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UNCOUNTABLE ABSORBING SYSTEMS RELATED TO THE HAUSDORFF DIMENSION

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Given any $n \in \mathbb{N}$, we describe the topology of the system $\{HD_{>\gamma}(\mathbb{I}^n)\}_{\gamma\in[0,n)}$ consisting of spaces of compact subsets of Hausdorff dimension $> \gamma$ in the n-dimensional cube $\mathbb{I}^n$.

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Описана топология системы $\{HD_{>\alpha}(\mathbb{I}^n)\}_{\alpha\in[0,n)}$, где $n \in \mathbb{N}$ и $HD_{>\alpha}(\mathbb{I}^n)$ — гиперпространство компактных подмножеств n-мерного куба хаусдорфовой размерности $> \alpha$.

1. Introduction. The problem of topological characterization (identification) of topological objects is one of the classical problems of topology. A classical result of this sort is the Curtis-Schori Theorem [3] asserting that for each non-degenerate Peano continuum X the hyperspace $\exp(X)$ of non-empty compact subsets of X is homeomorphic to the Hilbert cube $Q = [-1, 1]^\omega$. Later, D. Curtis [4] characterized topological spaces X whose hyperspace $\exp(X)$ is homeomorphic to the pseudointerior $s = (-1, 1)^\omega$ of the Hilbert cube Q.
In [7] T. Dobrowolski and L. Rubin indentified the topology of the subspace $D_{\le n}(Q) \subset \exp(Q)$ consisting of compact subsets of Q having covering (or cohomological) dimension $\le n$. They constructed a homeomorphism $h\colon \exp(Q) \to Q^\omega$ such that $h(D_{\le n}(Q)) = Q^n \times s^{\omega\setminus n}$ for all $n = \{0, \dots, n-1\} \in \omega$. In this case we say that the system $\{D_{\le n}(Q)\}_{n\in\omega}$ is homeomorphic to the system $\{Q^n \times s^{\omega\setminus n}\}_{n\in\omega}$. This results was later generalized by H. Gladdines ([10]) to products of Peano continua. Finally, R. Cauty ([5]) has characterized spaces X for which the system $\{D_{\le n}(X)\}_{n\in\omega}$ in $\exp(X)$ is homeomorphic to $\{Q^n \times s^{\omega\setminus n}\}_{n\in\omega}$ in Q^ω as Peano continua whose any non-empty open subset contains compact subsets of an arbitrary finite dimension.

In [11], [14] the author initiated the study of the subspace $HD_{\le\gamma}(\mathbb{I}^n)$ of the hyperspace $\exp(\mathbb{I}^n)$ of a finite-dimensional cube $\mathbb{I}^n = [0, 1]^n$, consisting of all compact subsets of $\mathbb{I}^n$ with Hausdorff dimension $\le \gamma$. Unlike the (integer-valued) topological dimension, the Hausdorff dimension of a metric compactum can take on any non-negative real value γ. So, the system $\{HD_{\le\gamma}(\mathbb{I}^n)\}_{0\le\gamma<n}$ that naturally appears in this situation is uncountable. In [14] it was proved that for each increasing sequence $\{\gamma_k\}_{k\in\omega} \subset [0, n)$ of real numbers the system $\{HD_{\le\gamma_k}(\mathbb{I}^n)\}_{k\in\omega}$ in $\exp(\mathbb{I}^n)$ is homeomorphic to $\{Q^k \times s^{\omega\setminus k}\}_{k\in\omega}$ in Q^ω.

In this paper we shall recognize the topological structure of the whole uncountable system $\{HD_{\le\gamma}(\mathbb{I}^n)\}_{0\le\gamma<n}$. Observe that each real number γ determines a partition $\mathbb{Q} = \mathbb{Q}_{\le\gamma} \cup \mathbb{Q}_{>\gamma}$

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of the set $\mathbb{Q}$ of rational numbers into two rays $\mathbb{Q}_{\geq\gamma} = \{q \in \mathbb{Q}\colon q \geq \gamma\}$ and $\mathbb{Q}_{<\gamma} = \{q \in \mathbb{Q}\colon q < \gamma\}$.

Theorem 1. *For every $n \in \mathbb{N}$ the system $\{HD_{\leq\gamma}(I^n)\}_{0\leq\gamma<n}$ in the hyperspace $\exp(\mathbb{I}^n)$ is homeomorphic to the system $\{Q^{\mathbb{Q}_{\leq\gamma}} \times s^{\mathbb{Q}_{>\gamma}}\}_{0\leq\gamma<n}$ in the Hilbert cube $Q^{\mathbb{Q}}$.*

This theorem will be proved in Section 5. In Section 2, the basic definitions and the facts of the theory of absorbing sets in the Hilbert cube are considered. Section 3 is devoted to the construction of a model uncountable linearly ordered absorbing system in the Hilbert cube. In Section 4, the system of hyperspaces of compacta of given Hausdorff dimension in the finite-dimensional cube is considered in the case when the values of the dimension run over the set $[0, n)$. The main result consists of a description of the topology of this system. (Theorem 3, Section 5).

