\beta \in T_{y_1}$. But $\alpha = \alpha\gamma$, where $\gamma = \begin{pmatrix} y_1 & y_2 & \cdots & y_i \\ y_1 & y_2 & \cdots & y_i \end{pmatrix} \in T_{y_1}$, and hence we have

$$(\alpha)h = (\alpha\gamma)h = (\alpha)h \cdot (\gamma)h = (\alpha)h \cdot (0)h = (\alpha \cdot 0)h = (0)h = e.$$

This completes the proof of the theorem. □

Theorem 8. *An infinite semigroup of matrix units does not embed into a Tychonoff topological semigroup with the pseudo-compact square.*

Proof. Suppose to the contrary: there exists a Tychonoff topological semigroup S with the pseudo-compact square $S \times S$ which contains the infinite semigroup of matrix units B_λ for some $\lambda \geq \omega$. By Theorem 1.3 [6] for any topological semigroup S with the pseudocompact square $S \times S$ the semigroup operation $\mu: S \times S \rightarrow S$ extends to a continuous semigroup operation $\beta\mu: \beta S \times \beta S \rightarrow \beta S$ and the map $\beta: S \rightarrow \beta S$ is a homeomorphism “into”. Therefore the restriction $\beta|_{B_\lambda}: B_\lambda \rightarrow \beta S$ is an embedding of a semigroup B_λ into a compact topological semigroup βS . This contradicts Theorem 10 [21]. The obtained contradiction implies the statement of the theorem. □

Since the semigroup of matrix units is congruence-free, Theorem 8 implies the following:

Theorem 9. *Any continuous homomorphism of an infinite semigroup of matrix units into a Tychonoff topological semigroup with the pseudo-compact square is annihilating.*

Since the semigroup of matrix units B_λ is isomorphic to the semigroup \mathcal{S}_λ^1 and \mathcal{S}_λ^1 is a subsemigroup of \mathcal{S}_λ^n for all cardinals $\lambda \geq 1$, Theorem 9 implies the following:

Corollary 4. *Let λ be an infinite cardinal and n be a positive integer. Then there exists no a Tychonoff topological semigroup S with the pseudo-compact square which contains \mathcal{S}_λ^n .*

The proof of the following theorem is similar to Theorem 7.

Theorem 10. *Let λ be an infinite cardinal and n be a positive integer. Then every continuous homomorphism of the topological semigroup \mathcal{S}_λ^n into a Tychonoff topological semigroup with the pseudo-compact square is annihilating.*

3. H -closed semigroup topologies on \mathcal{S}_λ^n . Let X be a set of cardinality λ . Fix any positive integer n and put

$$\exp_n(\lambda) = \{A \subseteq X : |A| \leq n\}.$$

Then $\exp_n(\lambda)$ with the operation “ \cap ” is a semilattice. Further by $\exp_n(\lambda)$ we denote the semilattice $(\exp_n(\lambda), \cap)$.

Proposition 2. *For any cardinal $\lambda \geq 1$ and any positive integer n the band $E(\mathcal{S}_\lambda^n)$ of the semigroup \mathcal{S}_λ^n is isomorphic to the semilattice $\exp_n(\lambda)$.*

Proof. An isomorphism $h: E(\mathcal{S}_\lambda^n) \rightarrow \exp_n(\lambda)$ we define by the formula: $h(\alpha) = \text{dom } \alpha$. \square

An element e of a topological semilattice E is called a *local minimum* if there exists an open neighbourhood $U(e)$ of e such that $U(e) \cap \downarrow e \subseteq \uparrow e$ [25].

