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SOLVABILITY OF DISCRETE APPROXIMATIONS FOR LINEAR AND NONLINEAR DIFFERENTIAL-OPERATOR EQUATIONS IN BANACH SPACES WITHIN A PROJECTION-ALGEBRAIC APPROACH

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The projection-algebraic approach to discrete approximations, proposed before in work [3], for linear and nonlinear differential operator equations in Banach spaces is analyzed. The convergence analysis of the corresponding finite-dimensional expressions, based on the functional-analytic properties of discrete approximations and methods of operator theory in Banach spaces, is studied. Based on a generalized Leray-Schauder type fixed-point theorem the projection-algebraic scheme of discrete approximations is investigated, its solvability and convergence for a special class of nonlinear operator equations are analyzed. Application of the results to the Lagrangian functional interpolation scheme of the projection-algebraic method of discrete approximations in case of linear differential operator equations is presented.

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Развивается проекционно-алгебраический метод дискретных аппроксимаций для линейных дифференциальных уравнений в банаховых пространствах, дается анализ сходимости конечномерных аппроксимаций, основывающийся на функционально-алгебраическом подходе к дискретным аппроксимациям и методах теории операторов в банаховых пространствах. На основании обобщенного утверждения типа Лере-Шаудера о неподвижной точке, рассмотрена проекционно-алгебраическая схема дискретных аппроксимаций та дан анализ её разрешимости и сходимости для специального класса нелинейных операторных уравнений. Рассмотрены приложения полученных результатов к лагранжевой функционально-интерполяционной схеме проекционно-алгебраического метода дискретных аппроксимаций.

1. Introduction. Let X and Y be Banach functional spaces. There is considers a differential operator equation

$$Au = f(u), \tag{1}$$

where, in general, $f\colon X \to Y$ is some nonlinear continuous mapping and $A\colon X \to Y$ is a closed linear differential expression

$$A := \sum_{|\beta|=0}^{m} a_\beta(x) \frac{\partial^\beta}{\partial x^\beta} \tag{2}$$

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in an open region $\Omega \subset \mathbb{R}^q$ with smooth coefficients $a_\beta \in C^\infty(\Omega;\mathbb{R})$, $q, m \in \mathbb{Z}_+$, defined on a domain $D(A) \subset X$ and satisfying, in general, the condition $\overline{\mathrm{Range}(A)} = Y$ or the condition $\mathrm{Range}(A) = Y$. Aiming to construct discrete approximations of equation (1), suitable for effective computer calculations we develop further a projective-algebraic approach, based on functional-analytic Lie-algebraic scheme of works [1, 2, 3, 16, 17]. Making use of general properties [12, 13, 14, 15] of approximating finite-dimensional Banach subspaces, we present first necessary convergence conditions of the projection-algebraic discrete approximation method [3] for a constant mapping $f \colon X \to Y$ both in general case and in the case of a functional-interpolating scheme of discrete approximations, adapted for finite-dimensional quasi-representations of basic Heisenberg-Weil operator algebra. Next, employing results of works [16, 17] on the existence of solutions to nonlinear operator equations, based on a generalized Leray-Schauder type theorem, we described sufficiently effective solvability and convergence conditions of the projective-algebraic method of discrete approximations for the case of nonlinear continuous A-compact mapping $f \colon X \to Y$ and closed linear surjective operator $A \colon X \to Y$.

A projection-algebraic approach to discrete approximations of differential operator equations has been proposed by F. Calogero in 1983 for calculating the eigenvalues of linear differential operators [5] in Hilbert spaces. There were also some other works devoted to applications of this approach to analytic solving some class of differential equations [6, 11]. In 1988 an important paper introducing mathematical backgrounds of a new analytical-numerical method, based on the projection-algebraic approach to discrete approximations, for solving linear and nonlinear partial differential operator equations appeared [3]. At the same time comprehensive numerical calculations, fulfilled in [7, 8], have shown a very high velocity of convergence demonstrated by this projection-algebraic approach. Moreover, many numerical tests for a nonhomogeneous heat equation were performed [4], making use of the projection-algebraic approach suggested in [3]. Here we develop further and investigate this projection-algebraic approach to discrete approximations concerning the important problems of its realisability and convergence for some class of linear and nonlinear differential operator equations in Banach spaces.

2. Functional-operator aspects of the projection-algebraic method of discrete approximations for nonlinear operator equations in Banach spaces. Consider a nonlinear operator equation (1), where $A \colon X \to Y$ is a closed surjective linear differential operator with a domain $D(A) \subset X$ (not necessary dense), $f \colon X \to Y$ is an arbitrary nonlinear continuous mapping with domain $D(f) = S_r(0) \cap D(A) \subset X$ (here $S_r(0) \subset X$ is the sphere of radius $r > 0$ centered at zero), and which satisfy such three conditions:

i) the A-compactness, that is for any bounded sets $U \subset D(f)$ and $V \subset Y$ the closure $\overline{f(U \cap A^{-1}(V)}) \subset Y$ is compact;

ii) the dimension $\dim A \geq 1$;

iii) there exist positive numbers $k_f < k_A \in \mathbb{R}_+$, where by definition

$$k_A^{-1} := \sup_{\|v\|_Y=1} \inf_{u \in D(A)} \{\|u\|_X \colon Au = v\} < \infty, \quad k_f \colon = \sup_{u \in S_r(0)} \frac{1}{r}\|f(u)\| < \infty. \qquad (3)$$