2. Preliminaries. A typical metric will be denoted by d. By $\mathrm{diam}(A)$ we denote the diameter of a subset A in a metric space. Given a cover $\mathcal{U}$ of a metric space, we define $\mathrm{mesh}(\mathcal{U})$ as $\sup\{\mathrm{diam}(U) \mid U \in \mathcal{U}\}$. For $x \in X$ and $\varepsilon > 0$ the set $O_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ is the *open ε-ball* centered at x. Further, all spaces are separable metrizable, all maps are continuous.

As usual, by $\mathbb{I}$ we denote the interval $[0, 1]$, by $\mathbb{I}^k$ the k-dimensional cube and by $\partial\mathbb{I}^k$ its geometrical boundary. Further, by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ we denote the spaces of natural, rational and real numbers respectively. We recall that $Q = [-1, 1]^\omega$ is the Hilbert cube, $s = (-1, 1)^\omega$ is its *pseudointerior* and $B(Q) = Q\backslash s$ stands for its *pseudoboundary.*

The class of absolute neighborhood retracts is denoted by ANR. A closed subset A of $X \in ANR$ is called a *Z-set* in X if for every continuous function $\varepsilon\colon X \to (0, \infty)$ there exists a map $f\colon X \to X\backslash A$ which is ε-close to the identity in the sense that $d(x, f(x)) < \varepsilon(x)$ for every $x \in X$. A subset of X is called a *σZ-set in X* if A can be written as the countable union of Z-sets in X. An embedding $g\colon Y \to X$ is called a *Z-embedding* if its image $g(Y)$ is a Z-set in X.

2.1. Hyperspaces. Let X be a metric space. The *hyperspace* of X is the space $\exp X$ of nonempty compact subsets of X endowed with the Vietoris topology. A base of this topology consists of the sets

$$\langle V_1, ..., V_n\rangle = \Big\{A \in \exp X \mid A \subset \bigcup_{i=1}^{n} V_i \text{ and for every } i \in \{1, 2, ..., n\}\ A \cap V_i \neq \varnothing\Big\},$$

where $V_1, ..., V_n$ run over the topology of X. If the topology of X is generated by a metric d then the Vietoris topology on $\exp(X)$ is generated by the Hausdorff metric $d_H(A, B) = \inf\{\varepsilon > 0 | A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}$.

For $n \in \mathbb{N}$ by $\exp_n X$ we denote the subspace of $\exp X$ consisting of the sets of cardinality $\leq n$. Let $\exp_\omega X = \bigcup\{\exp_n X \mid n \in \mathbb{N}\}$.

2.2. Hausdorff dimension. Given a compact metric space X and two non-negative real numbers s, ε, consider the number

$$\mathcal{H}^s_\varepsilon(X) = \inf_{\mathcal{B}} \sum_{B\in\mathcal{B}} (\mathrm{diam} B)^s,$$

where the infimum is taken over all covers $\mathcal{B}$ of X with $\mathrm{mesh}(\mathcal{B}) < \varepsilon$.

The limit $\mathcal{H}^s(X) = \lim\limits_{\varepsilon\to 0} \mathcal{H}^s_\varepsilon(X)$ is called the s-dimensional Hausforff measure of X. It is knowns that there is a unique finite or infinite number $\dim_H(X)$ called the *Hausdorff dimension* of X and denoted by $\dim_H(X)$ such that $\mathcal{H}^s(X) = \infty$ for all $s < \dim_H(X)$ and $\mathcal{H}^s(X) = 0$ for all $s > \dim_H(X)$, see [9], [8].

Proposition 1 (see [11]). *Let X be a complete separable metric space. For every $\alpha \geq 0$ the set $HD_{\leq\alpha}(X) = \{A \in \exp(X) \mid \dim_H(A) \leq \alpha\}$ is a G_δ-subset of* $\exp(X)$.

2.3. Absorbing systems. We briefly recall some definitions of the theory of absorbing systems; see [13], [10], [1], [2] for details.

Let Γ be an ordered set and let $\mathcal{M}_\gamma$ be a class of metric spaces for $\gamma \in \Gamma$. Put $\mathcal{M}_\Gamma = \{M_\gamma\}_{\gamma\in\Gamma}$. An $\mathcal{M}_\Gamma$-*system* in the space X is an order preserving indexed collection $\{\mathcal{A}_\gamma\}_{\gamma\in\Gamma}$ of subset of X such that $\mathcal{A}_\gamma \in \mathcal{M}_\gamma$ for every γ.

An $\mathcal{M}_\Gamma$-system $\mathfrak{X} = \{\mathcal{X}_\gamma\}_{\gamma\in\Gamma}$ in $X \in ANR$ is called *strongly* $\mathcal{M}_\Gamma$-*universal* in X if for every $\mathcal{M}_\Gamma$-system $\{\mathcal{A}_\gamma\}_{\gamma\in\Gamma}$ in Q every map $f\colon Q \to X$ that restricts to a Z-embedding on some compact subset K of Q can be approximated by a Z-embedding $g\colon Q \to X$ such that $g|K = f|K$ and for every $\gamma \in \Gamma$ we have $g^{-1}(\mathcal{X}_\gamma)\backslash K = \mathcal{A}_\gamma\backslash K$.