Lemma 3. *Let n be a positive integer and $\lambda \geq 1$. If τ is a semigroup topology on $\exp_n(\lambda)$, then any idempotent of $\exp_n(\lambda)$ is a local minimum and hence $\uparrow e$ is an open subset of $(\exp_n(\lambda), \tau)$.*

Proof. We observe that the statement of the lemma is trivial in case when e is the zero of the semigroup $\exp_n(\lambda)$. We suppose that $|e| = k$ for some $k = 1, 2, \dots, n$. Since the set $\exp_{k-1}(\lambda)$ is a subsemilattice of $\exp_n(\lambda)$, Theorem 9 of [37] implies that the set $U(e) = \exp_n(\lambda) \setminus \exp_{k-1}(\lambda)$ is an open neighbourhood of e in $\exp_n(\lambda)$. This implies that e is a local minimum in $\exp_n(\lambda)$. The continuity of semilattice operation in $\exp_n(\lambda)$ implies that there exists an open neighbourhood $V(e)$ of e such that $V(e) \cdot e \subseteq U(e)$. This implies that $V(e) \subseteq \uparrow e$. Then Theorem VI-1.13(iii) of [14] implies that $\uparrow e$ is an open subset of $(\exp_n(\lambda), \tau)$. \square

We define the family \mathcal{B} of non-empty subsets in $\exp_n(\lambda)$ as follows:

$$\mathcal{B} = \{U(e; e_1, \dots, e_i) = \uparrow e \setminus (\uparrow e_1 \cup \dots \cup \uparrow e_i) : e, e_1, \dots, e_i \in \exp_n(\lambda) \text{ such that } e < e_1, \dots, e < e_i, i \in \mathbb{N}\}. \quad (1)$$

Proposition 3. *The topology τ_c generated by the base \mathcal{B} is the unique compact Hausdorff topology on $\exp_n(\lambda)$ such that $(\exp_n(\lambda), \tau_c)$ is a topological semilattice.*

Proof. Let $e, f \in \exp_n(\lambda)$. If $e \leq f$ then $V(e) \cap V(f) = \emptyset$ for open neighbourhoods $V(e) = U(e; f)$ and $V(f) = \uparrow f$ of e and f , respectively. If the idempotents e and f are incomparable, then we put $g = e \cup f$ and hence we have that $U(e; g) \cap U(f; g) = \emptyset$. Therefore τ_c is a Hausdorff topology on $\exp_n(\lambda)$. We observe that since any chain in $\exp_n(\lambda)$ has $\leq n$ elements, τ_c is a compact topology on $\exp_n(\lambda)$.

Next we show that the semilattice operation \cap on $(\exp_n(\lambda), \tau_c)$ is continuous. Let e and f be arbitrary elements from $\exp_n(\lambda)$. If $e = f$ then

$$U(e; e_1, \dots, e_k) \cdot U(e; e_1, \dots, e_k) \subseteq U(e; e_1, \dots, e_k)$$

for all $U(e; e_1, \dots, e_k) \in \mathcal{B}$. If $e < f$ and $U(e; e_1, \dots, e_k) \in \mathcal{B}$ is an open neighbourhood of e , then

$$U(e; e_1, \dots, e_k, f) \cdot U(f; f_1, \dots, f_m) \subseteq U(e; e_1, \dots, e_k)$$

for all $f_1, \dots, f_m \in \uparrow f \setminus \{f\}$. If the idempotents e and f are incomparable, then we have $g = e \cap f < e, f$. Put $h = e \cup f$. Then for any open neighbourhood $U(g; g_1, \dots, g_k) \in \mathcal{B}$ of g we have

$$U(e; h) \cdot U(f; h) = \{g\} \subseteq U(g; g_1, \dots, g_k).$$

Hence $(\exp_n(\lambda), \tau_c)$ is a topological semilattice.

The uniqueness of the topology τ_c follows from Lemma 3. □

Proposition 4. *For a topological semilattice $\exp_n(\lambda)$ the following conditions are equivalent:*

- (i) $\exp_n(\lambda)$ is compact;
- (ii) $\exp_n(\lambda)$ is countably compact.

Proof. Since the statement of the proposition is trivial when the semilattice $\exp_n(\lambda)$ is finite, we suppose that the cardinal λ is infinite.

We observe that the implication (i) \Rightarrow (ii) is trivial.