If $\widetilde{X_N} \subset \widetilde{X}_{N+1} \subset X$ and $\widetilde{Y_N} \subset \widetilde{Y}_{N+1} \subset Y$, $N \in \mathbb{Z}_+$, are suitable finite-dimensional Banach subspaces, $P_N^{(x)} \colon X \to \widetilde{X}_N$, $N \in \mathbb{Z}_+$, and $P_N^{(y)} \colon Y \to \widetilde{Y}_N, N \in \mathbb{Z}_+$, are the corresponding projectors, one considers the following sequence of equations

$$P_N^{(y)} A\widetilde{u}_N = P_N^{(y)} f(\widetilde{u}_N) \tag{4}$$

on elements $\widetilde{u}_N \in \widetilde{X}_N$, $N \in \mathbb{Z}_+$, which are suitable approximations to a searched solution of equation (1), being in general non-unique, as $\dim\ker A \geq 1$. Note here that the projection method is often called “realizable”, if the set $\mathcal{M} \subset X$ of solutions of equation (1) is nonempty, and for enough large $N \in \mathbb{Z}_+$ there are nonempty sets $\mathcal{M}_N \subset \widetilde{X}_N$ of solutions to equations (4). The method is called “convergent” if it is realizable and there is fulfilled the condition

$$\lim_{N\to\infty} \sup_{\widetilde{u}_N \in \widetilde{\mathcal{M}}_N} \inf_{u\in\mathcal{M}} \|\widetilde{u}_N - u\|_X = 0. \tag{5}$$

It is obvious that for practical applications the realisability criteria of the projection method and its convergence are very important, thereby we will study them making use of results [16, 17], where these solvability questions were analyzed in detail for a certain class of nonlinear operator equations by means of a one generalized Leray-Schauder type fixed point theorem in Banach spaces. Namely, the following theorem, obtained in [16, 17] and giving a description of the solution set $\mathcal{M}$ of nonlinear operator equation (1), holds.

Theorem 1. *Let conditions i) and ii), formulated above, hold. Then at the necessary condition* $\dim\ker A \geq 1$ *equation (1) possesses the nonempty solution set* $\mathcal{M}$, *whose topological dimension* $\dim\mathcal{M} \geq \dim\ker A - 1$.

Below we will assume that all of the necessary conditions of Theorem 1 are fulfilled. Then the following result characterizes the realizability of the projection-algebraic method of discrete approximations (4).

Theorem 2. *Let conditions i), ii) be fulfilled and additionally*

$$\lim_{N\to\infty} \sup_{v\in \mathrm{Range}(A)\cap\mathrm{Range}(f)} \|P_N^{(y)} v - v\|_Y = 0. \tag{6}$$

Then for large enough integers $N \in \mathbb{Z}_+$ *the solution sets* $\widetilde{\mathcal{M}}_N \subset \widetilde{X}_N$ *are nonempty and the convergence condition (5) holds.*

Proof. Put, by definition, that

$$k_f^{(N)} := \sup_{\widetilde{u}_N \in S_r(0)} \frac{1}{r} \|P_N^{(y)} f(\widetilde{u}_N)\|_{\widetilde{Y}_N}, \tag{7}$$

$$k_A^{(N),-1} := \sup_{\widetilde{v}_N \in \widetilde{Y}_N} \frac{1}{\|\widetilde{v}_N\|_{\widetilde{Y}_N}} \inf_{\widetilde{u}_N \in P_N^{(x)} D(A)} \{\|\widetilde{u}_N\|_{\widetilde{X}_N} : P_N^{(y)} A\widetilde{u}_N = \widetilde{v}_N\}, \tag{8}$$

and choose such integer $N_0 \in \mathbb{Z}_+$, that $\dim\ker(P_{N_0}^{(y)} A) \geq 1$, and

$$k_f \leq k_f^{(N_0)} < k_A^{(N_0)} \leq k_A. \tag{9}$$

Then based on expressions (7) and (8) from condition (9) we obtain that for all $N \geq N_0$ the following inequalities $k_f \leq k_f^{(N)} < k_A^{(N)} \leq k_A$ hold. But this means that, owing to the generalized Leray-Schauder type fixed point theorem [16, 17], the sequence of equations (5) possesses solutions for all $N \geq N_0$, that is all solution sets $\widetilde{\mathcal{M}}_N \subset \widetilde{X}_N$, $N \geq N_0$, are nonempty, and the projection-algebraic method itself is realizable.

Take now some $\varepsilon > 0$ and consider the neighborhood

$$U_\varepsilon(\mathcal{M}) := \Big\{u \in D(f)\colon \inf_{\overline{u} \in \mathcal{M} \subset D(f)} \|\overline{u} - u\|_X < \varepsilon\Big\}. \tag{10}$$

It is evident that the closed set $D(f) \backslash U_\varepsilon(\mathcal{M})$ does not contain solutions to equation (1), and for some $\alpha_\varepsilon > 0$ the quantity

$$\inf_{\overline{u} \in D(f) \backslash U_\varepsilon(\mathcal{M})} \|Au - f(u)\|_Y = \alpha_\varepsilon > 0. \tag{11}$$

Choose now, based on (6), an integer $N_\varepsilon \in \mathbb{Z}_+$ in such a way that for all $N \geq N_\varepsilon$

$$\sup_{u \in D(f)} (\|Au - P_N^{(y)} Au\|_Y + \|f(u) - P_N^{(y)} f(u)\|_Y) < \alpha_\varepsilon. \tag{12}$$

Then for all $u \in D(f) \backslash U_\varepsilon(\mathcal{M})$ the following inequality

$$\|P_N^{(y)} Au - P_N^{(y)} f(u)\|_Y \geq \|Au - f(u)\|_Y - \\ -(\|Au - P_N^{(y)} Au\|_Y + \|f(u) - P_N^{(y)} f(u)\|_Y) > \alpha_\varepsilon - \alpha_\varepsilon = 0,$$

holds, that is for $N \geq N_\varepsilon$ there holds the imbedding $\widetilde{\mathcal{M}}_N \subset U_\varepsilon(\mathcal{M})$. Since $\varepsilon > 0$ is chosen enough small, the condition $\widetilde{\mathcal{M}}_N \subset U_\varepsilon(\mathcal{M})$ for all $N \geq N_\varepsilon$ is equivalent to that of convergence for (5), proving the theorem. □