An $\mathcal{M}_\Gamma$-system $\mathfrak{X}$ is called $\mathcal{M}_\Gamma$-*absorbing* in X if the set $\bigcup_{\gamma\in\Gamma}\mathcal{X}_\gamma$ is contained in a σ-compact σZ-set of X and $\mathfrak{X}$ is strongly $\mathcal{M}_\Gamma$-universal in X.

By $\mathcal{F}_\sigma$ we denote the class of σ-compact spaces.

We shall now consider a special case when the system $\mathfrak{X}$ is a decreasing system of absorbers (so Γ is linearly ordered by the relation $\geq$) and we assume that all the classes $\mathcal{M}_\gamma$ are equal to the class $\mathcal{F}_\sigma$. In this situation, we shall use the term $\mathcal{F}_\sigma$-absorbing system.

3. Uncountable absorbing systems in the Hilbert cube. In this section we consider the special case of decreasing $\mathcal{F}_\sigma$-absorbing systems in the Hilbert cube, namely systems $\{\mathcal{X}_\gamma\}_{\gamma\in\Gamma}$ indexed by a linearly ordered set Γ and catisfying the condition:

($*$) for every $\gamma \in \Gamma$, $\mathcal{X}_\gamma = \bigcup\limits_{\gamma'>\gamma,\gamma'\in\Gamma} \mathcal{X}_{\gamma'}$.

Observe that for any countable set A the pseudoboundary $B(Q^A) = Q^A \setminus s^A$ can be written as the union

$$B(Q^A) = \bigcup_{\alpha\in A}\Big(\prod_{\alpha'\neq\alpha,\alpha'\in A} Q_{\alpha'} \times B(Q)_\alpha\Big).$$

For every real number γ consider the subset

$$\mathcal{X}_\gamma = Q^{\mathbb{Q}_{\leq\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}}),$$

in the Hilbert cube $Q^{\mathbb{Q}}$. Here $\mathbb{Q}_{\leq\gamma} = \{q \in \mathbb{Q}\colon q \leq \gamma\}$ and $\mathbb{Q}_{>\gamma} = \{q \in \mathbb{Q}\colon q > \gamma\}$.

For every $n \in \mathbb{N}$ the $\mathcal{F}_\sigma$-system $\{\mathcal{X}_\gamma\}_{\gamma\in[0,n)}$ in $Q^{\mathbb{Q}}$ defined in this way is decreasing uncountable and ordered by the order induced from $\mathbb{R}$ and satisfies condition ($*$).

Proposition 2. *For every $n \in \mathbb{N}$ the system $\{\mathcal{X}_\gamma\}_{\gamma\in[0,n)}$ is $\mathcal{F}_\sigma$-absorbing in $Q^{\mathbb{Q}}$.*

Proof. Since $\bigcup_{\gamma\in[0,n)} X_\gamma$ is a σZ-set in $Q^{\mathbb{Q}}$, it suffices to check that the system $\{\mathcal{X}_\gamma\}_{\gamma\in[0,n)}$ is strongly $\mathcal{F}_\sigma$-universal in $Q^{\mathbb{Q}}$.

Let $\delta\colon \mathbb{Q} \to \mathbb{N}$ be a bijection. For every $\gamma \in \mathbb{Q}$ let ρ_γ be an admissible metric on Q, bounded by the number $2^{-\delta(\gamma)}$. Define, for any $x, y \in Q^{\mathbb{Q}}$,

$$d(x, y) = \sup\{\rho_\gamma(x_\gamma, y_\gamma)|\gamma \in \mathbb{Q}\}.$$

It is easy to verify that the function $d\colon Q^{\mathbb{Q}} \times Q^{\mathbb{Q}} \to [0, \infty)$ defined in this way is a metric. Observe that d is continuous, since we consider $Q^{\mathbb{Q}}$ endowed with the product topology and the functions ρ_γ are continuous on $Q_{\mathbb{Q}} \times Q_{\mathbb{Q}}$. Then the metric d is an admissible metric on $Q^{\mathbb{Q}}$.

Consider a map $f\colon Q \to Q^{\mathbb{Q}}$ that restricts to a Z-embedding on some compact set $K \subseteq Q$ and a decreasing system $\{\mathcal{A}_\gamma\}_{\gamma\in[0,n)}$ of σ-compact subsets of Q for which the condition

(0) $\mathcal{A}_\gamma = \bigcup_{\gamma' \in (\gamma,n)} \mathcal{A}_{\gamma'}$ for every$\gamma \in [0,n)$

holds. We may assume that f is a Z-embedding. Write $Q\backslash K$ as a union of sequence $\{F_i\}_{i=1}^{\infty}$ of compacta with $F_i \subseteq \operatorname{int}(F_{i+1})$ for every i and $F_0 = \varnothing$. Let $\varepsilon > 0$ and put $\varepsilon_i = \min\{\frac{\varepsilon}{2^i}, \frac{1}{2}d(f[K], f[F_i])\}$ for every i. For any $\gamma \in \mathbb{Q}$, consider now the corresponding component $f_\gamma\colon Q \to Q$ of f. We shall construct a sequence $\alpha_i\colon Q \to Q$, $i = 0, 1, 2, \dots$ of continuous functions such that for every i:

(1) $\hat{\rho}_\gamma(\alpha_i, \alpha_{i-1}) < \varepsilon_i, \alpha_i \mid F_{i-1} = \alpha_{i-1} \mid F_{i-1}$;

(2) $\alpha_i \mid Q\backslash F_{i+1} = f_\gamma \mid Q\backslash F_{i+1}$ and $\alpha_i \mid F_i$ is a Z-embedding;

(3) if $\gamma \in [0,n)$, then $\alpha_i^{-1}[B(Q)] = \mathcal{A}_\gamma \cap F_i$.

