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HYPERSPACE AS INTERSECTION OF INCLUSION HYPERSPACES AND IDEMPOTENT MEASURES

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We prove that the hyperspace $\exp X$ of a compact space X can be obtained as the intersection of the space GX of inclusion hyperspaces over X and the space IX of idempotent measures on X.

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Доказано, что гиперпространство может быть представлено в виде пересечения гиперпространств включения и идемпотентных мер.

0. The algebraic aspect of the theory of functors in categories of topological spaces and continuous maps was investigated rather recently. It is based, mainly, on the existence of monad (or triple) structure in the sense of S. Eilenberg and J. Moore ([1]).

Many classical constructions lead to monads: hyperspaces, spaces of probability measures, superextensions etc. There were many investigations of monads in topological categories (see [2], [3]). But it seems that the main difficulty to obtain general results in the theory of monads is the different nature of functors.

A functional representation of the hyperspace functor exp is given in [4]. This representation essentially uses the linear structure on function spaces.

The hyperspace functor could be included in the hyperspace monad ([3]). From the algebraic point of view hyperspaces are free Lawson semilattices. A functional representation of the hyperspace functor exp which involves the semilattice structure on function spaces is given in [5].

A suficiently wide class of so-called Lawson monads was introduced in [6]. Lawson monads have a functional representation, i.e., their functorial part FX can be naturally imbedded in $\mathbb{R}^{CX}$. The class of Lawson monads includes the inclusion hyperspace monad $\mathbb{G}$ ([6]) and the idempotent measure monad ([7]). The main goal of this paper is to show that hyperspace monad can be represented as intersection of inclusion hyperspaces and idempotent measures.

The paper is organized as follows: in Section 1 we give some necessary definitions, in Section 2 we obtain the main result.

1. By $\mathcal{C}omp$ we denote the category whose objects are compacta (compact Hausdorff spaces) and morphisms are continuous mappings.

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We start with the definition of the monad of order-preserving functionals $\mathbb{O}$ ([8]) which contains all the above mentioned monads as submonads.

Let X be compactum. By CX we denote the Banach space of all continuous functions $\phi\colon X \to \mathbb{R}$ with the usual sup-norm. We consider CX with natural order, linear structure and lattice operations – pointwise minimum and maximum. For each $c \in \mathbb{R}$ we denote by c_X the constant function from CX defined by the formula $c_X(x) = c$ for each $x \in X$.

A functional $\nu\colon CX \to \mathbb{R}$ is called *weakly additive* if for each $c \in \mathbb{R}$ and $\phi \in CX$ we have $\nu(\phi + c_X) = \nu(\phi) + c$; *normed* if $\nu(1_X) = 1$; *order-preserving* if for each ϕ, $\psi \in CX$ with $\phi \le \psi$ we have $\nu(\phi) \le \nu(\psi)$. (We do not suppose a priori that ν is linear.) It is shown in [8] that each order-preserving weakly additive functional is continuous.

For a compactum X by OX we will denote the set of all order-preserving weakly additive normed functionals. We consider $O(X)$ as the subspace of the space $C_p(C(X))$ of all continuous functions on $C(X)$ equipped with point-wise convergence topology. A base of this topology consists of sets of the form

$$(\mu; \phi_1, \dots, \phi_n; \varepsilon) = \big\{\mu' \in C_p(C(X))\colon |\mu'(\phi_i) - \mu(\phi_i)| < \varepsilon \text{ for each } i \in \{1, \dots, k\}\big\},$$

where $\mu \in C_p(C(X))$, $\phi_1, \dots, \phi_k \in C(X)$, $\varepsilon > 0$. For each compactum X the space $O(X)$ is compact ([8]).

Let X, Y be compacta and let $f\colon X \to Y$ be a continuous map. Define the map $O(f)\colon O(X) \to O(Y)$ by the formula $(O(f)(\mu))(\phi) = \mu(\phi \circ f)$, where $\mu \in O(X)$ and $\phi \in C(Y)$.

It is easy to check that $O(f)$ is well-defined, continuous, and $O(f \circ g) = O(f) \circ O(g)$. Thus, O is a covariant functor on the category $\mathcal{C}omp$. Let's define the mapping $\nu X : O^2(X) \to O(X)$ by the formula $\nu X(\alpha)(g) = \alpha(\widetilde{g})$, where $\alpha \in O^2(X)$, $g \in C(X)$ and the mapping $\widetilde{g}\colon O(X) \to \mathbb{R}$ is defined by the formula $\widetilde{g}(\gamma) = \gamma(g)$, $\gamma \in O(X)$. It is easy to check that νX is correctly defined and continuous. Let us define a map $\xi X : X \to OX$ by the formula $\xi X(x)(\phi) = \phi(x)$, $\phi \in C(X)$, $x \in X$.

