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Marek Golasiński

EVALUATION FIBRATIONS AND PATH-COMPONENTS OF THE MAP SPACE $M(\mathbb{S}^{n+k},\mathbb{S}^n)$ FOR $0\le k\le 7$

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Let $M(\mathbb{S}^m,\mathbb{S}^n)$ be the space of maps of the m-sphere $\mathbb{S}^m$ into the n-sphere $\mathbb{S}^n$ with $m,n\ge 1$. We estimate the number of homotopy types of path-components of $M(\mathbb{S}^{n+k},\mathbb{S}^n)$ and fibre homotopy types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+k}(\mathbb{S}^n)$ with $0\le k\le 7$. Further, the number of strongly homotopy types of $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k},\mathbb{S}^n)\to\mathbb{S}^n$ for $0\le k\le 7$ is determined.

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Пусть $M(\mathbb{S}^m,\mathbb{S}^n)$ обозначает пространство отображений из m-сферы $\mathbb{S}^m$ в n-сферу $\mathbb{S}^n$ $(m,n\ge 1)$. Оценивается число гомотопических типов компонент линейной связности $M(\mathbb{S}^{n+k},\mathbb{S}^n)$ и послойно гомотопических типов расслоений вычисления $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k},\mathbb{S}^n)\to\mathbb{S}^n$ для $\alpha\in\pi_{n+k}(\mathbb{S}^n)$ и $0\le k\le 7$. Также найдено число сильных гомотопических типов отображений $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k},\mathbb{S}^n)\to\mathbb{S}^n$ для $0\le k\le 7$.

Introduction. For a pair of spaces X and Y, let $M(X,Y)$ be the space of all continuous maps of X into Y equipped with the compact-open topology. The space $M(X,Y)$ is generally disconnected with path-components in one-to-one correspondence with the set $[X,Y]$ of (free) homotopy classes of maps of X into Y.

For $x_0\in X$, consider the evaluation map

$$\omega\colon M(X,Y)\to Y$$

defined by $\omega(f)=f(x_0)$ for $f\in M(X,Y)$. Let $M_\alpha(X,Y)$ denote the path-component of $M(X,Y)$ which contains the maps in $\alpha\in[X,Y]$. By [6, Theorem 13.1], the evaluation map $\omega_\alpha\colon M_\alpha(X,Y)\to Y$ obtained by restricting ω to $M_\alpha(X,Y)$ is a Hurewicz fibration provided X is locally compact. Then, the natural classification problems arise:
(1) Divide the set of path-components in $M(X,Y)$ into homotopy types.
(2) Divide the set of evaluation fibrations $\omega_\alpha\colon M_\alpha(X,Y)\to Y$ into fibre- and strongly fibre-homotopy types for $\alpha\in[X,Y]$.

Certainly, for the space $M(\mathbb{S}^m,\mathbb{S}^n)$ of maps of the m-sphere $\mathbb{S}^m$ into the n-sphere $\mathbb{S}^n$ with $m,n\ge 1$, the path-components can be enumerated by the homotopy group $\pi_m(\mathbb{S}^n)$. By [5], there is a strong relation between evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^m,\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_m(\mathbb{S}^n)$ and the Whitehead product $[\alpha,\iota_n]$. This was used in [5, Theorem 5.1 and Theorem 5.2] to tackle a complete homotopy classification of path-components of $M(\mathbb{S}^m,\mathbb{S}^n)$ for $m=n,n+1$.

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The purpose of this note is to present the classification problems above for $m = n + k$ with $0 \le k \le 7$. Section 1 summarizes [4, 5] and follows [7] to connect in Theorem 1.1 these classification problems for $M(\mathbb{S}^m, \mathbb{S}^n)$ with the m-th Gottlieb group $G_m(\mathbb{S}^n)$ considered in [2, 3] and then studied in [1].