Put $\alpha_0 = f_\gamma$ and assume that α_i has been constructed. Using the strong universality of Q, we can find a Z-embedding $\beta\colon\ F_{i+1} \to Q$, close to $\alpha_i \mid F_{i+1}$, with $\beta \mid F_i = \alpha_i \mid F_i$. If $\gamma \in [0,n)$, then we additionally assume that $\beta^{-1}[B(Q)] = \mathcal{A}_\gamma \cap F_{i+1}$ (because of the $\mathcal{F}_\sigma$-absorbing property of $B(Q)$ in Q). Using the fact that $Q \in AR$, extend β to a map $\alpha_{i+1}\colon Q \to Q$ such that $\alpha_{i+1} \mid \overline{Q\backslash F_{i+2}} = f_\gamma \mid \overline{Q\backslash F_{i+2}}$ and α_{i+1} is sufficiently close to α_i.

The sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ is obviously a Cauchy sequence and therefore the function $g_\gamma = \lim_{i\to\infty} \alpha_i$ is continuous. It is easy to verify that g_γ has the following properties:

(4) $\hat{\rho}_\gamma(g_\gamma, f_\gamma) < \varepsilon$;

(5) if $x \in F_{i+1}\backslash F_i$ then $\rho_\gamma(g_\gamma(x), f_\gamma(x)) < d(f[K], f[F_{i+1}])$;

(6) $g_\gamma \mid K = f_\gamma \mid K$, $g_\gamma \mid F_i$ is a Z-embedding for every i;

(7) $g_\gamma^{-1}[B(Q)]\backslash K = \mathcal{A}_\gamma\backslash K$ if $\gamma \in [0,n)$.

Define $g = (g_\gamma)_{\gamma\in\mathbb{Q}}\colon Q \to Q^{\mathbb{Q}}$. Note that g is one-to-one, and hence is an embedding. The set $g[Q]$ is contained in the σZ-set

$$f[K] \cup \bigcup_{i=0}^{\infty} \Big(\prod_{\gamma\neq\gamma'} Q_\gamma \times g_{\gamma'}[F_i] \Big)$$

and is therefore a Z-set. The maps f and g are ε-close and $f \mid K = g \mid K$.

Let $x \in Q\backslash K$. If $x \in \mathcal{A}_\gamma$ for some $\gamma \in [0,n)$ then by construction and by condition (0), $g_{\gamma'}(x) \in B(Q)$ for all rational $\gamma' \in [0, \gamma + \delta)$ for some $\delta > 0$, that is, $(g(x))_{\gamma'} \in B(Q)$ for some $\gamma' > \gamma$ and therefore, $g(x) \in \mathcal{X}_\gamma$. On the other hand, if $g(x) \in \mathcal{X}_\gamma$ for some $\gamma \in [0,n)$ then $g_{\gamma'}(x) \in B(Q)$ for some $\gamma' \in (\gamma, n)$ and therefore, $x \in \mathcal{A}_{\gamma'} \subset \mathcal{A}_\gamma$.

Let now $x \in \mathcal{A}_\gamma$ for some irrational $\gamma \in [0,n)$. Taking into account condition (0) and the fact that the system $\{\mathcal{A}_\gamma\}_{\gamma\in[0,n)}$ is decreasing we may assert that there exists a rational γ', $\gamma < \gamma' < n$ for which $x \in \mathcal{A}_{\gamma''}$ for all rational $\gamma'' \leq \gamma'$. We see that $g_{\gamma'}(x) \in B(Q)$ and it follows that $g(x) \in \mathcal{X}_{\gamma'} \subset \mathcal{X}_\gamma$. On the other hand, if $g(x) \in \mathcal{X}_\gamma$ for some irrational $\gamma \in [0,n)$ then $g_{\gamma'}(x) \in B(Q)$ for some rational $\gamma' > \gamma$, $\gamma' \in \Gamma$ and therefore $x \in \mathcal{A}_{\gamma'} \subset \mathcal{A}_\gamma$. □

4. Main result. Let $n \in \mathbb{N}$ and $\gamma \in [0,n)$. We recall that

$$HD_{>\gamma}(\mathbb{I}^n) = \{A \in \exp(\mathbb{I}^n) \mid \dim_H(A) > \gamma\}$$

stands for the set of all non-empty compacta in $\mathbb{I}^n$ with the Hausdorff dimension $> \gamma$. It is clear that $HD_{>\gamma}(\mathbb{I}^n) = \bigcup_{\alpha>\gamma} HD_{>\alpha}(\mathbb{I}^n)$. By Proposition 1, $HD_{>\gamma}(\mathbb{I}^n)$ is an F_σ-set in $\exp(\mathbb{I}^n)$. Thus, we obtain a decreasing uncountable $\mathcal{F}_\sigma$-system of hyperspaces related to the Hausdorff dimension $\{HD_{>\gamma}(\mathbb{I}^n)\}_{\gamma\in[0,n)}$ in the hyperspace $\exp(\mathbb{I}^n)$ of finite-dimensional cube $\mathbb{I}^n$.