It was proved in [8] that ξX and νX are the components of natural transformations $\xi\colon \mathrm{Id}_{\mathcal{C}omp} \to O$ and $\nu\colon O^2 \to O$ and the triple $\mathbb{O} = (O, \xi, \nu)$ forms a monad on the category $\mathcal{C}omp$.

M.M. Zarichnyi considered the space of idempotent measures IX as a subspace of OX defined as follows: $IX = \{\alpha \in OX \mid \alpha(\max\{\phi, \psi\}) = \max\{\alpha(\phi), \alpha(\psi)\}$ for each ϕ, $\psi \in CX\}$. It appears that the construction I determines a submonad $\mathbb{I}$ of $\mathbb{O}$ ([7]).

Finally let us describe the hyperspace monad $\mathbb{H}$ and the inclusion hyperspaces monad $\mathbb{G}$.

For a compactum X by $\exp X$ we denote the set of non-void compact subsets of X provided with the Vietoris topology. A base of this topology consists of the sets of the form

$$\langle U_1, \dots, U_n\rangle = \Big\{A \in \exp\ X \mid A \subset \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \varnothing \text{ for each } i \in \{1, \dots, n\}\Big\}$$

where $U_1, \dots, U_n$ are open in X. The space $\exp\ X$ is called the hyperspace of X.

For a continuous mapping $f\colon X \to Y$ the mapping $\exp\ f\colon \exp\ X \to \exp\ Y$ is defined by the formula $\exp\ f(A) = fA \in \exp\ Y$, $A \in \exp\ X$. It is easy to see that this defines a functor $\exp\colon \mathcal{C}omp \to \mathcal{C}omp$ (the hyperspace functor). Define the natural transformations $s\colon \mathrm{Id}_{\mathcal{C}omp} \to \exp$ and $u\colon \exp^2 \to \exp$ as follows: $sX(x) = \{x\}$ for each $x \in X$; $uX(\mathcal{A}) = \cup\mathcal{A}$, $\mathcal{A} \in \exp^2\ X$. Then $\mathbb{H} = (\exp, s, u)$ is a monad ([2]). M.M. Zarichnyi has remarked that the monad $\mathbb{H}$ could be represented as a submonad of $\mathbb{I}$ ([7]).

An element $\mathcal{A} \in \exp^2 X$ is called an *inclusion hyperspace* if for each $A \in \mathcal{A}$ and $B \in \exp X$ with $A \subset B$ we have $B \in \mathcal{A}$. Let us denote by $GX = \{\mathcal{A} \in \exp^2 X \mid \mathcal{A}$ is inclusion hyperspace$\}$. We consider GX as a subset of $\exp\exp X$. For a map $f\colon X \to Y$ define a map $Gf\colon GX \to GY$ by the formula $Gf(\mathcal{A}) = \{A \in \exp Y \mid f(B) \subset A$ for some $B \in \mathcal{A}\}$, $\mathcal{A} \in GX$. It is easy to check that G is a covariant functor on the category $\mathcal{C}omp$.

Define natural transformations $\eta\colon I_{\mathcal{C}omp} \to G$ and $\mu\colon G^2 \to G$ as follows: $\eta X(x) = \{A \in \exp X \mid x \in A\}$, $x \in X$ and $\mu X(\widetilde{\mathcal{A}}) = \bigcup\{\bigcap \alpha \mid \alpha \in \widetilde{\mathcal{A}}\}$, where $\widetilde{\mathcal{A}} \in G^2X$. It is shown in [9] that the triple $\mathbb{G} = (G, \eta, \mu)$ is a monad on the category $\mathcal{C}omp$.

2. The main goal of this section is to represent monads $\mathbb{H}$ and $\mathbb{G}$ as submonads of $\mathbb{O}$ in such way that the image of $\exp X$ will be intersection of the image of GX and IX.

Let us recall that a natural transformation $\psi\colon T \to T'$ is called a *morphism* from a monad $\mathbb{T} = (T, \eta, \mu)$ into a monad $\mathbb{T}' = (T', \eta', \mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \psi T' \circ T\psi$. If all the components ψX of ψ are monomorphisms then ψ is called an *embedding* of the monad $\mathbb{T}$ in the monad $\mathbb{T}'$.

First of all consider a natural transformation $j\colon \mathbb{H} \to \mathbb{G}$ defined as follows $jX(A) = \{B \in \exp X \mid B \cap A \neq \varnothing\}$.

Lemma 1. *The natural transformation $j\colon \mathbb{H} \to \mathbb{G}$ is an embedding of the monad $\mathbb{H}$ in the monad $\mathbb{G}$.*

Proof. It is obvious that jX is an embedding for each compactum X. Let us show that j is a morphism of monads.

