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FUNCTORS OF FINITE DEGREE AND ASYMPTOTIC DIMENSION

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We prove that any finitary weakly normal functor of finite degree preserves the class of metric spaces of finite asymptotic dimension.

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Доказано, что каждый финитный слабо нормальный функтор конечной степени сохраняет класс собственных метрических пространств конечной асимптотической размерности.

1. Introduction. The asymptotic dimension asdim of a metric space was defined by Gromov for studying asymptotic invariants of discrete groups ([4]). This dimension can be considered as an asymptotic counterpart of the Lebesgue covering dimension dim.

In [5] E. Shchepin has introduced the class of normal functors in the category **Comp** of compact Hausdorff spaces. In the last decades, many results were obtained for normal and close to normal functors. Concerning the dimension dim, one has the following result by Basmanov ([1, 2]): the finitary functors of finite degree preserve the class of finite-dimensional spaces. Moreover, if the dimension of the space X does not exceed m and the degree of a functor F does not exceed n, then the dimension of the space $F(X)$ does not exceed nm.

O. Shukel' introduced a counterpart of the notion of normal functor in the asymptotic category and proved that such functors preserve the class of metric spaces of asymptotic dimension zero ([6]). In this note we prove the counterpart of Basmanov result for asymptotic dimension.

2. Preliminaries. First of all we need the construction of a functor of finite degree for metric spaces. Fix a natural number n. Let $\mathcal{K}_n$ denote the category of sets of cardinality $\leq n$ and Set_f the category of finite sets.

Let $F\colon \mathcal{K}_n \to \mathrm{Set}_f$ be a functor possessing the following properties: 1) $F(\varnothing) = \varnothing$; 2) $F(\{*\}) = \{*\}$; 3) if $i\colon X \to Y$ is an embedding, then so is $F(i)\colon F(X) \to F(Y)$ (In the sequel, for any subset $X \subset Y$ we shall identify $F(X)$ with the subset $F(i)(F(X))$ of $F(Y)$). 4) $F(A \cap B) = F(A) \cap F(B)$; (This property helps us to define the *support* $\mathrm{supp}(a) = \cap\{A \subset X \mid a \in F(A)\}$ for each element $a \in F(X)$). 5) if $f\colon X \to Y$ is an onto map, then so is $F(f)\colon F(X) \to F(Y)$.

Given a functor $F\colon \mathcal{K}_n \to \mathrm{Set}_f$ as above and a metric space (X, d), define a metric space $(F(X), \widehat{d})$ as follows.

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Consider the family $\exp_f X$ of nonempty finite subsets in X, partially ordered by the inclusion relation. We define the set $F(X)$ to be the direct limit of the direct system $\{F(A),\ F(\iota_{AB});\ \exp_f X\}$ (here, for sets $A \subset B$ in $\exp_f X$ by $\iota_{AB}\colon A \to B$ the inclusion mapis denoted). For every $A \in \exp_f X$, we identify $F(A)$ with the corresponding subset of $F(X)$ along the map $F(\iota_A)$, where $\iota_A\colon A \to X$ is the limit inclusion map. For any $a \in F(X)$, there exists a unique minimal $A \in \exp_f X$ such that $a \in F(A)$. Then we say that A is the *support* of a and write $\mathrm{supp}(a) = A$. Note that this notion of support agrees with that defined above. Note that the cardinality of the supports does not exceed n. We express this by saying that the degree of F is at most n.

Given two maps $f, g\colon Y \to X$ where (X, d) is a metric space, we define the distance between f and g as $\sup_{y\in Y} d(f(y), g(y))$. We keep the notation d for this distance.

We recall the definion of a metric $\widehat{d}$ on $F(X)$ introduced in [6]. Given $a, b \in F(X)$, we let

$$\widehat{d}(a,b) = \inf\Big\{ \sum_{i=1}^m d(f_{2i-1}, f_{2i})\colon\ f_{2i-1}, f_{2i}\colon A_i \to X \text{ are such that there exist } c_i \in F(A_i), \\ \mathrm{supp}(c_i) = A_i,\ i \in \{1,\dots,m\},\ \text{with } a = F(f_1)(c_1),\ F(f_2)(c_1) = F(f_3)(c_2), \dots, \\ F(f_{2m-1})(c_m) = F(f_{2m-2})(c_{m-1}),\ F(f_{2m})(c_m) = b\Big\}.$$

Hereafter, we say that $f_1, \dots, f_{2m}$ and $c_1, \dots, c_m$ form a *chain* connecting a and b. The number m is then called the *length* of this chain.

It was shown in [6] that the function $\widehat{d}\colon F(X)\times F(X) \to \mathbb{R}$ is a metric on $F(X)$. Moreover, in the definition of the metric $\widehat{d}$ we can restrict ourselves by considering chains of length $\le N$ where N depends only on the functor F, see [6].

The construction F could be completed to a functor in the asymptotic category.

Definition. If $F\colon \mathcal{K}_n \to \mathrm{Set}_f$ is a functor as above, we say that the obtained functor in the *asymptotic category* (for which we preserve the notation F) is a finitary weakly normal functor of degree $\le n$.

Let us remark that only normal functors (with additional condition of inverse image preserving) were considered in [6]. But all the arguments are also valid for weakly normal functors, i.e. the functors that do not necessarily satisfy this condition.

A family $\mathcal{A}$ of subsets of a metric space is called *uniformly bounded* if there exists a number $C > 0$ such that $\operatorname{diam} A \le C$ for each $A \in \mathcal{A}$; $\mathcal{A}$ is called r-disjoint for some $r > 0$ if $d(A_1, A_2) \ge r$ for each $A_1,\ A_2 \in \mathcal{A}$ such that $A_1 \ne A_2$.