We shall prove the implication (ii) \Rightarrow (i) by induction. By Lemma 1 every countably compact semigroup topology on $\exp_1(\lambda)$ is compact. We suppose that the assertion of the proposition holds for $k = 1, 2, \dots, n - 1$ and we shall prove that it is true for $k = n$. Suppose the contrary: there exists a semigroup topology τ on $\exp_n(\lambda)$ such that $(\exp_n(\lambda), \tau)$ is a countably compact non-compact topological semilattice. Let $\mathcal{C} = \{U_\alpha\}_{\alpha \in \mathcal{D}}$ be an open cover of the topological semilattice $(\exp_n(\lambda), \tau)$ which does not contain a finite subcover. Since by assumption of induction the subsemilattice $\exp_{n-1}(\lambda)$ is compact, there exists a finite subfamily $\mathcal{C}_0 = \{U_{\alpha_1}, \dots, U_{\alpha_n}\}_{\alpha_1, \dots, \alpha_n \in \mathcal{D}}$ of \mathcal{C} such that \mathcal{C}_0 is an open cover of $\exp_{n-1}(\lambda)$. Since the topological space $(\exp_n(\lambda), \tau)$ is non-compact and by Lemma 3 any idempotent $\varepsilon \in \exp_n(\lambda) \setminus \exp_{n-1}(\lambda)$ is an isolated point in $(\exp_n(\lambda), \tau)$, we have that $A = \exp_n(\lambda) \setminus (U_{\alpha_1} \cup \dots \cup U_{\alpha_n})$ is a closed discrete infinite subspace of $(\exp_n(\lambda), \tau)$. But Theorem 3.10.4 of [12] implies that A is a countably compact space, a contradiction. The obtained contradiction implies the statement of the proposition. □

The following example shows that there exists a semigroup topology τ^c on the semigroup \mathcal{I}_λ^n such that $E(\mathcal{I}_\lambda^n)$ is a compact semilattice.

Example 1. We identify the semilattice $E(\mathcal{I}_\lambda^n)$ with the semilattice $\exp_n(\lambda)$. Let τ_c be the topology on $E(\mathcal{I}_\lambda^n)$ determined by the base \mathcal{B} (see: (1)). We define the topology τ^c on \mathcal{I}_λ^n as follows:

- (i) at any idempotent $e \in \mathcal{I}_\lambda^n$ the base of the topology τ^c coincides with the base of τ_c ;
- (ii) all non-idempotent elements of the semigroup \mathcal{I}_λ^n are isolated points.

Proposition 5. $(\mathcal{I}_\lambda^n, \tau^c)$ is a topological inverse semigroup.

Proof. Since all non-idempotent elements of the semigroup \mathcal{I}_λ^n are isolated points in $(\mathcal{I}_\lambda^n, \tau^c)$, Proposition 3 implies that it is completely to show that the semigroup operation in $(\mathcal{I}_\lambda^n, \tau^c)$ is continuous in the following two cases:

$$\text{a) } \alpha \cdot \varepsilon; \quad \text{and} \quad \text{b) } \varepsilon \cdot \alpha, \quad \text{for } \alpha \in \mathcal{I}_\lambda^n \setminus E(\mathcal{I}_\lambda^n) \quad \text{and} \quad \varepsilon \in E(\mathcal{I}_\lambda^n).$$

By Corollary 8 [18] an idempotent ε of the semigroup \mathcal{I}_λ^n is an isolated point in \mathcal{I}_λ^n in the case $\text{rank } \alpha = n$. Therefore we can assume that $\text{rank } \varepsilon < n$.

In case a) let $\{x_1, \dots, x_s\} = \{x \in \text{ran } \alpha : x \notin \text{dom } \varepsilon\}$. Put $\varepsilon_1 = \varepsilon \cup \{x_1\}, \dots, \varepsilon_s = \varepsilon \cup \{x_s\}$. Then

$$\{\alpha\} \cdot U(\varepsilon; \varepsilon_1, \dots, \varepsilon_s) = \{\alpha \cdot \varepsilon\}.$$

In case b) let $\{y_1, \dots, y_p\} = \{x \in \text{dom } \alpha : x \notin \text{ran } \varepsilon\}$. Put $\varepsilon_1 = \varepsilon \cup \{y_1\}, \dots, \varepsilon_p = \varepsilon \cup \{y_p\}$. Then

$$U(\varepsilon; \varepsilon_1, \dots, \varepsilon_p) \cdot \{\alpha\} = \{\varepsilon \cdot \alpha\}.$$

Therefore the semigroup operation is continuous on $(\mathcal{S}_\lambda^n, \tau^c)$.