Theorem 2. *For every $n \in \mathbb{N}$ the system $\{HD_{>\gamma}(\mathbb{I}^n)\}_{\gamma\in[0,n)}$ is $\mathcal{F}_\sigma$-absorbing in* $\exp(\mathbb{I}^n)$.

By using basic facts from the theory of absorbing systems in the Hilbert cube, for the proof of this theorem we need to verify the following conditions:

(I) the set $HD_{>\gamma}(\mathbb{I}^n)$ is an $\mathcal{F}_\sigma$-set in $\exp(\mathbb{I}^n)$ for every $\gamma \in [0, n)$;

(II) the set $\bigcup\limits_{\gamma\in[0,n)} HD_{>\gamma}(\mathbb{I}^n)$ is contained in some σ-compact σZ-subset of $\exp(\mathbb{I}^n)$;

(III) the system $\{HD_{>\gamma}(\mathbb{I}^n)\}_{\gamma\in[0,n)}$ is strongly $\mathcal{F}_\sigma$-universal in $\exp(\mathbb{I}^n)$.

Proof of Theorem 2. Let us verify conditions (I)-(III).

Condition (I), as mentioned above, follows from Proposition 1.

Condition (II). It is enough to verify that $HD_{>0}(\mathbb{I}^n)$ is a σZ-set in $\exp(\mathbb{I}^n)$. By Condition (I) we conclude that $HD_{>0}(\mathbb{I}^n)$ is an $\mathcal{F}_\sigma$-set in $\exp(\mathbb{I}^n)$. By using properties of the Hausdorff dimension (see [9], [8]) we obtain the inclusion $HD_{>0}(\mathbb{I}^n) \subseteq \exp(\mathbb{I}^n)\setminus \exp_\omega(\mathbb{I}^n)$. The set $\exp(\mathbb{I}^n)\setminus \exp_\omega(\mathbb{I}^n)$ is homotopy negligible in $\exp(\mathbb{I}^n)$ (see [1]) therefore we can conclude that $HD_{>0}(\mathbb{I}^n)$ is a σZ-set in $\exp(\mathbb{I}^n)$.

Now prove Condition (III). We shall use the vector addition and scalar multiplication operations that $\mathbb{I}^n$ inherits from $\mathbb{R}^n$.

Consider the sequence of compact subsets $\{B_i\}_{i=1}^\infty$ in $\mathbb{I}^n$ defined as follows:

$$\begin{aligned}
&B_1 = \tfrac{1}{2}\cdot \mathbb{I}^n ,\\
&B_2 = \tfrac{1}{2^2}\cdot \mathbb{I}^n + \tfrac{1}{2}\cdot y_0 ,\\
&\dots\\
&B_k = \tfrac{1}{2^k}\cdot \mathbb{I}^n + \left(\tfrac{1}{2} + \dots + \tfrac{1}{2^{k-1}}\right)\cdot y_0 ,\\
&\dots ,
\end{aligned}$$

where $y_0 = (1, 1, \dots, 1)$. It is easy to see that the sets B_i are reduced copies of cube $\mathbb{I}^n$ which are located on its diagonal and the sequence $\{B_i\}_{i=1}^\infty$ converges to the vertex $y_0 = (1, 1, \dots, 1)$.

Let α_i be some embedding of the segment $[-1, 1]$ in B_i. For every $x \in Q$, $x = (x_i)_{i=1}^\infty$ define an element $\hat{x} \in Q$ as

$$\hat{x} = (\underbrace{x_1}, \underbrace{x_1, x_2}, \underbrace{x_1, x_2, x_3}, \underbrace{x_1, x_2, x_3, x_4}, \dots).$$

Define the map ξ by the formula

$$\xi(x) = \bigcup_{i=1}^{\infty} \alpha_i(\hat{x}_i) \cup \{y_0\}.$$

Continuity of ξ follows from the continuity of the used maps.

Obviously, for every $x \in Q$, $\xi(x)$ is a compact subset in $\mathbb{I}^n$. On the other hand, $\xi(x)$ is a countable subset in $\mathbb{I}^n$ therefore, from properties of Hausdorff dimension (see [9],[8]), $\dim_H(\xi(x)) = 0$.

Choose any two points $x, x' \in Q$, $x = (x_i)_{i=1}^\infty$, $x' = (x'_i)_{i=1}^\infty$. If $x \neq x'$ then there exists $i \in \mathbb{N}$ such that $x_i \neq x'_i$. The way we have constructed the point $\hat{x}$ guarantees that there is $j \in \mathbb{N}$ such that $\alpha_j(\hat{x}_j) \neq \alpha_j(\hat{x}'_j)$. Consequently, $\xi(x) \neq \xi(x')$. This implies that ξ is an embedding.