Section 2 makes use of [1] to take up the systematic study of the quotient sets $\pi_{n+k}(\mathbb{S}^n)/\pm G_{n+k}(\mathbb{S}^n)$ with $0 \le k \le 7$. Then, our basic results stated in Theorems: 2, 3, 4 and 5 estimate the number of homotopy types of path-components of $M(\mathbb{S}^{n+k}, \mathbb{S}^n)$ and fibre-homotopy types of evaluations fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ for $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ with $0 \le k \le 7$. Further, the number of strong fibre-homotopy types of $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ with $0 \le k \le 7$ is determined. At the end, Corollary 1 concludes a list of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k}, \mathbb{S}^n) \to \mathbb{S}^n$ which are fibre-homotopy equivalent but not strongly fibre-homotopy equivalent for $\alpha \in \pi_{n+k}(\mathbb{S}^n)$ with some $0 \le k \le 7$.

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1. Preliminaries. A *fibration* $p\colon E \to B$ with a fibre F means a Hurewicz fibration together with a fixed homotopy equivalence $i\colon F \to p^{-1}(b_0)$ over the base point $b_0 \in B$. Recall that for fibrations $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ a based map $f\colon E_1 \to E_2$ is:
(1) a *fibre homotopy equivalence* (**fhe**) if there exists $g\colon E_2 \to E_1$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identities by based homotopies F and G satisfying $p_1 \circ F(e_1, t) = p_1(e_1)$ and $p_2 \circ G(e_2, t) = p_2(e_2)$ for $e_1 \in E_1$, $e_2 \in E_2$ and $t \in [0, 1]$;
(2) a *strong fibre homotopy equivalence* (**sfhe**) if it is a fibre homotopy equivalence and $i_2' \circ f \circ i_1$ is homotopic to the identity map id_F, where i_2' is an arbitrary homotopy inverse of i_2.

Let now X be a connected and pointed space. The m-th *Gottlieb group* $G_m(X)$ [2, 3] of the space X is the subgroup of the m-th homotopy group $\pi_m(X)$ containing all elements which can be represented by a map $f\colon \mathbb{S}^m \to X$ such that $f \vee \mathrm{id}_X\colon \mathbb{S}^m \vee X \to X$ extends (up to homotopy) to a map $F\colon \mathbb{S}^m \times X \to X$. Observe that $G_m(X) = \pi_m(X)$ provided X is an H-space.

Given $\alpha \in \pi_m(\mathbb{S}^n)$, we deduce that $\alpha \in G_m(\mathbb{S}^n)$ if and only if the Whitehead product $[\alpha, \iota_n] = 0$, where ι_n denotes the homotopy class of $\mathrm{id}_{\mathbb{S}^n}$. In other words, $G_m(\mathbb{S}^n) = \ker[\iota_n, -]$ for the map $[\iota_n, -]\colon \pi_m(\mathbb{S}^n) \longrightarrow \pi_{m+n-1}(\mathbb{S}^n)$ with $m \ge 1$. Write $\sharp$ for the order of a group element. Then, by [1, Section 2], from this interpretation of Gottlieb groups of spheres, we obtain

$$G_k(\mathbb{S}^n) = (\sharp[\iota_n, \alpha])\pi_k(\mathbb{S}^n), \text{ if } \pi_k(\mathbb{S}^n) \text{ is a cyclic group with a generator } \alpha. \tag{1}$$

It follows that $G_k(\mathbb{S}^n) = \pi_k(\mathbb{S}^n)$ for $\sharp[\iota_n, \alpha] = 1$ and $G_k(\mathbb{S}^n) = 0$ for $\sharp[\iota_n, \alpha] = \infty$. Furthermore, because of H-structures on the spheres $\mathbb{S}^n$ for $n = 1, 3, 7$, it holds $G_m(\mathbb{S}^n) = \pi_m(\mathbb{S}^n)$ for any $m \ge 1$.

Given a group G and its subgroup $G' < G$, write $G/\pm G'$ for the quotient set of G by the relation $\sim$ defined as follows: for $x, y \in G$, $x \sim y$ if and only if $xy \in G'$ or $xy^{-1} \in G'$. Writing $\simeq$ for the homotopy equivalence relation, [5, Theorem 2.3] and [4, Theorem 1, Theorem 2] lead to

Theorem 1. *Let $m, n \ge 1$. Then:*
(1) *there are surjections:*

(i) $\pi_m(\mathbb{S}^n)/\pm G_m(\mathbb{S}^n) \longrightarrow \{M_\alpha(\mathbb{S}^m,\mathbb{S}^n);\ \alpha\in\pi_m(\mathbb{S}^n)\}/\simeq$;
(ii) $\pi_m(\mathbb{S}^n)/\pm G_m(\mathbb{S}^n) \longrightarrow \{\omega_\alpha\colon M_\alpha(\mathbb{S}^m,\mathbb{S}^n)\to\mathbb{S}^n;\ \alpha\in\pi_m(\mathbb{S}^n)\}/fhe$;