In the case when the sequences of subspaces $\widetilde{X}_N \subset \widetilde{X}_{N+1} \subset X, N \in \mathbb{Z}_+$ and $\widetilde{Y}_N \subset \widetilde{Y}_{N+1} \subset Y, N \in \mathbb{Z}_+$, are chosen Hilbert and, moreover

$$\bigcup_{N \in \mathbb{Z}_+} \widetilde{X}_N = X, \qquad \bigcup_{N \in \mathbb{Z}_+} \widetilde{Y}_N = Y, \tag{13}$$

with projectors $P_N^{(x)}\colon X \to \widetilde{X}_N, P_N^{(y)}\colon Y \to \widetilde{Y}_N, N \in \mathbb{Z}_+$, being operators of orthogonal projection, the norms $\|P_N^{(x)}\| = 1$, $\|P_N^{(x)}\| = 1, N \in \mathbb{Z}_+$, and for all $u \in X, v \in Y$

$$\lim_{N \to \infty} \|u - P_N^{(x)} u\|_Y = 0, \qquad \lim_{N \to \infty} \|v - P_N^{(y)} v\|_Y = 0. \tag{14}$$

We assume further that conditions (5), (6) are fulfilled and $\dim \ker A \geq 1$. Then an analog of Theorem 2 on the realisability of the projection-algebraic scheme of discrete approximations for nonlinear operator equation (1) in Hilbert spaces holds.

Theorem 3. *For enough large $N \in \mathbb{Z}_+$ solution sets $\widetilde{\mathcal{M}}_N \subset \widetilde{X}_N$ are nonempty and the convergence condition (4) holds.*

Proof. It is clear that we need only to state the condition (6). Having assumed the contrary, one can find such a subsequence of indices $N_k \in \mathbb{Z}_+$ for $k \in \mathbb{Z}_+$, as well as elements $u_k \in D(f), k \in \mathbb{Z}_+$, for which there exists $\varepsilon > 0$, that

$$\|P_{N_k}^{(y)} f(u_k) - f(u_k)\|_Y > \varepsilon. \tag{15}$$

Since for all $k \in \mathbb{Z}_+$ elements $f(u_k) \in \mathrm{Range}(A)$, owing to the A-compactness of the mapping $f\colon D(f) \to Y$ there exists the limit $\lim_{k\to\infty} f(u_k) = \overline{v} \in Y$. Making now use of the existence of limits (14), we obtain:

$$\lim_{k\to\infty} \|P_{N_k}^{(y)} f(u_k) - f(u_k)\|_Y \leq \lim_{k\to\infty} \|P_{N_k}^{(y)} f(u_k) - P_{N_k}^{(y)}\overline{v}\|_Y + \\ + \lim_{k\to\infty} \|P_{N_k}^{(y)}\overline{v} - \overline{v}\|_Y + \lim_{k\to\infty} \|\overline{v} - f(u_k)\|_Y = 0,$$

contradicting the initial inequality (15), thereby proving the theorem. □

If the mapping $f\colon D(f) \subset X \to Y$ is constant, the operator $A\colon D(A) \subset X \to Y$ is densely defined and $\overline{\mathrm{Range}\, A} = Y$, one can state additional convergence conditions of the projection-algebraic method of discrete approximations for equation (1), to which we proceed below.

3. Functional-interpolating properties of the projection-algebraic method of discrete approximations for linear differential operator equations.

3.1 Lie-algebraic preliminaries. Here we present some preliminaries from the theory of Lie algebraic structures [18] and the Lagrangian interpolation [13] necessary for the further exposition. We assume here, for brevity, that the mapping $f\colon X \to Y$ is taken constant and the differential operator $A(x;\partial)\colon X \to Y$ belongs to a suitable operator closure of the universal enveloping algebra $U(\mathcal{G})$ of the Heisenberg-Weil algebra $\mathcal{G} = \bigoplus_{j=\overline{1,q}} \mathcal{G}_j$, $\mathcal{G}_j := \{x_j, \partial_{x_j}, 1\}$, $j = \overline{1,q}$, of differential operations [12, 18], where $x_j\colon X \to X$ is the operator of multiplication on the independent variable $x_j \in \mathbb{R}$ and ∂_{x_j} is the suitable operator of differentiation with respect to variable x_j, $j = \overline{1,q}$, and $1\colon X \to X$ is the identity operator. The Lie brackets of the Lie algebra $\mathcal{G}$ are defined as follows: $[a,b] := a \cdot b - b \cdot a$ for any elements $a, b \in U(\mathcal{G})$, where " $\cdot$ " means the usual superposition of operators.

As it was mentioned above, within the projection-algebraic method we try to find the corresponding representations of all elements involved in (2), both functions and operators [3]. We define a sequence of linear mappings $\Phi_N^{(x)} := \pi_N^{(x)} P_N^{(x)}\colon X \to X_N$, $\Phi_N^{(y)} := \pi_N^{(y)} P_N^{(y)}\colon \ \to Y_N$, $N \in \mathbb{Z}_+^q$, where $P_N^{(x)}\colon X \to \widetilde{X}_N \subset X$ and $P_N^{(y)}\colon Y \to \widetilde{Y}_N \subset Y$ are suitable projection operators upon finite dimensional functional subspaces $\widetilde{X}_N \subset X$, $N \in \mathbb{Z}_+^q$, and $\widetilde{Y}_N \subset Y$, $N \in \mathbb{Z}_+^q$, respectively, of polynomial in variables $x_j \in \mathbb{R}$, $j = \overline{1,q}$, vector-functions, and $\pi_N^{(x)}: \widetilde{X}_N \rightleftharpoons X_N$, $\pi_N^{(y)}: \widetilde{Y}_N \rightleftharpoons X_N$ are the corresponding isomorphisms between functional subspaces $\widetilde{X}_N \subset X$, $\widetilde{Y}_N \subset Y$, and some finite dimensional Euclidean spaces X_N, Y_N, respectively, satisfying conditions $\dim \widetilde{X}_N = \dim X_N = \dim Y_N = \dim \widetilde{Y}_N$ for all $N \in \mathbb{Z}_+^q$. In order to define the projectors $P_N^{(x)}: X \to \widetilde{X}_N \subset X$ and $P_N^{(y)}: Y \to \widetilde{Y}_N \subset Y$ more precisely, we consider a lattice Θ of an open cube $\Omega := K \subset \mathbb{R}^q$ with nodes, being the mesh points with respect to variables $x \in K \subset \mathbb{R}^q$, that is