Let $\varepsilon > 0$ and $f\colon Q \to \exp(\mathbb{I}^n)$ be maps that restricts to a Z-embedding on some compact subset K of Q. Without loss of generality we may assume that f is a Z-embedding because $\exp(\mathbb{I}^n)$ is homeomorphic to the Hilbert cube Q. Define a map $\mu\colon Q \to [0, 1]$ by the formula

$$\mu(x) = \frac{1}{3}\cdot \min\{\varepsilon, d_H(f(x), f[K])\}.$$

Since, as mentioned above, the set $\exp(\mathbb{I}^n)\setminus\exp_\omega(\mathbb{I}^n)$ is homotopy negligible in $\exp(\mathbb{I}^n)$, there exists a homotopy $H\colon \exp(\mathbb{I}^n)\times\mathbb{I}\to\exp(\mathbb{I}^n)$ such that:

(1) $H_0 = 1_{\exp(\mathbb{I}^n)}$;

(2) for every $t\in(0,1]$, $H_t(\exp(\mathbb{I}^n))\subseteq\exp_\omega(\mathbb{I}^n)$.

It is clear that we may additionally assume that (for details see [13])

(3) for every $t\in[0,1]$, $\hat{d}_H(H_t, 1_{\exp(\mathbb{I}^n)})\le 2t$;

(4) for every $t\in(0,1]$, $H_t(\exp(\mathbb{I}^n))\subseteq\exp_\omega([0,1-3t/4]^n)$.

For every $x\in Q$ let $F(x)=H(f(x),\mu(x))$. Hence if $\mu(x)>0$ then $F(x)$ is a finite approximation of the set $f(x)$.

Let $\beta_i\colon\mathbb{I}^n\to B_i$ be similitude homeomorphisms. For some $\lambda\in[0,1]$ and $y\in\mathbb{I}^n$, define the map $(\beta_i)^\lambda_y=\lambda\beta_i+y+\lambda y_0$, where $y_0=(1,1,...,1)$. Observe that for $\lambda\in(0,1]$ the map $(\beta_i)^\lambda_y$ is a similitude homeomorphism and for $\lambda=0$, $(\beta_i)^0_y$ is a constant map.

Let $\Gamma=[0,n)\cap\mathbb{Q}$, $\alpha\colon\Gamma\times\mathbb{N}\to\mathbb{N}$ be a bijection. Denote $N_\gamma=\{\alpha(\gamma,j)\mid j\in\mathbb{N}\}$ for every $\gamma\in\Gamma$. Then $\alpha(\gamma,p)$ is pth element in N_γ. Choose a decreasing system of σ-compact subsets $\{\mathcal{A}_\gamma\}_{\gamma\in[0,n)}$ in Q such that for every $\gamma\in[0,n)$ the following holds: $\mathcal{A}_\gamma=\bigcup\limits_{\gamma'\in(\gamma,n)}\mathcal{A}_{\gamma'}$. For every $\gamma\in\Gamma$ write $\mathcal{A}_\gamma=\bigcup\limits_{p=1}^{\infty}A^p_\gamma$, where $A^1_\gamma\subseteq A^2_\gamma\subseteq\ldots$ and A^p_γ is a compact subset in Q.

For every $\gamma\in(0,n]$ there exists a set $C\in\exp(\mathbb{I}^n)$ such that $\dim_H(C)=\gamma$. For $\gamma\in\Gamma$ and $p\in\mathbb{N}$ let a set $C_{\alpha(\gamma,p)}\in\exp(\mathbb{I}^n)$ be such that $\dim_H(C_{\alpha(\gamma,p)})=\gamma$. For an arbitrary natural i define the map $\varphi_i\colon\mathbb{I}\to\exp(\mathbb{I}^n)$ by the formula $\varphi_i(t)=H(C_i,t)$. Then for the map φ_i the following holds: $\varphi_i(0)=C_i$ and $\varphi_i((0,1])\subseteq\exp_\omega(\mathbb{I}^n)$.

Now define the map $g\colon Q\to\exp(\mathbb{I}^n)$ by the formula

$$g(x)=F(x)\cup\bigcup_{y\in F(x)}[(\mu(x)/4\cdot\xi(x)+y)\cup$$

$$\cup\bigcup_{\gamma\in\Gamma}\bigcup_{p=1}^{\infty}(\beta_{\alpha(\gamma,p)})^{\mu(x)/4}_y(\varphi_{\alpha(\gamma,p)}(d(x,A^p_\gamma)))\cup\{y+\mu(x)/2\cdot y_0\}\cup$$

$$\cup(\mu(x)/4\cdot\xi(x)+y+\mu(x)/2\cdot y_0)].$$

We claim that g is as required, i. e., g is an approximation of f with the properties stated in the definition of strong $\mathcal{F}_\sigma$-universality.

CLAIM 1: The map g is well-defined, is continuous and satisfies $g\mid K=f\mid K$. Moreover, for every $x\in Q$,

$$d_H(f(x),g(x))\le\frac{11}{12}\min\{\varepsilon,d(f(x),f[K])\}.$$

(a) Let $x\in Q$. Then by (4), $F(x)\subseteq[0,1-3\mu(x)/4]^n$. For every $y\in F(x)$, the set $\mu(x)/4\cdot\xi(x)+y$ is contained in the cube $[0,\mu(x)/4]^n+y$; the set

$$\bigcup_{\gamma\in\Gamma}\bigcup_{p=1}^{\infty}(\beta_{\alpha(\gamma,p)})^{\mu(x)/4}_y(\varphi_{\alpha(\gamma,p)}(d(x,A^p_\gamma)))\cup\{y+\mu(x)/2\cdot y_0\}$$

is contained in the cube $[0,\mu(x)/4]^n+y+\mu(x)/4\cdot y_0$; and the set $\mu(x)/4\cdot\xi(x)+y+\mu(x)/2\cdot y_0$ in the cube $[0,\mu(x)/4]^n+y+\mu(x)/2\cdot y_0$. Therefore, the set attached to y is contained in the cube $[0,3\mu(x)/4]^n+y$, that implies $g(x)\subseteq\mathbb{I}^n$.