(2) *there is a bijection*

$$\pi_m(\mathbb{S}^n)/G_m(\mathbb{S}^n) \xrightarrow{\approx} \{\omega_\alpha\colon M_\alpha(\mathbb{S}^m,\mathbb{S}^n)\to\mathbb{S}^n;\ \alpha\in\pi_m(\mathbb{S}^n)\}/sfhe.$$

We point out that a generalization of the results above has been stated in [7].

2. Main results. As it is well-known, $[\iota_n,\iota_n]=0$ if and only if $n=1,3,7$ and $\sharp[\iota_n,\iota_n]=2$ for n odd and $n\neq 1,3,7$, and it is infinite provided n is even. Thus, by [1, 2], we obtain $G_n(\mathbb{S}^n)=\pi_n(\mathbb{S}^n)\cong\mathbb{Z}$ for $n=1,3,7$, $G_n(\mathbb{S}^n)=2\pi_n(\mathbb{S}^n)\cong 2\mathbb{Z}$ for n odd and $n\neq 1,3,7$, and $G_n(\mathbb{S}^n)=0$ for n even, where $\mathbb{Z}$ denotes the additive group of integers. Further, by [1, Proposition 1.2], the infinite direct summand of $G_{4n-1}(\mathbb{S}^{2n})$ is $\{3[\iota_{2n},\iota_{2n}]\}$ unless $n=1$.

Observe that in light of the structure of $\pi_m(\mathbb{S}^n)$ (see e.g., [8]) and by the above, $\pi_m(\mathbb{S}^n)/\pm G_m(\mathbb{S}^n)$ is finite, unless $m=n$ and consequently in view of [4, Theorem 2], [5, Theorem 2.3] and Theorem 1, we can complete [5, Theorem 5.1] as follows:

Remark 1. (1) If $m\neq n$ then the sets:

$$\{M_\alpha(\mathbb{S}^m,\mathbb{S}^n);\ \alpha\in\pi_m(\mathbb{S}^n)\}/\simeq,\quad \{\omega_\alpha\colon M_\alpha(\mathbb{S}^m,\mathbb{S}^n)\to\mathbb{S}^n;\ \alpha\in\pi_m(\mathbb{S}^n)\}/\mathit{fhe}\ \text{ and}$$

$\{\omega_\alpha\colon M_\alpha(\mathbb{S}^m,\mathbb{S}^n)\to\mathbb{S}^n;\ \alpha\in\pi_m(\mathbb{S}^n)\}/\mathit{sfhe}$ are finite.
(2) If n is even then two evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^n,\mathbb{S}^n)\to\mathbb{S}^n$ and $\omega_\beta\colon M_\beta(\mathbb{S}^n,\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha,\beta\in\pi_n(\mathbb{S}^n)$ are **fhe** (resp. **sfhe**) if and only if $\alpha=\pm\beta$ (resp. $\alpha=\beta$).
(3) If n is odd then the evaluation fibration $\omega_\alpha\colon M_\alpha(\mathbb{S}^n,\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_n(\mathbb{S}^n)$ is **fhe** (resp. **sfhe**) to $\omega_0\colon M_0(\mathbb{S}^n,\mathbb{S}^n)\to\mathbb{S}^n$ are **fhe** (resp. **sfhe**) if and only if $\deg\alpha$ is even and to $\omega_{\iota_n}\colon M_{\iota_n}(\mathbb{S}^n,\mathbb{S}^n)\to\mathbb{S}^n$ if and only if $\deg\alpha$ is odd. Furthermore, $\omega_0\colon M_0(\mathbb{S}^n,\mathbb{S}^n)\to\mathbb{S}^n$ and $\omega_{\iota_n}\colon M_{\iota_n}(\mathbb{S}^n,\mathbb{S}^n)\to\mathbb{S}^n$ are **fhe** (resp. **sfhe**) if and only if $n=1,3,7$.