$$\Theta := \{x_{(i)} \in \Omega\colon (i) \in \mathbb{Z}_+^q\}. \tag{16}$$

Then, by definition,

$$P_N^{(x)} u := \sum_{(i)} L_{(i)}(x) u(x_{(i)}), \quad P_N^{(y)} v := \sum_{(i)} L_{(i)}(x) v(x_{(i)}), \tag{17}$$

for arbitrary continuous functions $u \in \widetilde{X}_N \subset X$ and $v \in \widetilde{Y}_N \subset Y$, with

$$L_{(i)}(x) := \bigotimes_{j=\overline{1,q}} l_j(x_j|x_{(i)}),\ (i) \in \mathbb{Z}_+^q,$$

being basis Lagrange polynomials, normalized by the unity operator:

$$\sum_{(i)} L_{(i)}(x) := I := \bigotimes_{j=\overline{1,q}} 1_j, \tag{18}$$

where the sign "$\bigotimes$" means here the usual tensor product of vectors.

Let $S_N^{(j)}, D_N^{(j)}$and $I_N^{(j)} \in \mathrm{Hom}(X_N;Y_N)$, $j=\overline{1,q}$, $N \in \mathbb{Z}_+^q$, be the corresponding matrix quasi-representations [3] of the Heisenberg-Weil algebra basis with respect to the Lagrange interpolation mappings (17). Then problem (1) can be represented as the following sequence of linear algebraic vector equations

$$A_N u_N = f_N\ , \tag{19}$$

where, by definition, $A_N := A(S_N;D_N)$ is a special selection [10] of the set-valued mapping $\Phi_N^{(y)} A \Phi_N^{(x),-1} \colon X_N \to Y_N$, $u_N := \Phi_N^{(x)} u \in X_N$, $f_N := \Phi_N^{(y)} f \in Y_N$, $N \in \mathbb{Z}_+^q$.

The sequence of algebraic vector equations (19) is main for our further studying approximate solutions to differential-operator equation (1) within the projection-algebraic method.

3.2. Convergence analysis. Consider now two families of chosen above finite-dimensional functional subspaces $\widetilde{X}_N \subset X$ and $\widetilde{Y}_N \subset Y$ for $N \in \mathbb{Z}_+$, such that

$$\begin{aligned} \widetilde{X}_N &\subset \widetilde{X}_{N+1}, & \widetilde{Y}_N &\subset \widetilde{Y}_{N+1}, \\ \overline{\bigcup_{N\in\mathbb{Z}_+} \widetilde{X}_N} &= X, & \overline{\bigcup_{N\in\mathbb{Z}_+} \widetilde{Y}_N} &= Y \end{aligned} \tag{20}$$

Assume, for brevity, that a region $\Omega \subset \mathbb{R}^q$ is bounded, for the space $X := L_p(\Omega;\mathbb{R})$ and domain $D(A) = W_p^{(m+s)}(\Omega)$ and $\mathrm{Range}(A) = W_p^{(s)}(\Omega) \subset L_p(\Omega;\mathbb{R}) := Y$, $p > q$, $s > 0$, expressions

$$\widetilde{X}_N := P_N^{(x)} W_p^{(m+s)}(\Omega), \widetilde{Y}_N := P_N^{(y)} W_p^{(s)}(\Omega), \tag{21}$$

where $P_N^{(x)}$ are linear operators, defined on continuous functions in a region $\Omega \subset \mathbb{R}^q$. Operators $P_N^{(x)}$ and $P_N^{(y)}$ are, as is well known, projectors for which there are satisfied the conditions

$$P_N^{(x)} P_N^{(x)} = P_N^{(x)}, \quad P_N^{(y)} P_N^{(y)} = P_N^{(y)} \tag{22}$$

for all $N \in \mathbb{Z}_+$.

Consider now for each $N \in \mathbb{Z}_+$ the following equation

$$P_N^{(y)} A \widetilde{u}_N = P_N^{(y)} f \tag{23}$$

on element $\widetilde{u}_N \in \widetilde{X}_N$, for which as $N \to \infty$

$$\lim_{N\to\infty} \|A\widetilde{u}_N - f\|_Y = 0, \tag{24}$$

where mapping $f \colon X \to Y$ is a constant element of the space Y. It is evidently that equation (23) possesses a unique solution $\widetilde{u}_N \in \widetilde{X}_N$, if for each $N \in \mathbb{Z}_+$ there holds the equality

$$P_N^{(y)} A \widetilde{X}_N = \widetilde{Y}_N. \tag{25}$$

Condition (25) is equivalent to the existence of the inverse finite-dimensional operator

$$P_N^{(y)} A P_N^{(x)} := A_N \colon \widetilde{X}_N \to \widetilde{Y}_N \tag{26}$$

for every $N \in \mathbb{Z}_+$.