(b) If $\mu(x) > 0$ then $g(x)$ is compact and non-empty, being a finite union of compact non-empty sets. Namely, with every $y \in F(x)$ we unite the sets $\mu(x)/4 \cdot \xi(x) + y$ and $\mu(x)/4 \cdot \xi(x) + y + \mu(x)/2 \cdot y_0$ that are compact and non-empty by the construction of ξ; and the set

$$\bigcup_{\gamma\in\Gamma}\bigcup_{p=1}^{\infty}(\beta_{\alpha(\gamma,p)})_y^{\mu(x)/4}(\varphi_{\alpha(\gamma,p)}(d(x, A_\gamma^p))) \cup \{y + \mu(x)/2 \cdot y_0\},$$

which is the union of infinitely many compacta (by the construction of the maps φ_i and β_i) that converge to the point $y + \mu(x)/2 \cdot y_0$.

If $\mu(x) = 0$ then $g(x) = f(x)$ which is also compact and non-empty. Thus, $g(x) \in \exp(\mathbb{I}^n)$ for every $x \in Q$.

(c) Continuity of g follows from the continuity of used maps.

(d) Fix an arbitrary $x \in Q$. By (3) we have

$$d_H(f(x), g(x)) \leq 2 \cdot \mu(x) + \mu(x)/4 + \mu(x)/4 + \mu(x)/4 = 11 \cdot \mu(x)/4,$$

therefore,

$$d_H(f(x), g(x)) \leq 11/12 \cdot \min\{\varepsilon, d_H(f(x), f[K])\}.$$

Last inequality implies that $g \mid K = f \mid K$.

CLAIM 2: The map g is injective.

Let us first observe that from Claim 1 and the fact that f is an embedding it follows that

$$g[Q\backslash K] \cap g[K] = \varnothing. \tag{$*$}$$

Now fix arbitrary $x, x' \in Q$. If both x and x' belong to K, then since $g \mid K = f \mid K$ and since f is an embedding, it is trivial that the equality $g(x) = g(x')$ implies the equality $x = x'$. If $x \notin K$ and $x' \in K$, then from $(*)$ it follows that $g(x) \neq g(x')$. Therefore, without loss of generality we may assume that $x, x' \in Q\backslash K$.

Let $g(x) = g(x')$. Our aim is to show that $x = x'$. We will first prove that $\mu(x) = \mu(x')$. Assume the contrary, that is, $\mu(x) < \mu(x')$. For certain $y \in F(x)$ consider in $\mathbb{I}^n$ the set $B_y = (\mu(x)/4) \cdot \mathbb{I}^n + y$. There exists a point $m \in g(x)$ such that $|m| \leq |p|$ for all $p \in g(x)$ (here $|\cdot|$ stands for the distance from origin in $\mathbb{I}^n$). Moreover, this point m is an element of $F(x) \cap F(x')$ (taking into account the construction of g and the equality $g(x) = g(x')$). For this point m, it is easy to see that the set $B_m \cap g(x)$ contains a copy of $\xi(x)$ and therefore is infinite, at that time when $B_m \cap g(x')$ is finite as finite union of finite sets. This contradiction establishes that $\mu(x) = \mu(x')$.

Now we consider the point $\hat{m} = (m_1, \ldots, m_n) \in g(x)$ such that $|p| \leq |\hat{m}|$ for all $p \in g(x)$. Since $\mu(x) = \mu(x')$, we have

$$m^* = (m_1 - 3\mu(x)/4, \ldots, m_n - 3\mu(x)/4) \in F(x) \cap F(x').$$

Since $F(x)$ and $F(x')$ are finite, $\hat{m}$ is maximal, and we have that there are a neighborhood U of $\hat{m}$ and $\delta \in (0, 1]$ such that

$$\begin{aligned} U \cap g(x) &= m^* + \mu(x)/2 \cdot y_0 + \mu(x)/4(\xi(x) \cap O_\delta(y_0)) = \\ &= m^* + \mu(x')/2 \cdot y_0 + \mu(x')/4(\xi(x') \cap O_\delta(y_0)). \end{aligned}$$

Since the coordinates of x appear infinitely often in the coordinates of $\hat{x}$ (at pregiven places), and the same is true for x', it now easily follows that $x = x'$.

CLAIM 3: For every $\gamma \in [0, n)$ we have

$$g^{-1}[HD_{>\gamma}(\mathbb{I}^n)]\backslash K = \mathcal{A}_\gamma\backslash K.$$

First, observe that if $\mu(x) > 0$ then

$$F(x) \cup \bigcup_{y \in F(x)} (\mu(x)/4 \cdot \xi(x) + y)$$

is a finite union of countable sets. Therefore, for every $x \in Q\backslash K$ the Hausdorff dimension of this set is equal to zero.