By [1, Lemma 1.1], $G_m(\mathbb{S}^2)=\pi_m(\mathbb{S}^2)$ for $m\geq 3$. Hence, by Theorem 1, we can state

Remark 2. (1) All path-components of $M(\mathbb{S}^m,\mathbb{S}^2)$ for $m\geq 3$ are homotopy equivalent.
(2) All evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^m,\mathbb{S}^2)\to\mathbb{S}^2$ for $\alpha\in\pi_m(\mathbb{S}^2)$ with $m\geq 3$ have the same **fhe** and **sfhe** type.

Because $G_m(\mathbb{S}^n)=\pi_m(\mathbb{S}^n)$ for $n=1,3,7$, it follows from the above that to examine $M(\mathbb{S}^m,\mathbb{S}^n)$, we are allowed to assume in the sequel that $n\neq 1,2,3,7$ and certainly $1\leq n\leq m$.

To state further results, write $\eta_2\in\pi_3(\mathbb{S}^2)$, $\nu_4\in\pi_7(\mathbb{S}^4)$ and $\sigma_8\in\pi_{15}(\mathbb{S}^8)$ for the Hopf maps, respectively. We set $\eta_n=E^{n-2}\eta_2\in\pi_{n+1}(\mathbb{S}^n)$ for $n\geq 2$, $\nu_n=E^{n-4}\nu_4\in\pi_{n+3}(\mathbb{S}^n)$ for $n\geq 4$ and $\sigma_n=E^{n-8}\sigma_8\in\pi_{n+7}(\mathbb{S}^n)$ for $n\geq 8$.

According to [1, Section 2], we know the following results:

$$[\iota_n,\eta_n]=0 \text{ if and only if } n\equiv 3\,(\mathrm{mod}\ 4) \text{ or } n=2,6;$$

$$[\iota_n,\eta_n^2]=0 \text{ if and only if } n\equiv 2,3\,(\mathrm{mod}\ 4) \text{ or } n=5,$$

where $\eta_n^2=\eta_n\circ\eta_{n+1}$. Hence, by (1), the group $G_{n+k}(\mathbb{S}^n)$ for $k=1,2$ are completely determined. In view of the structure of $\pi_{n+1}(\mathbb{S}^n)$ nad $\pi_{n+2}(\mathbb{S}^n)$ (see e.g., [8]) and Theorem 1, the following extends [5, Theorem 5.2(iii)].

Theorem 2. *Consider $M(\mathbb{S}^{n+1},\mathbb{S}^n)$ for $n\geq 3$ and $M(\mathbb{S}^{n+2},\mathbb{S}^n)$ for $n\geq 2$.*
(1) *The space $M(\mathbb{S}^{n+1},\mathbb{S}^n)$ has two path-components. These components of $M(\mathbb{S}^{n+1},\mathbb{S}^n)$ have the same homotopy type if and only if $n=6$ or $n\equiv 3\,(\mathrm{mod}\,4)$.*

The space $M(\mathbb{S}^{n+2},\mathbb{S}^n)$ has two path-components. These components has the same homotopy type if and only if $n=5$ or $n\equiv 2,3\,(\mathrm{mod}\,4)$.
(2) *The evaluation fibrations $\omega_0\colon M_0(\mathbb{S}^{n+1},\mathbb{S}^n)\to\mathbb{S}^n$ and $\omega_{\iota_n}\colon M_{\iota_n}(\mathbb{S}^{n+1},\mathbb{S}^n)\to\mathbb{S}^n$ are* **fhe** *if and only if $n=6$ or $n\equiv 3\,(\mathrm{mod}\,4)$ and are not* **sfhe**.

The evaluation fibrations $\omega_0\colon M_0(\mathbb{S}^{n+2},\mathbb{S}^n)\to\mathbb{S}^n$ and $\omega_{\iota_n}\colon M_{\iota_n}(\mathbb{S}^{n+2},\mathbb{S}^n)\to\mathbb{S}^n$ are **fhe** *if and only if $n=5$ or $n\equiv 2,3\,(\mathrm{mod}\,4)$ and are not* **sfhe**.