Give now the useful definition of an arbitrary limiting-dense family of subspaces $\{\mathcal{B}_N \subset \mathcal{B} \colon N \in \mathbb{Z}_+\}$ of a Banach space B.

Definition 1. A family of subspaces $\{\mathcal{B}_N \subset \mathcal{B} \colon N \in \mathbb{Z}_+\}$ is called *limiting-dense in* $\mathcal{B}$, if for each $g \in \mathcal{B}$ the following equality

$$\rho(g, \mathcal{B}_N) := \inf_{\widetilde{w}_N \in \mathcal{B}_N} \|g - \widetilde{w}_N\|_{\mathcal{B}} \to 0 \tag{27}$$

holds as $N \to \infty$.

For further analysis we will need the following convergence theorem for our approximation process slightly generalizing the corresponding statement from [14].

Theorem 4. *Let a linear operator $A \colon X \to Y$ with a dense domain $D(A) \subset X$ be invertible and satisfy the condition $\overline{\mathrm{Range}(A)} = Y$, where X and Y are Banach spaces. Assume also that a family of subspaces $\left\{A\widetilde{X}_N \in Y \colon\ N \in \mathbb{Z}_+\right\}$ is limiting-dense and projection operators $P_N^{(y)} \colon Y \to \widetilde{Y}_N \subset Y$ satisfy the condition*

$$\|P_N^{(y)}\| \le c_N^{(y)} \tag{28}$$

for some positive sequence $c_N^{(y)} \in \mathbb{R}_+$, $N \in \mathbb{Z}_+$. Then for each element $f \in Y$ equations

$$P_N^{(y)} A u = P_N^{(y)} f \tag{29}$$

have the unique solutions $\widetilde{u}_N \in \widetilde{X}_N$ for all $N \in \mathbb{Z}_+$, where

$$\lim_{N\to\infty} \|A\widetilde{u}_N - f\|_Y = 0, \tag{30}$$

iff:

i) condition (25) is satisfied;

ii) there exists such a positive sequence $\tau_N^{(y)} \in \mathbb{R}_+$, $N \in \mathbb{Z}_+$, that for each element $\widetilde{v}_N \in A\widetilde{X}_N$, $N \in \mathbb{Z}_+$

$$\|P_N^{(y)} \widetilde{v}_N\|_{\widetilde{Y}_N} \ge \tau_N^{(y)} \|\widetilde{v}_N\|_Y; \tag{31}$$

iii) for every $f \in Y$

$$\overline{\lim_{N\to\infty}} \left[\left(1 + c_N^{(y)} \tau_N^{(y),-1}\right) \rho(f, A\widetilde{X}_N)\right] = 0. \tag{32}$$

Proof. First assume that for each element $f \in Y$ equation $P_N^{(y)} A u = P_N^{(y)} f$, $N \in \mathbb{Z}_+$, has the unique solution $\widetilde{u}_N \in \widetilde{X}_N$, as well as $\|A\widetilde{u}_N - f\|_Y \to 0$ as $N \to \infty$. Then owing to the inequality

$$\rho(f, A\widetilde{X}_N) = \inf_{w_N \in A\widetilde{X}_N} \|f - \widetilde{w}_N\|_Y \le \|f - A\widetilde{u}_N\|_Y$$

one can infer that $\lim_{N\to\infty}\rho(f, A\widetilde{X}_N)=0$, that is the family of subsets $\{A\widetilde{X}_N\in Y\colon N\in\mathbb{Z}_+\}$ is limiting-dense in Y. Define now $N\in\mathbb{Z}_+$ and consider $P_N^{(y)}Au=\widetilde{f}_N\in\widetilde{Y}_N$. It is clear that there exists such an element $f\in Y$ for which $P_N^{(y)}f=\widetilde{f}_N$ ensuring, owing to the assumptions of the Theorem, the unique solution $\widetilde{u}_N\in\widetilde{X}_N$. Equivalently this means that $P_N^{(y)}A\widetilde{X}_N=\widetilde{Y}_N$, proving condition $i)$ of the theorem.

Since the mapping $P_N^{(y)}\colon Y\to\widetilde{Y}_N\subset Y$ is a projector, one can consider its constraint $\overline{P}_N^{(y)}:=P_N^{(y)}|_{A\widetilde{X}_N}\colon A\widetilde{X}_N\to\widetilde{Y}_N$ for each $N\in\mathbb{Z}_+$. The operator $\overline{P}_N^{(y)}\colon A\widetilde{X}_N\subset Y\to\widetilde{Y}_N$, owing to (28), is bounded and one-to-one mapping. Then based on the Banach inverse operator theorem [12, 9] we obtain that there exists the bounded inverse operator $\overline{P}_N^{(y),-1}\colon\widetilde{Y}_N\to A\widetilde{X}_N\subset Y$.

Let now $\widetilde{u}_N\in\widetilde{X}_N$ be the corresponding approximated solution of the equation $P_NAu=P_Nf$. Then the following equality $A\widetilde{u}_N=\overline{P}_N^{(y),-1}P_Nf$ holds, from which and the condition (30) one obtains that

$$\lim_{N\to\infty}\|\overline{P}_N^{(y),-1}P_Nf-f\|_Y=0 \tag{33}$$

for any $f\in Y$. But this means that $\lim_{N\to\infty}\overline{P}_N^{(y),-1}P_Nf=f$ for every given element $f\in Y$. Making use of the classical Banach-Schteinhaus theorem [12, 13, 9] we obtain that

$$\sup_{N\in\mathbb{Z}_+}\|\overline{P}_N^{(y),-1}P_N^{(y)}\|_Y\le c^{(y)}<\infty \tag{34}$$

for some bounded value $c^{(y)}\in\mathbb{R}_+$. Thus for each element $\widetilde{w}_N=P_N^{(y)}j_N\widetilde{w}_N\in\widetilde{Y}_N$, where $j_N\colon\widetilde{Y}_N\to Y$ is the corresponding imbedding operator, one finds that