The continuity of the inversion in $(\mathcal{S}_\lambda^n, \tau^c)$ follows from the facts that the band $E(\mathcal{S}_\lambda^n)$ is a compact open subsemigroup of $(\mathcal{S}_\lambda^n, \tau^c)$ and all non-idempotent elements of the semigroup \mathcal{S}_λ^n are isolated points in $(\mathcal{S}_\lambda^n, \tau^c)$. \square

Theorem 11. *Let $(\mathcal{S}_\lambda^n, \tau)$ be a topological semigroup. If the band $E(\mathcal{S}_\lambda^n)$ is a compact subset of $(\mathcal{S}_\lambda^n, \tau)$, then $(\mathcal{S}_\lambda^n, \tau)$ is an absolutely H -closed topological semigroup.*

Proof. We shall prove the assertion of the theorem by induction. By Theorem 2 the topological semigroup \mathcal{S}_λ^1 with compact band is absolutely H -closed. We suppose that the assertion of the theorem holds for $n = 1, 2, \dots, k-1$ and we shall prove that it is true for $n = k$.

Suppose the contrary: the topological semigroup $(\mathcal{S}_\lambda^n, \tau)$ is not absolutely H -closed. Then there exist a Hausdorff topological semigroup T and continuous homomorphism $h: \mathcal{S}_\lambda^n \rightarrow T$ such that $(\mathcal{S}_\lambda^n)h$ is not a closed subsemigroup of T . Since the closure $\text{cl}_G(L)$ of a subsemigroup L of topological semigroup G is a subsemigroup in G (see [9, Vol. 1, p. 9]), without loss of generality we can assume that $(\mathcal{S}_\lambda^n)h$ is a dense subsemigroup of T and $T \setminus (\mathcal{S}_\lambda^n)h \neq \emptyset$. Then by Lemma 8 [21], zero of the semigroup $(\mathcal{S}_\lambda^n)h$ is zero in T and we denote it by 0_T . Let $x \in T \setminus (\mathcal{S}_\lambda^n)h$. Then $0_T \cdot x = 0_T$. The continuity of the semigroup operation in T implies that for any open neighbourhood $W(0_T)$ of zero 0_T in T there exist open neighbourhoods $U(0_T)$ and $V(0_T)$ of 0_T in T and an open neighbourhood $V(x)$ of x in T such that

$$V(0_T) \cdot V(x) \subseteq U(0_T) \subseteq W(0_T), \quad V(0_T) \subseteq U(0_T) \quad \text{and} \quad U(0_T) \cap V(x) = \emptyset.$$

By the assumption of the induction and Theorem 9 of [37] we have that $V(x) \cap ((\mathcal{S}_\lambda^{n-1})h \cup (E(\mathcal{S}_\lambda^n))h) = \emptyset$. We remark that for any idempotent ε_0 of the semigroup \mathcal{S}_λ^n with $\text{ran } \varepsilon_0 = 1$ the set $\mathcal{S}_\lambda^n(\varepsilon_0) = \{\chi \in \mathcal{S}_\lambda^n : \chi\varepsilon_0 = \varepsilon_0\chi = \varepsilon_0\}$ is a subsemigroup of \mathcal{S}_λ^n and simple observations show that $\mathcal{S}_\lambda^n(\varepsilon_0)$ is algebraically isomorphic to the semigroup $\mathcal{S}_\lambda^{n-1}$. Then the compactness of the band $E(\mathcal{S}_\lambda^n)$ implies that there exist a finite subset of idempotents $\{\varepsilon_1, \dots, \varepsilon_k\}$ in \mathcal{S}_λ^n with $\text{ran } \varepsilon_1 = \dots = \text{ran } \varepsilon_k = 1$, an open neighbourhood $\tilde{V}(0)$ of the zero 0 of the semigroup \mathcal{S}_λ^n and an open neighbourhood $O(x)$ of x in T such that