The asymptotic dimension of a metric space X does not exceed $n \in \mathbb{N} \cup \{0\}$ (written $\operatorname{asdim} X \le n$) if for every $D > 0$ there exists a uniformly bounded cover $\mathcal{U}$ of X such that $\mathcal{U} = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_n$, where all $\mathcal{U}_i$ are D-disjoint [4].

3. Main result. Let F be a finitary weakly normal functor of degree $\le n$. The main result of this note is the following one.

Theorem 1. *If* $\operatorname{asdim} X \le m$, *then* $\operatorname{asdim} F(X) \le mn$.

Proof. Consider a space X with $\operatorname{asdim} X \le m$. By [6] there is a constant N (depending only on the functor F) such that for every $a, b \in F(X)$ the distance $\widehat{d}(a, b)$ is attained by chains of length $\le N$.

Given any $D > 0$ we can choose ND-disjoint uniformly bounded families $\mathcal{U}_0, \dots, \mathcal{U}_{mn}$ such that any $m + 1$ families cover X (see Lemma 36 [3]). It means in particular, that for each $x_1, \dots, x_n \in X$ there exists $i \in \{0, \dots, mn\}$ such that $\{x_1, \dots, x_n\} \subset X_i$ where by $X_i = \bigcup \mathcal{U}_i$. Hence we have $F(X) = \bigcup_{i=0}^{mn} F(X_i)$.

Given a set Y, a family $\mathcal{U}$ of subsets of Y and two maps $f, g\colon X \to Y$, we say that f and g are $\mathcal{U}$-near if for every $x \in X$ there exists a set $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$.

Fix any $s \in \{0, \dots, mn\}$. We say that $a, b \in F(X_s)$ are $\mathcal{U}_s$-*chainable* if there exist elements $c_i \in F(A_i)$, $\operatorname{supp}(c_i) = A_i$, $i \in \{1, \dots, k\}$, and maps $f_{2i-1}, f_{2i} : A_i \to X_s$, $i \in \{1, \dots, k\}$ such that $a = F(f_1)(c_1)$, $F(f_{2i})(c_i) = F(f_{2i+1})(c_{i+1})$, for every $i < k$, $F(f_{2k})(c_k) = b$, and the maps f_{2i-1}, f_{2i} are $\mathcal{U}_s$-near for all $i \le k$.

For any $a \in F(X_s)$ the set $\{b \in F(X_s)\colon\ a \text{ and } b \text{ are } \mathcal{U}_s\text{-chainable}\}$ is called the $\mathcal{U}_s$-*component* of a. Obviously, every two $\mathcal{U}_s$-components are either disjoint or coincide. Denote by $\widehat{\mathcal{U}}_s$ the family of all $\mathcal{U}_s$-components of the points in $F(X_s)$.

We obtain $F(X_s) = \bigcup \widehat{\mathcal{U}}_s$, hence the families $\widehat{\mathcal{U}}_0, \dots, \widehat{\mathcal{U}}_{mn}$ cover $F(X)$. We have to show that each family $\widehat{\mathcal{U}}_s$ is D-disjoint and uniformly bounded.

Consider $a, b \in F(X_s)$ with $\widehat{d}(a, b) < D$. Then there exist $c_i \in F(A_i)$, $\operatorname{supp}(c_i) = A_i$, and $f_{2i-1},\ f_{2i}\colon A_i \to X$, $i \in \{1, \dots, k\}$, such that

$$a = F(f_1)(c_1),\quad F(f_{2i})(c_i) = F(f_{2i+1})(c_{i+1}),\ i \in \{1, \dots, k-1\},\ F(f_{2k})(c_k) = b$$

and $\sum_{i=1}^{k} d(f_{2i-1}, f_{2i}) < D$. We can suppose that $k \le N$.

Given $x \in \operatorname{supp} F(f_{2i})(c_i)$, $i \in \{1, \dots, k-1\}$, we can find $y_{i+1} \in A_{i+1}, \dots, y_k \in A_k$ such that $f_{2i+1}(y_{i+1}) = x$ and $f_{2j}(y_j) = f_{2j+1}(y_{j+1})$ for all $j \in \{i+1, \dots, k-1\}$. Put $r_i(x) = f_{2k}(y_k) \in \operatorname{supp}(b)$. Define functions $\phi_{2i-1}, \phi_{2i}\colon A_i \to X_s$, $i \in \{1, \dots, k\}$ as follows

$$\phi_1 = f_1, \phi_2 = r_1 \circ f_2, \dots, \phi_{2i} = r_i \circ f_{2i}, \phi_{2i+1} = r_i \circ f_{2i+1}, \dots,\quad \phi_{2k-1} = r_{k-1} \circ f_{2k-1}, \phi_{2k} = f_{2k}.$$

Evidently $d(\phi_{2i-1}, \phi_{2i}) < D$. Thus we obtain a chain with values in X_s connecting a and b such that $\sum_{i=1}^{k} d(\phi_{2i-1}, \phi_{2i}) < ND$. Since the family $\mathcal{U}_s$ is ND-disjoint, ϕ_{2i-1} and ϕ_{2i} are $\mathcal{U}_s$-close for each $i \in \{1, \dots, k\}$. Hence a and b belong to the same element of the family $\widehat{\mathcal{U}}_s$. Thus the family $\widehat{\mathcal{U}}_s$ is D-disjoint. The proof of the fact that each family $\widehat{\mathcal{U}}_s$ is uniformly bounded is the same as in [6]. □

Remark. For the functor $F = SP^n_G$ of symmetric power the inequality $\operatorname{asdim} F(X) \le n \operatorname{asdim}(X)$ from Theorem 1 was obtained in [7] using the characterization theorem for the asymptotic dimension in terms of approximation by simplicial complexes.

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