$$\|\overline{P}_N^{(y),-1}\widetilde{w}_N\|_Y=\|\overline{P}_N^{(y),-1}P_N^{(y)}j_N\widetilde{w}_N\|_Y\le\|\overline{P}_N^{(y),-1}P_N\|_Y\|j_N\widetilde{w}_N\|_Y\le c^{(y)}\|j_N\|\|\widetilde{w}_N\|_Y \tag{35}$$

for all $N\in\mathbb{Z}_+$. The result (35) means that the norm of the operator $\overline{P}_N^{(y),-1}\colon\widetilde{Y}_N\to A\widetilde{X}_N\subset Y$ is for all $N\in\mathbb{Z}_+$ bounded, that is

$$\|\overline{P}_N^{(y),-1}\|\le c^{(y)}\|j_N\|. \tag{36}$$

Choose now an arbitrary element $\widetilde{v}_N\in A\widetilde{X}_N\subset Y$ and calculate $\widetilde{w}_N:=\overline{P}_N^{(y)}\widetilde{v}_N\in\widetilde{Y}_N$. Then, making use of inequality (36), we obtain

$$\|\widetilde{v}_N\|_Y=\|\overline{P}_N^{(y),-1}\widetilde{w}_N\|_Y\le c^{(y)}\|j_N\|\|\widetilde{w}_N\|_Y:=\tau_N^{(y),-1}\|P_N\widetilde{v}_N\|_{\widetilde{Y}_N}, \tag{37}$$

where quantities $\tau_N^{(y),-1}:=c^{(y)}\|j_N\|>0$ are bounded for all $N\in\mathbb{Z}_+$. But this means that the condition $ii)$ of our theorem is fulfilled concerning each element $\widetilde{v}_N\in A\widetilde{X}_N$, that is $\|P_N^{(y)}\widetilde{v}_N\|\ge\tau_N^{(y)}\|\widetilde{v}_N\|_Y$, $N\in\mathbb{Z}_+$.

Sufficiency of conditions $i)-iii)$ we will prove in the next way. Let us solve the equation $P_NAu=P_Nf$ for $N\in\mathbb{Z}_+$, whose solution $\widetilde{u}_N\in\widetilde{X}_N$ is unique. Then it can be represented as $\widetilde{u}_N=A^{-1}\overline{P}_N^{(y),-1}P_Nf$, where, as above, the linear mapping $\overline{P}_N^{(y)}:=P_N^{(y)}|_{A\widetilde{X}_N}\colon A\widetilde{X}_N\to\widetilde{Y}_N$ is the corresponding reduction upon $A\widetilde{X}_N\subset Y$ of the projection operator $P_N^{(y)}\colon Y\to Y$ upon the subspace $\widetilde{Y}_N\subset Y$. Since, based on condition $ii)$, we have $\|\overline{P}_N^{(y)-1}\|_Y\le\tau_N^{(y),-1}$, the norm $\|\overline{P}_N^{(y),-1}P_N^{(y)}\| \le c_N^{(y)}\tau_N^{(y),-1}$ for all $N \in \mathbb{Z}_+$. Whence for any element $\widetilde{w}_N \in A\widetilde{X}_N \subset Y$ we obtain

$$\begin{aligned} &\|A\widetilde{u}_N - f\|_Y = \|\overline{P}_N^{(y),-1}P_Nf - f\|_Y \le \\ &\le \inf_{\widetilde{w}_N \in A\widetilde{X}_N} \left(\|\overline{P}_N^{(y),-1}P_Nf - \overline{P}_N^{(y),-1}P_N^{(y)}\widetilde{w}_N\|_Y + \|\widetilde{w}_N - f\|_Y\right) \le \\ &\le \inf_{\widetilde{w}_N \in A\widetilde{X}_N} \left(\|\overline{P}_N^{(y),-1}P_Nf - \overline{P}_N^{(y),-1}P_N^{(y)}\widetilde{w}_N\|_Y + \|\widetilde{w}_N - f\|_Y\right) \le \\ &\le \inf_{\widetilde{w}_N \in A\widetilde{X}_N} \left(c_N^{(y)}\tau_N^{(y),-1} + 1\right)\rho(f, \widetilde{w}_N) = \left(c^{(y)}\tau_N^{(y),-1} + 1\right)\rho(f, A\widetilde{X}_N), \end{aligned} \tag{38}$$

where we took into account that $\overline{P}_N^{(y),-1}P_N^{(y)}\widetilde{w}_N = \widetilde{w}_N$ for all $\widetilde{w}_N \in A\widetilde{X}_N \subset Y$. But owing to the assumption *iii)* this means the existence of the limit $\lim_{N\to\infty}\|A\widetilde{u}_N - f\|_Y = 0$ for an arbitrary element $f \in Y$, finishing the proof. □

Remark. We note here that some analog of Theorem 4, but alternative from it, was earlier stated in [14].

As an obvious corollary from the proof of Theorem 4 in the case when dim $\widetilde{X}_N =$ dim $\widetilde{Y}_N < \infty$ for all $N \in \mathbb{Z}_+$we obtain that condition *i)* in form (25) follows from *ii)*. Moreover, the next statement about the convergence of the solutions $\widetilde{u}_N \in \widetilde{X}_N$ as $N \to \infty$ to element $u \in X$ holds.

Theorem 5. *Let all the conditions of Theorem 4 be fulfilled, in particular, the operator* $A\colon X \to Y$ *is surjective and, moreover, closed. (This means that* $\|A^{-1}\| < \infty$ *owing to the classical statement [12, 9] about the closed operator). Then the obtained sequence of solutions* $\widetilde{u}_N \in \widetilde{X}_N$ *to the equation* $P_N^{(y)}Au = P_N^{(y)}f$ *as* $N \to \infty$ *is the corresponding approximation to the solution of the equation* $Au = f$ *subject to the norm* $\|\cdot\|_X$.