$$\begin{aligned} (\tilde{V}(0))h \subseteq V(0_T), \quad \tilde{V}(0) \cap (\mathcal{S}_\lambda^n(\varepsilon_1) \cup \dots \cup \mathcal{S}_\lambda^n(\varepsilon_k)) &= \emptyset, \quad O(x) \subseteq V(x) \quad \text{and} \\ (O(x) \cap (\mathcal{S}_\lambda^n)h)h^{-1} \cap (\mathcal{S}_\lambda^n(\varepsilon_1) \cup \dots \cup \mathcal{S}_\lambda^n(\varepsilon_k)) &= \emptyset. \end{aligned}$$

The compactness of the band $E(\mathcal{S}_\lambda^n)$ and the infiniteness of the set $(O(x) \cap (\mathcal{S}_\lambda^n)h)h^{-1}$ imply that there exists $\alpha \in (O(x) \cap (\mathcal{S}_\lambda^n)h)h^{-1}$ such that $\alpha \in \tilde{V}(0) \cdot \alpha$. Then

$$(\tilde{V}(0))h \cdot (\alpha)h \subseteq V(0_T) \cdot O(x) \subseteq U(0_T) \quad \text{and} \quad ((\tilde{V}(0))h \cdot (\alpha)h) \cap O(x) \neq \emptyset.$$

This contradicts to the assumption $U(0_T) \cap V(x) = \emptyset$. The obtained contradiction implies that $T \setminus (\mathcal{S}_\lambda^n)h = \emptyset$, and hence \mathcal{S}_λ^n is an absolutely H -closed topological semigroup. \square

Proposition 4 and Theorem 11 imply the following:

Corollary 5. *Let $(\mathcal{S}_\lambda^n, \tau)$ be a topological semigroup. If the band $E(\mathcal{S}_\lambda^n)$ is a countably compact subset of $(\mathcal{S}_\lambda^n, \tau)$, then $(\mathcal{S}_\lambda^n, \tau)$ is an absolutely H -closed topological semigroup.*

4. Countably compact topological Brandt λ^0 -extensions.

Definition 3 ([22]). Let \mathcal{S} be some class of topological monoids with zero. Let λ be any cardinal ≥ 1 , and $(S, \tau) \in \mathcal{S}$. Let τ_B be a topology on $B_\lambda^0(S)$ such that

- a) $(B_\lambda^0(S), \tau_B) \in \mathcal{S}$;
- b) $\tau_B|_{S_{\alpha,\alpha}} = \tau$ for some $\alpha \in \lambda$.

Then $(B_\lambda^0(S), \tau_B)$ is called a *topological Brandt λ^0 -extension of (S, τ) in \mathcal{S}* . If \mathcal{S} coincides with the class of all topological semigroups, then $(B_\lambda^0(S), \tau_B)$ is called a *topological Brandt λ^0 -extension of (S, τ)* .

A topological Brandt λ^0 -extension $(B_\lambda^0(S), \tau_B)$ is called *compact* (resp., *countably compact*) if the topological space $(B_\lambda^0(S), \tau_B)$ is compact (resp., countably compact) [24]. Gutik and Repovš in [24] described the structures of compact topological Brandt λ^0 -extensions.

We need the following lemma from [24]:

Lemma 4 ([24]). *For any topological monoid (S, τ) with zero and for any finite cardinal $\lambda \geq 1$ there exists a unique topological Brandt λ^0 -extension $(B_\lambda^0(S), \tau_B)$ and the topology τ_B generated by the base $\mathcal{B}_B = \bigcup\{\mathcal{B}_B(t) : t \in B_\lambda^0(S)\}$, where:*

- (i) $\mathcal{B}_B(t) = \{(U(s))_{\alpha,\beta} \setminus \{0_S\} : U(s) \in \mathcal{B}_S(s)\}$, when $t = (\alpha, s, \beta)$ is a non-zero element of $B_\lambda^0(S)$, $\alpha, \beta \in \lambda$;
- (ii) $\mathcal{B}_B(0) = \{\bigcup_{\alpha,\beta \in \lambda} (U(0_S))_{\alpha,\beta} : U(0_S) \in \mathcal{B}_S(0_S)\}$, when 0 is the zero of $B_\lambda^0(S)$,

and $\mathcal{B}_S(s)$ is a base of the topology τ at the point $s \in S$.