In view of [1, Section 2], $G_7(\mathbb{S}^4)\cong 3\mathbb{Z}\oplus\mathbb{Z}_2$ and

$$\sharp[\iota_n,\nu_n]=\begin{cases}1, & \text{if } n\equiv 7 \text{ (mod 8) or } n=2^i-3 \text{ for } i\geq 3;\\ 2, & \text{if } n\equiv 1,3,5 \text{ (mod 8) and } n\geq 9 \text{ and } n\neq 2^i-3;\\ 12, & \text{if } n\equiv 2 \text{ (mod 4) and } n\geq 6 \text{ or } n=4,12;\\ 24, & \text{if } n\equiv 0 \text{ (mod 4) and } n\geq 8 \text{ unless } n=12.\end{cases}$$

Thus, (1) leads to a complete description of $G_{n+3}(\mathbb{S}^n)$ for $n\geq 5$. By means of the structure of $\pi_{n+3}(\mathbb{S}^n)$ (see e.g., [8]) and Theorem 1, we obtain

Theorem 3. (1) *The number of homotopy types of path-components of the space $M(\mathbb{S}^{n+3},\mathbb{S}^n)$ and* **fhe** *types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+3},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+3}(\mathbb{S}^n)$ is:*
(i) *at most ten, if $n=4$;*
(ii) *only one, if $n\equiv 7$ (mod 8) or $n=2^i-3$ for $i\geq 3$;*
(iii) *two if $n\equiv 1,3,5$ (mod 8) and $n\geq 9$, and $n\neq 2^i-3$;*
(iv) *at most seven, if $n\equiv 2$ (mod 4) and $n\geq 6$ or $n=4,12$;*
(v) *at most thirteen, if $n\equiv 0$ (mod 4) and $n\geq 8$ unless $n=12$.*
(2) *The number of* **sfhe** *types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+3},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+3}(\mathbb{S}^n)$ is:*
(i) *eighteen, if $n=4$;*
(ii) *one, if $n\equiv 7$ (mod 8) or $n=2^i-3$ for $i\geq 3$;*
(iii) *two, if $n\equiv 1,3,5$ (mod 8) and $n\geq 9$, and $n\neq 2^i-3$;*
(iv) *twelve, if $n\equiv 2$ (mod 4) and $n\geq 6$ or $n=4,12$;*
(v) *twenty four, if $n\equiv 0$ (mod 4) and $n\geq 8$ unless $n=12$.*

We recall [1, Proposition 2.1, Proposition 2.3]:

Proposition 1. (1) *$G_{n+4}(\mathbb{S}^n)=\pi_{n+4}(\mathbb{S}^n)$; $G_{n+5}(\mathbb{S}^n)=\pi_{n+5}(\mathbb{S}^n)$ unless $n=6$ and $G_{11}(\mathbb{S}^6)=3\pi_{11}(\mathbb{S}^6)\cong 3\mathbb{Z}$.*
(2) *$G_{n+6}(\mathbb{S}^n)=\pi_{n+6}(\mathbb{S}^n)$ if $n\equiv 4,5,7$ (mod 8) or $n=2^i-5$ for $i\geq 4$ and $G_{n+6}(\mathbb{S}^n)=0$ otherwise.*
(3) *$G_{n+7}(\mathbb{S}^n)=0$ if $n=4,6$, $G_{12}(\mathbb{S}^5)=\pi_{12}(\mathbb{S}^5)$ and $G_{15}(\mathbb{S}^8)\cong 3\mathbb{Z}\oplus\mathbb{Z}_2$.*

Further, in view of [1, Section 2]:

$$\sharp[\iota_n,\sigma_n]=\begin{cases}1, & \text{if } n=11 \text{ or } n\equiv 15 \pmod{16};\\ 2, & \text{if } n \text{ is odd and } n\geq 9 \text{ unless } n=11 \text{ and } n\equiv 15 \pmod{16};\\ 120, & \text{if } n=8;\\ 240, & \text{if } n \text{ is even and } n\geq 10.\end{cases}$$

Hence, by means of (1), the group $G_{n+7}(\mathbb{S}^n)$ for $n\geq 9$ has been fully described as well.