Proof. Assume that $u_N \in X_N$ is a solution to the equation $P_N^{(y)}Au_N = P_N^{(y)}f$ for all $N \in \mathbb{Z}_+$. Then one can estimate the difference $(u - \widetilde{u}_N\ ) \in X$ subject to the norm in the Banach space X:

$$\begin{aligned} \|\widetilde{u}_N - u\|_X = \|\widetilde{u}_N - A^{-1}f\|_X = \|A^{-1}A\widetilde{u}_N - A^{-1}f\|_X = \\ = \|A^{-1}(A\widetilde{u}_N - f)\|_X \le \|A^{-1}\|\ \|A\widetilde{u}_N - f\|_Y. \end{aligned} \tag{39}$$

Based now on inequality (38) we receive that $\lim_{N\to\infty}\|A\widetilde{u}_N - f\|_Y = 0$. As the inverse operator A^{-1} is closed and, therefore, bounded, the right hand side of inequality (39) tends to zero as $N \to \infty$. Thereby we state that $\lim_{N\to\infty}\|\widetilde{u}_N - u\|_X = 0$, finishing the proof. □

3.3. The special case: Lagrangian interpolation. For further studying the projection-algebraic method of discrete approximations of solutions to linear differential-operator equations we once more assume that $\Omega := K \subset \mathbb{R}^q$ is a q-dimensional cube. Define also the finite-dimensional functional subspaces $\widetilde{X}_N \in X$ and $\widetilde{Y}_N \in Y$ for $N \in \mathbb{Z}_+$, making use of the Lagrange projection operators, described above.

Let $X := L_p(K;\mathbb{R})$, $D(A) = W_p^{(m+s)}(K;\mathbb{R})$, $Y := L_p(K;\mathbb{R})$, Range$(A) = W_p^{(s)}(K;\mathbb{R})$, $p > q$, $s \ge 1$, and $s - q/p > 0$. Based on the Sobolev imbedding theorem [13] there holds the property $W_p^{(m)}(K;\mathbb{R}) \subset C(K;\mathbb{R})$. So, we can construct the suitable subspaces $\widetilde{Y}_N \subset Y$, $N \in \mathbb{Z}_+$, as follows:

$$\widetilde{Y}_N = P_N^{(y)}W_p^{(s)}(K;\mathbb{R}), \tag{40}$$

where the projector $P_N^{(y)}: Y \to \widetilde{Y}_N$ is the classical Lagrange interpolation operator, defined as

$$P_N^{(y)} f(x) := \sum_{(\alpha)\in\mathbb{Z}_+^q}^{\prod_{j=1}^q \alpha_j = N} f(x_{(\alpha)}) l_{(\alpha)}(x), \tag{41}$$

where $x_{(\alpha)} \in K$ are suitable nodes in the cube $K \subset \mathbb{R}^q$, multi-index $(\alpha) \in \mathbb{Z}_+^q$, and $l_{(\alpha)}(x) := \Pi_{j=1}^q l_{\alpha_j}(x_j)$, $l_{(\alpha)}(x_{(\beta)}) = \delta_{(\alpha),(\beta)}$, are basic q-dimensional Lagrange polynomials. As we have chosen subspaces $\widetilde{X}_N \in X$ and $\widetilde{Y}_N \in Y$ finite-dimensional, condition (25) follows from *ii*) of Theorem 4. Now it is necessary to consider the inequality

$$\|P_N^{(y)} \widetilde{v}_N\|_{\widetilde{Y}_N} \geq \tau_N^{(y)} \|\widetilde{v}_N\|_Y \tag{42}$$

for all $\widetilde{v}_N \in A\widetilde{X}_N \in Y$, where $\tau_N^{(y)} < \infty$, $N \in \mathbb{Z}_+$. Condition (42) means, that the projector $P_N^{(y)}$ is invertible upon the subspace $A\widetilde{X}_N \subset Y$, since, evidently, $\ker P_N^{(y)} = \{0\}$. Now from the definition $\widetilde{v}_N = A\widetilde{w}_N$, where $\widetilde{w}_N \in \widetilde{X}_N$, and from the fact that the operator $A\colon X \to Y$ is invertible, one follows that also $\ker(P_N^{(y)} A) = \{0\}$. Additionally, from the condition dim $\widetilde{X}_N =$ dim $\widetilde{Y}_N$ one follows that $\mathrm{Range}(P_N^{(y)} A)|_{\widetilde{X}_N} = \widetilde{Y}_N$.

Consider now the subspace $C(K;\mathbb{R})$ of continuous functions on the cube $K \subset \mathbb{R}^q$ and the corresponding Lagrange interpolation projector (41), whose nodes are roots of Hermite's polynomials. We define such quantities:

$$\delta_N\colon = \min_{k=\overline{1,q}, j=\overline{1,N_k}} \{(x_k^{(j+1)} - x_k^{(j)})\colon j = \overline{1, N_k}\}, \quad N := \prod_{j=\overline{1,q}} N_j, \tag{43}$$

$$\lambda_N^{(y)}\colon = \|P_N^{(y)}\| \leq c_q^{(1)}(K)(\log N)^q,$$

where $c_q^{(1)}(K) > 0$ for all $N \in \mathbb{Z}_+$. Let now a functional $x^* \in C^*(K;\mathbb{R})$, $x \in K$, is determined in such a way that $x^*(f) := f(x)$ for any function $f \in C(K;\mathbb{R})$. Then, as it is well known [13], for the interpolation, whose nodes are chosen as roots of Hermite's polynomials, the following inequality