Proposition 6 describes the structures of countably compact Brandt λ^0 -extensions of topological monoids.

Proposition 6. *A topological Brandt λ^0 -extension $B_\lambda^0(S)$ of a topological monoid (S, τ) with zero is countably compact if and only if the cardinal $\lambda \geq 1$ is finite and (S, τ) is a countably compact topological semigroup. Moreover, for any countably compact topological monoid (S, τ) with zero and for any finite cardinal $\lambda \geq 1$ there exists a unique countably compact topological Brandt λ^0 -extension $(B_\lambda^0(S), \tau_B)$ and the topology τ_B generated by the base $\mathcal{B}_B = \bigcup\{\mathcal{B}_B(t) : t \in B_\lambda^0(S)\}$, where:*

- (i) $\mathcal{B}_B(t) = \{(U(s))_{\alpha,\beta} \setminus \{0_S\} : U(s) \in \mathcal{B}_S(s)\}$, when $t = (\alpha, s, \beta)$ is a non-zero element of $B_\lambda^0(S)$, $\alpha, \beta \in \lambda$;
- (ii) $\mathcal{B}_B(0) = \{\bigcup_{\alpha,\beta \in \lambda} (U(0_S))_{\alpha,\beta} : U(0_S) \in \mathcal{B}_S(0_S)\}$, when 0 is the zero of $B_\lambda^0(S)$,

and $\mathcal{B}_S(s)$ is a base of the topology τ at the point $s \in S$.

Proof. Since by Theorem 6 the infinite semigroup of matrix units does not embed into a countably compact topological semigroup, the countable compactness of the topological Brandt λ^0 -extension $(B_\lambda^0(S), \tau_B)$ of a topological semigroup (S, τ) implies that the cardinal λ is finite. Then by Theorem 1.7(e) of [9, Vol. 1], $(\alpha, 1_S, \alpha)B_\lambda^0(S)(\alpha, 1_S, \alpha) = S_{\alpha,\alpha}$ is a countably

compact semigroup for any $\alpha \in \lambda$, and hence (S, τ) is a countably compact topological semigroup. The converse follows from Lemma 1 [22] and the assertion that the finite union of countably compact spaces is a countably compact space.

Lemma 4 implies the last assertion of the proposition. \square

Lemma 4 and Proposition 6 imply

Theorem 12. *Every countably compact (and hence compact) topological Brandt λ^0 -extension $(B_\lambda^0(S), \tau_B)$ of a topological inverse semigroup (S, τ) is a topological inverse semigroup.*

Definition 4 ([24]). Let λ be any cardinal ≥ 2 . We shall say that a semigroup S has the \mathcal{B}_λ^* -property if S satisfies the following conditions:

- 1) T does not contain the semigroup of $\lambda \times \lambda$ -matrix units;
- 2) T does not contain the semigroup of 2×2 -matrix units B_2 such that the zero of B_2 is the zero of T .

Gutik and Repovš in [24] proved the following:

Theorem 13 ([24]). *Let λ_1 and λ_2 be any finite cardinals such that $\lambda_2 \geq \lambda_1 \geq 1$. Let $B_{\lambda_1}^0(S)$ and $B_{\lambda_2}^0(T)$ be topological Brandt λ_1^0 - and λ_2^0 -extensions of topological monoids S and T with zero, respectively. Let $h: S \rightarrow T$ be a continuous homomorphism such that $(0_S)h = 0_T$ and $\varphi: \lambda_1 \rightarrow \lambda_2$ an one-to-one map. Let e be a non-zero idempotent of T , H_e a maximal subgroup of T with unity e and $u: \lambda_1 \rightarrow H_e$ a map. Then $I_h = \{s \in S: (s)h = 0_T\}$ is a closed ideal of S and the map $\sigma: B_{\lambda_1}^0(S) \rightarrow B_{\lambda_2}^0(T)$ defined by the formulae*

$$((\alpha, s, \beta))\sigma = \begin{cases} ((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi), & \text{if } s \notin S \setminus I_h; \\ 0_2, & \text{if } s \in I_h, \end{cases}$$

and $(0_1)\sigma = 0_2$, is a non-trivial continuous homomorphism from $B_{\lambda_1}^0(S)$ into $B_{\lambda_2}^0(T)$. Moreover if for the semigroup T the following conditions hold:

- (i) every idempotent of T lies in the center of T ;
- (ii) T has $\mathcal{B}_{\lambda_1}^*$ -property,

then every non-trivial continuous homomorphism from $B_{\lambda_1}^0(S)$ into $B_{\lambda_2}^0(T)$ can be so constructed.

Next we define a category of countably compact topological monoids and pairs of finite sets and a category of countably compact topological semigroups.

Let S and T be countably compact topological monoids with zeros. Let $\mathbf{CHom}_0(S, T)$ be a set of all continuous homomorphisms $\sigma: S \rightarrow T$ such that $(0_S)\sigma = 0_T$. We put

$$\mathbf{E}_1^{\text{top}}(S, T) = \{e \in E(T): \text{there exists } \sigma \in \mathbf{CHom}_0(S, T) \text{ such that } (1_S)\sigma = e\}$$

and define the family

$$\mathcal{H}_1^{\text{top}}(S, T) = \{H(e): e \in \mathbf{E}_1^{\text{top}}(S, T)\},$$

where by $H(e)$ we denote the maximal subgroup with the unity e in the semigroup T . Also by \mathbf{CCTB} we denote the class of all countably compact topological monoids S with zero such that S has \mathcal{B}^* -property and every idempotent of S lies in the center of S .

We define a category $\mathcal{ITCB}_{\text{fin}}$ as follows:

- (i) $\mathbf{Ob}(\mathcal{TCCB}_{\text{fin}}) = \{(S, I) : S \in \mathcal{CTB} \text{ and } I \text{ is a finite set}\}$, and if S is a trivial semigroup then we identify (S, I) and (S, J) for all finite sets I and J ;
- (ii) $\mathbf{Mor}(\mathcal{TCCB}_{\text{fin}})$ consists of triples $(h, u, \varphi) : (S, I) \rightarrow (S', I')$, where

$$\begin{aligned} h : S &\rightarrow S' \text{ is a continuous homomorphism such that } h \in \mathbf{CHom}_0(S, S'), \\ u : I &\rightarrow H(e) \text{ is a map, for } H(e) \in \mathcal{H}_1^{\text{top}}(S, S'), \\ \varphi : I &\rightarrow I' \text{ is an one-to-one map,} \end{aligned}$$

with the composition

$$(h, u, \varphi)(h', u', \varphi') = (hh', [u, \varphi, h', u'], \varphi\varphi'),$$

where the map $[u, \varphi, h', u'] : I \rightarrow H(e)$ is defined by the formula

$$(\alpha)[u, \varphi, h', u'] = ((\alpha)\varphi)u' \cdot ((\alpha)u)h' \quad \text{for } \alpha \in I.$$

Straightforward verification shows that $\mathcal{TCCB}_{\text{fin}}$ is the category with the identity morphism $\varepsilon_{(S, I)} = (\text{Id}_S, u_0, \text{Id}_I)$ for any $(S, I) \in \mathbf{Ob}(\mathcal{TCCB}_{\text{fin}})$, where $\text{Id}_S : S \rightarrow S$ and $\text{Id}_I : I \rightarrow I$ are identity maps and $(\alpha)u_0 = 1_S$ for all $\alpha \in I$.

We define a category $\mathcal{B}_{\text{fin}}^*(\mathcal{TCCS})$ as follows:

- (i) let $\mathbf{Ob}(\mathcal{B}_{\text{fin}}^*(\mathcal{TCCS}))$ be all finite topological Brandt λ^0 -extensions of countably topological monoids S with zeros such that S has \mathcal{B}^* -property and every idempotent of S lies in the center of S ;
- (ii) let $\mathbf{Mor}(\mathcal{B}_{\text{fin}}^*(\mathcal{TCCS}))$ be homomorphisms of finite topological Brandt λ^0 -extensions of countably compact topological monoids S with zeros such that S has \mathcal{B}^* -property and every idempotent of S lies in the center of S .