Consider any $x \in X$. Then $jX \circ s(x) = \{B \in \exp X \mid x \in B\} = \eta X(x)$. Now, let $\mathcal{A} \in \exp^2(X)$. Then $\mu X \circ jGX \circ \exp jX(\mathcal{A}) = \mu X \circ jGX(\{jX(A) \mid A \in \mathcal{A}\}) = \mu X(\{\alpha \in \exp GX \mid jX(A) \in \alpha$ for some $A \in \mathcal{A}\}) = \bigcup\{jX(A) \mid A \in \mathcal{A}\} = \{B \in \exp X \mid B \cap A \neq \varnothing$ for some $A \in \mathcal{A}\} = \{B \in \exp X \mid B \cap (\cup\mathcal{A}) \neq \varnothing\} = jX \circ sX(\mathcal{A})$. The lemma is proved. □

We say that a functional $\nu \in OX$ *weakly preserves* min (resp. max) if for each $\phi \in CX$ and $c \in \mathbb{R}$ we have $\nu(\min\{\phi, c_X\}) = \min\{\nu(\phi), c\}$ (resp. $\nu(\max\{\phi, c_X\}) = \max\{\nu(\phi), c\}$). Let us consider the map $gX\colon GX \to OX$ defined as follows $gX(\mathcal{A})(\phi) = \sup\{\inf \phi A \mid A \in \mathcal{A}\}$, $\mathcal{A} \in GX$ and $\phi \in CX$. It was proved in [6] that $g\colon \mathbb{G} \to \mathbb{O}$ is an embedding of monads. Moreover, $gX(GX) = \{\alpha \in OX \mid \alpha$ weakly preserves max and min$\}$.

Let us denote $HX = gX \circ jX(\exp X)$.

Lemma 2. *For each $\alpha \in HX$ α preserves* max.

Proof. Consider any $A \in \exp X$, ϕ_1, $\phi_2 \in CX$. We have $gX \circ jX(A)(\phi_i) = \sup_{x\in A} \phi(x)$ for $i \in \{1, 2\}$. Since $\sup_{x\in A}(\max\{\phi_1(x), \phi_2(x)\}) = \max\{\sup_{x\in A} \phi_1(x), \sup_{x\in A} \phi_1(x)\}$, the lemma is proved. □

Theorem 1. *$HX = IX \cap gX(GX)$.*

Proof. The inclusion $\subset$ follows from Lemma 2. Let us show the inverse inclusion. Consider any $\nu \in IX \cap gX(GX)$. Suppose the contrary $\nu \notin HX$. There exists $\mathcal{A} \in G(X) \setminus jX(\exp X)$ such that $gX(\mathcal{A}) = \nu$. Hence, there exists $A \in \mathcal{A}$ such that A contains at least two distinct points and A is a minimal element of $\mathcal{A}$ with respect to inclusion. Let x, $y \in A$ be two distinct elements. Consider two open subset V_1, V_2 of X such that $x \in V_1$, $y \in V_2$ and $\operatorname{cl} V_1 \cap \operatorname{cl} V_2 = \varnothing$. Put $B_i = A \setminus V_i$ for $i \in \{1, 2\}$. Since A is a minimal element in $\mathcal{A}$, we have that $B_i \notin \mathcal{A}$. Then there exist two open subset O_1 and O_2 of X such that $B_i \subset O_i$ for $i \in \{1,2\}$ and $C \notin \mathcal{A}$ for each $C \subset O_i$. Certainly we have $A \subset O_1 \cup O_2$.

Consider two disjoint open subsets U_i of X such that $V_i \subset \operatorname{cl} V_i \subset U_i$ for $i \in \{1,2\}$. Define two continuous functions $\psi_i \colon X \to [0,1]$ such that $\psi_i(z) = 0$ for each $z \in \operatorname{cl} V_i \cup (X \setminus (O_1 \cup O_2))$ and $\psi_i(y) = 1$ for each $y \in X \setminus U_i$.

Consider any $B \in \mathcal{A}$. Let us show that for each $i \in \{1,2\}$ there exist $b_i \in B$ such that $\psi_i(b_i) = 0$. Consider two cases. If $B \subset O_1 \cup O_2$, then $B \nsubseteq O_i$ for $i \in \{1,2\}$. The we can choose $b_i \in B \cap V_i$. In the opposite case we can choose $b \in B \setminus (O_1 \cup O_2)$ and put $b_1 = b = b_2$. We have $\psi_i(b_i) = 0$ in both cases.

Then $\nu(\psi_i) = \sup\{\inf \psi_i B \mid B \in \mathcal{A}\} = 0$ for each $i \in \{1,2\}$. But $\max\{\psi_1, \psi_2\}(x) = 1$ for each $x \in A$. Hence $\nu(\max\{\psi_1, \psi_2\}) = 1$ and we obtain a contradiction. The theorem is proved. □

As a conclusion we obtain a functional representation of exp distinct of the representations given in [4] and [5].

Corollary. *HX consists of all functionals which are normed, weakly additive, preserve* max *and weakly preserve* min.

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