$$\|x^* P_N^{(y)}\|_{C^*(K;\mathbb{R})} = \sum_{\alpha\in\mathbb{Z}_+^q}^{\prod_{j=1}^q \alpha_j = N} |l_\alpha(x)| := \lambda_N^{(y)} \leq c_q^{(2)}(K) N^{-s} (\log N)^q \tag{44}$$

holds for all $x \in K$ and $N \in \mathbb{Z}_+$. Based on the Jackson type inequality [13] for each function $f \in C^{(s)}(K;\mathbb{R})$ we have also

$$E_N[f] \leq c_q^{(2)} N^{-s} \|f^{(s)}\|_{C(K;\mathbb{R})}. \tag{45}$$

Thus, making use of (44) and (45), for $f \in C^{(s)}(K;\mathbb{R})$ we obtain such an estimation:

$$|(x^* P_N^{(y)})(f) - x^*(f)| := |P_N^{(y)} f(x) - f(x)| \leq$$

$$\leq \sum_{\alpha\in\mathbb{Z}_+^q}^{\prod_{j=1}^q \alpha_j = N} |l_\alpha(x)|\, |f(x_\alpha) - p_N(x_\alpha) + p_N(x_\alpha) - f(x)| \leq c_q^{(3)}(K) \frac{(\log N)^q}{N^s} \|f^{(s)}\|_{C(K;\mathbb{R})}, \tag{46}$$

where the constant $c_q^{(3)}(K) := c_q^{(1)}(K) c_q^{(2)}(K) < \infty$, and p_N is the best approximation polynomial of degree $N \in \mathbb{Z}_+$ on the cube $K \subset \mathbb{R}^q$. Assuming now that differential operator

(2) is bounded as an operator $A\colon C^{(s+m)}(K;\mathbb{R})\to C^{(s)}(K;\mathbb{R})$, and for $s\geq 1$ there is fulfilled the Kato-Rellich condition [12]

$$\|u\|_{p,m+s}\leq c_q^{(4)}(K)(\|Au\|_{p,s}+\|u\|_{p,s}) \tag{47}$$

for some constant $c_q^{(4)}>0$ and every $u\in W_p^{(m+s)}(K;\mathbb{R})$, based on the estimation (46) from inequality (47) we obtain the existence of such a constant $\tau_N^{(y)}>0$, that the main inequality (42) holds. Thereby, owing to Theorem 4 we make the inference of the existence of unique approximated solutions to equations (29) from subspaces $\widetilde{X}_N\subset W_p^{(m+s)}(K;\mathbb{R})$ for each $N\in\mathbb{Z}_+$, which approximate the exact solution to differential equation (1). This means, that the system of finite-dimensional functional equations

$$\widetilde{A}_N\,\widetilde{u}_N:=P_q^{(y)}AP_N^{(x)}\widetilde{u}_N=\widetilde{f}_N:=P_N^{(y)}f \tag{48}$$

for all $N\in\mathbb{Z}_+$ may be represented in an equivalent form as a general system of the algebraic vector equations

$$A_Nu_N=f_N, \tag{49}$$

where $u_N:=\pi_N^{(x)}\widetilde{u}_N, f_N:=\pi_N^{(y)}\widetilde{f}_N$, $A_N:=\pi_N^{(y)}\widetilde{A}_N\pi_N^{(x),-1}$, with $\pi_N^{(x)}\colon\widetilde{X}_N\to X_N\simeq\mathbb{R}^{(N)}$, $\pi_N^{(y)}\colon\widetilde{Y}_N\to Y_N\simeq\mathbb{R}^{(N)}$ being the corresponding canonical isomorphisms [2, 3] between finite–dimensional subspaces for all $N\in\mathbb{Z}_+$. Thus, the following theorem holds.

Theorem 6. *Let the differential operator (2) defined above is invertible and satisfies condition (47) for some $s\geq 1$. Then there exists the unique solution $u\in W_p^{(m+s)}(K;\mathbb{R})$ of equation (1), which is the corresponding limit of approximated solutions to finite-dimensional equations (49), constructed by means of the projection-algebraic method of discrete approximations.*

Concerning the effective construction of finite-dimensional operators $A_N\colon X_N\to Y_N$, $N\in\mathbb{Z}_+$, we use the functional-algebraic properties of discrete approximations of Heisenberg-Weil algebra basis operators $\mathcal{G}(q):=\bigoplus_{j=\overline{1,q}}\{1,x_j,\partial/\partial x_j\}$, which were discussed above and studied in works [1, 2, 3]. Then expression (49) becomes a usual system of algebraic vector equations, whose matrix $A_N\colon X_N\to Y_N$, owing to the homomorphism property [18] of the universal algebra $\mathcal{U}(\mathcal{G})$ representations, is given in the form

$$A_N=\sum_{|\beta|=0}^{m}a_\beta(S_N)D_N^\beta, \tag{50}$$

where S_N and $D_N\colon X_N\to X_N$ are the corresponding finite-dimensional tensor quasi-representations of basis elements (or generators) of the Heisenberg-Weil algebra $\mathcal{G}(q)$. Solving equations (49) with matrices (50), we obtain vector solutions $u_N\in X_N\simeq\mathbb{R}^{(N)}$, which generate, respectively, the searched approximated functional solution $\widetilde{u}_N:=\pi_N^{(x),-1}u_N\in\widetilde{X}_N\subset X$ of our given equation (1). In case when additionally there are imposed some boundary conditions upon the solution $u\in X$ to equation (1), the developed projection-algebraic method can also be with success used and for this problem. Analogous results one can also obtain and for the case of evolution equations in partial derivatives in the form $du/dt-Au=f$, where $u\in X, f\in Y$ and parameter $t\in\mathbb{R}_+$. We plan these aspects of the problem to discuss in a separate investigation.

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