For each $(S, I_{\lambda_1}) \in \mathbf{Ob}(\mathcal{TCCB}_{\text{fin}})$ with non-trivial S , let $\mathbf{B}(S, I_{\lambda_1}) = B_{\lambda_1}^0(S)$ be the countably compact topological Brandt λ^0 -extension of the countably compact topological monoid S . For each $(h, u, \varphi) \in \mathbf{Mor}(\mathcal{TCCB}_{\text{fin}})$ with a non-trivial continuous homomorphism h , where $(h, u, \varphi) : (S, I_{\lambda_1}) \rightarrow (T, I_{\lambda_2})$ and $(T, I_{\lambda_2}) \in \mathbf{Ob}(\mathcal{TCCB}_{\text{fin}})$, we define a map $\mathbf{B}(h, u, \varphi) : \mathbf{B}(S, I_{\lambda_1}) = B_{\lambda_1}^0(S) \rightarrow \mathbf{B}(T, I_{\lambda_2}) = B_{\lambda_2}^0(T)$ as follows:

$$((\alpha, s, \beta))[\mathbf{B}(h, u, \varphi)] = \begin{cases} ((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi), & \text{if } s \notin S \setminus I_h; \\ 0_2, & \text{if } s \in I_h^*, \end{cases}$$

and $(0_1)[\mathbf{B}(h, u, \varphi)] = 0_2$, where $I_h = \{s \in S : (s)h = 0_T\}$ is a closed ideal of S and 0_1 and 0_2 are the zeros of the semigroups $B_{\lambda_1}^0(S)$ and $B_{\lambda_2}^0(T)$, respectively. For each $(h, u, \varphi) \in \mathbf{Mor}(\mathcal{TCCB}_{\text{fin}})$ with a trivial homomorphism h we define a map $\mathbf{B}(h, u, \varphi) : \mathbf{B}(S, I_{\lambda_1}) = B_{\lambda_1}^0(S) \rightarrow \mathbf{B}(T, I_{\lambda_2}) = B_{\lambda_2}^0(T)$ as follows: $(a)[\mathbf{B}(h, u, \varphi)] = 0_2$ for all $a \in \mathbf{B}(S, I_{\lambda_1}) = B_{\lambda_1}^0(S)$. If S is a trivial semigroup then we define $\mathbf{B}(S, I_{\lambda_1})$ to be a trivial semigroup.

A functor \mathbf{F} from a category \mathcal{C} into a category \mathcal{K} is called *full* if for any $a, b \in \mathbf{Ob}(\mathcal{C})$ and for any \mathcal{K} -morphism $\alpha : \mathbf{F}a \rightarrow \mathbf{F}b$ there exists a \mathcal{C} -morphism $\beta : a \rightarrow b$ such that $\mathbf{F}\beta = \alpha$, and \mathbf{F} called *representative* if for any $a \in \mathbf{Ob}(\mathcal{K})$ there exists $b \in \mathbf{Ob}(\mathcal{C})$ such that a and $\mathbf{F}b$ are isomorphic.

Theorem 4.1 [24] and Theorem 13 imply

Theorem 14. \mathbf{B} is a full representative functor from $\mathcal{TCCB}_{\text{fin}}$ into $\mathcal{B}_{\text{fin}}^*(\mathcal{TCCS})$.

Remark 2. We observe that the similar statements to Theorem 14 hold for the categories of countably compact topological inverse monoids, countably compact Clifford topological inverse monoids, countably compact Brandt topological semigroups, countably compact topological semilattices and finite sets and corresponding their countably compact topological Brandt λ^0 -extensions of countably compact topological monoids. Moreover in the case of countably compact topological semilattices the functor \mathbf{B} determines the equivalency of such categories. The last assertion follows from Proposition 4.3 [24].

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