In view of the structure of $\pi_{n+4}(\mathbb{S}^n)$, $\pi_{n+5}(\mathbb{S}^n)$ and $\pi_{n+6}(\mathbb{S}^n)$ (see e.g., [8]), Theorem 1 and Proposition 1, we obtain

Theorem 4. (1) *The number of homotopy types of path-components of the space $M(\mathbb{S}^{n+4},\mathbb{S}^n)$,* **fhe** *and* **sfhe** *types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+4},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+4}(\mathbb{S}^n)$ is one.*

(2) *The number of homotopy types of:*

(i) *path-components of the space $M(\mathbb{S}^{n+5},\mathbb{S}^n)$,* **fhe** *and* **sfhe** *types types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+5},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+5}(\mathbb{S}^n)$ is one unless $n=6$;*

(ii) *path-components of the space $M(\mathbb{S}^{11},\mathbb{S}^6)$ and* **fhe** *types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{11},\mathbb{S}^6)\to\mathbb{S}^6$ for $\alpha\in\pi_{11}(\mathbb{S}^6)$ is at most two;*

(iii) **sfhe** *types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{11},\mathbb{S}^6)\to\mathbb{S}^6$ for $\alpha\in\pi_{11}(\mathbb{S}^6)$ is three.*

(3) *The number of homotopy types of path-components of the space $M(\mathbb{S}^{n+6},\mathbb{S}^n)$,* **fhe** *and* **sfhe** *types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+6},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+6}(\mathbb{S}^n)$ is one if $n\equiv 4,5,7\,(\mathrm{mod}\,8)$ or $n=2^i-5$ for $i\geq 4$ and two otherwise.*

In view of the structure of $\pi_{n+7}(\mathbb{S}^n)$ (see e.g., [8]), Theorem 1, Proposition 1 and the relations above, we obtain

Theorem 5. *The number of homotopy types of path-components of the space $M(\mathbb{S}^{n+7},\mathbb{S}^n)$ and* **fhe** *types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+7},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+7}(\mathbb{S}^n)$ is:*

(i) *one, if $n=5,11$ or $n\equiv 15\,(\mathrm{mod}\,16)$;*

(ii) *two, if n is odd and $n\geq 9$ unless $n=5,11$ and $n\equiv 15\,(\mathrm{mod}\,16)$;*

(iii) *at most:*

(a) *eight, if $n=4$;*

(b) *thirty one if $n=6$;*

(c) *ninety, if $n=8$;*

(d) *one hundred twenty one, if n is even and $n\geq 10$.*

(2) *The number of* **sfhe** *types of evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+7},\mathbb{S}^n)\to\mathbb{S}^n$ for $\alpha\in\pi_{n+7}(\mathbb{S}^n)$ is:*

(i) *one, if $n=5,11$ or $n\equiv 15\,(\mathrm{mod}\,16)$;*

(ii) *fifteen, if $n=4$;*

(iii) *sixty, if $n=6$;*

(iv) *one hundred eighty, if $n=8$;*

(v) *two, if n is odd and $n\geq 9$ unless $n=5,11$ and $n\equiv 15\,(\mathrm{mod}\,16)$;*

(vi) *two hundred forty, if n is even and $n\geq 10$.*

At the end, we recall that in [5, Example 1], two **fhe** evaluation fibrations $\omega_\alpha\colon M_\alpha(\mathbb{S}^2\vee\mathbb{S}^2,\mathbb{S}^2)\to\mathbb{S}^2$ and $\omega_\beta\colon M_\beta(\mathbb{S}^2\vee\mathbb{S}^2,\mathbb{S}^2)\to\mathbb{S}^2$ for $\alpha,\beta\in[\mathbb{S}^2\vee\mathbb{S}^2,\mathbb{S}^2]$ being not **sfhe** are constructed.

From the results above, we obtain

Corollary 1. *There is a list of evaluation fibrations* $\omega_\alpha\colon M_\alpha(\mathbb{S}^{n+k},\mathbb{S}^n)\to\mathbb{S}^n$ *for* $\alpha\in\pi_{n+k}(\mathbb{S}^n)$ *and some* $0\le k\le 7$ *being* **fhe** *and not* **sfhe**.

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Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18, 87-100 Toruń, Poland
marek@mat.uni.torun.pl

Faculty of Mathematics and Computer Science
University of Warmia and Mazury
Żołnierska 14, 10-561 Olsztyn, Poland
marek@matman.uwm.edu.pl

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