Choose $x \in Q\backslash K$. If $x \notin \mathcal{A}_\gamma$ for every $\gamma \in [0, n)$ then by construction $g(x)$ is a countable set and therefore, $\dim_H(g(x)) = 0$.

Let for some $\gamma \in [0, n)$, $x \in \mathcal{A}_\gamma$. Since the system $\{\mathcal{A}_\gamma\}_{\gamma \in [0,n)}$ is decreasing, it is obvious that $x \in \mathcal{A}_{\gamma'}$ for all $\gamma' < \gamma$. Since

$$\mathcal{A}_\gamma = \bigcup_{\gamma' \in (\gamma, n)} \mathcal{A}_{\gamma'},$$

there exists $\delta > 0$ such that $x \in \mathcal{A}_{\gamma'}$ for all $\gamma' \in [\gamma, \gamma + \delta)$. For this x, we put $O(x) = [0, \gamma + \delta)$, the set of all γ' that $x \in \mathcal{A}_{\gamma'}$. For every $\gamma' \in O(x) \cap \mathbb{Q}$ the set

$$\bigcup_{p=1}^{\infty} (\beta_{\alpha(\gamma',p)})_y^{\mu(x)/4} (\varphi_{\alpha(\gamma',p)}(d(x, A^p_{\gamma'})))$$

is a finite union of finite sets and a countable union of sets of Hausdorff dimension $= \gamma'$. In this case, by properties of the Hausdorff dimension (see [9], [8]), the Hausdorff dimension of the set

$$\bigcup_{\gamma' \in O(x) \cap \mathbb{Q}} \bigcup_{p=1}^{\infty} (\beta_{\alpha(\gamma',p)})_y^{\mu(x)/4} (\varphi_{\alpha(\gamma',p)}(d(x, A^p_{\gamma'})))$$

is equal to $\sup\{\gamma' \mid \gamma' \in O(x) \cap \mathbb{Q}\} = \gamma + \delta$. On the other hand, for every $\gamma' \in \Gamma\backslash O(x)$, $x \notin \mathcal{A}_{\gamma'}$, therefore, the set

$$\bigcup_{\gamma' \in \Gamma\backslash O(x)} \bigcup_{p=1}^{\infty} (\beta_{\alpha(\gamma',p)})_y^{\mu(x)/4} (\varphi_{\alpha(\gamma',p)}(d(x, A^p_{\gamma'})))$$

is countable and its Hausdorff dimension is equal to zero. Therefore, if $x \in \mathcal{A}_\gamma$, then $g(x) \in HD_{>\gamma}(\mathbb{I}^n)$.

On the other hand, if $x \notin \mathcal{A}_\gamma$ for some $\gamma \in [0, n)$ then $x \notin \mathcal{A}_{\gamma'}$ for all $\gamma' \geq \gamma$, that is, the set

$$\bigcup_{\gamma' \geq \gamma, \gamma' \in \Gamma} \bigcup_{p=1}^{\infty} (\beta_{\alpha(\gamma',p)})_y^{\mu(x)/4} (\varphi_{\alpha(\gamma',p)}(d(x, A^p_{\gamma'})))$$

is countable and its Hausdorff dimension is equal to zero at that time when the set

$$\bigcup_{\gamma' < \gamma, \gamma' \in \Gamma} \bigcup_{p=1}^{\infty} (\beta_{\alpha(\gamma',p)})_y^{\mu(x)/4} (\varphi_{\alpha(\gamma',p)}(d(x, A^p_{\gamma'})))$$

by similar to the mentioned above, is of Hausdorff dimension $\leq \gamma$. This implies that $g(x) \notin HD_{>\gamma}(\mathbb{I}^n)$.

Equality $(*)$ finishes the proof of CLAIM 3.

CLAIM 4: The map g is a Z-embedding.

Since $g[K] = f[K]$ is a Z-set, it suffices to show that $g[Y]$ is a Z-set if $Y \subseteq Q\backslash K$ is compact. But this is clear because a map $g' \colon Q \to \exp(\mathbb{I}^n)$, defined as

$$g'(x) = \overline{O_\delta(g(x))},$$

maps Q into the complement of $g[Y]$ for every positive δ and is δ-close to g. □

5. Conclusions. By the West-Curtis-Shori theorem (see [13]), the hyperspace of finite-dimensional cube $\exp(\mathbb{I}^n)$ is homeomorphic to the Hilbert cube Q. Now combining the Uniqueness Theorem for absorbing systems in the Hilbert cube (see [13], [10]) with Proposition 2 and Theorem 2, we obtain the following result implying Theorem 1 announced in the Introduction.

Theorem 3. *For every* $n \in \mathbb{N}$ *there exists a homeomorphism* $h\colon \exp(\mathbb{I}^n) \to Q^{\mathbb{Q}}$ *such that* $h(HD_{>\gamma}(\mathbb{I}^n)) = Q^{\mathbb{Q}_{\le\gamma}} \times B(Q^{\mathbb{Q}_{>\gamma}})$ *and* $h(HD_{\le\gamma}(\mathbb{I}^n)) = Q^{\mathbb{Q}_{\le\gamma}} \times s^{\mathbb{Q}_{>\gamma}}$ *for every* $\gamma \in [0, n)$.

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