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SOME PROPERTIES OF THE FUNCTOR OF NON-EXPANDING FUNCTIONALS

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In the present paper we prove that the functor of non-expanding functionals is not open.

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Доказано, что функтор нерастягивающих функционалов неоткрыт.

1. The general theory of functors acting in the category **Comp** of compact Hausdorff spaces (compacta) and continuous mappings was founded by E. V. Shchepin [1]. He distinguished some elementary properties of such functors and defined the notion of normal functor that has become very fruitful. The classes of normal and weakly normal functors include many classical constructions: the hyperspace exp, the functor of probability measures P, the superextension λ, the functor of inclusion hyperspaces G and many other functors ([2], [3]).

In [4] the functor of order-preserving functionals O was introduced. It contains all functors mentioned above as subfunctors. The functor O is a weakly normal functor, in particular, it preserves weight of infinite compacta. The proof of this property uses the fact that any order-preserving functional is non-expanding. A. Stan'ko and J. Camargo introduced the weakly normal functor E of non-expanding functionals in their master thesis.

The investigations of topological properties of the functors exp, P, λ, G and O are based on the openness of these functors (see [5] and [6]). We show in this paper that the functor E does not have this property, although it is finitely open.

The paper is organized as follows: in Section 2 we give some necessary definitions, in Section 3 we obtain the main results.

2. We work in the category **Comp**, that is why we assume all spaces to be compact Hausdorff, and mappings to be continuous.

Let $F\colon$ **Comp** $\to$ **Comp** be a covariant functor. A functor F is called *monomorphic (epimorphic)* if it preserves monomorphisms (respectively epimorphisms).

A monomorphic functor F is said to have the *preimage-preserving* property if for each map $f\colon X \to Y$ and each closed subset $A \subset Y$ we have $(F(f))^{-1}(F(A)) = F(f^{-1}(A))$.

For a monomorphic functor F the *intersection-preserving* property is defined as follows: $F(\cap\{X_\alpha|\alpha \in \mathcal{A}\}) = \cap\{F(X_\alpha)|\alpha \in \mathcal{A}\}$ for every family $\{X_\alpha|\alpha \in \mathcal{A}\}$ of closed subsets of X.

A functor F is called *continuous* if it preserves the limits of inverse systems $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, \mathcal{A}\}$ over a directed set $\mathcal{A}$.

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Finally, a functor F is called *weight-preserving* if $w(X) = w(F(X))$ for every infinite compactum X.

A covariant functor $F\colon$ **Comp** $\to$ **Comp** is called *normal* if it is continuous, monomorphic, epimorphic, preserves weight, intersections, preimages, one-point spaces and the empty space. If a functor F satisfies all the properties mentioned above except the preimage-preserving property, it is called *weakly normal.*

A functor F is said to be (*finitely*) *open* if for any (finite) compact Hausdorff spaces X and Y and any open continuous surjective function $f\colon X \to Y$ the map $Ff\colon FX \to FY$ is open.

A functor F is called *bicommutative*, if for any bicommutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[q]{} & T \end{array}$$

the diagram

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle Fg}\downarrow & & \downarrow{\scriptstyle Fp} \\ FZ & \xrightarrow[Fq]{} & FT \end{array}$$

is also bicommutative.

Following the notation of [6], for any space X by $\pi^2(X)$ we denote the diagram

$$\begin{array}{ccc} X^4 & \xrightarrow{pr_{12}} & X^2 \\ {\scriptstyle pr_{13}}\downarrow & & \downarrow{\scriptstyle pr_1} \\ X^2 & \xrightarrow[pr_1]{} & X \end{array}$$

By $C(X)$, where X is a compactum, we denote the Banach space of all continuous real-valued functions with the sup-norm $\|\varphi\| = \sup\{|\varphi(x)|\colon x \in X\}$; also, the denotation $d(\varphi, \psi)$ will stand for the distance between two functions $\varphi, \psi \in C(X)\colon\ d(\varphi, \psi) = \|\varphi - \psi\|$. By c_X, where $c \in \mathbb{R}$, we denote the constant function: $c_X(x) = c$ for all $x \in X$.

Finally let us define the functor E. For any space X the set EX consists of all functionals $\nu\colon C(X) \to \mathbb{R}$ which satisfy the properties: 1) $\nu(c_X) = c$ for any $c \in \mathbb{R}$; 2) for any $\varphi, \psi \in C(X)$ the inequality $|\nu(\varphi) - \nu(\psi)| \le d(\varphi, \psi)$ holds. The functionals which satisfy 1) and 2) are called *non-expanding.* We will consider EX equipped with the topology generated by the base consisting of the sets

$$O(\mu; \varphi_1, \dots, \varphi_n; \varepsilon) = \{\nu \in EX\colon\ |\nu(\varphi_i) - \mu(\varphi_i)| < \varepsilon,\ i = \overline{1, n}\}.$$

For a map $f\colon X \to Y$ put $Ef(\nu)(\varphi) = \nu(\varphi \circ f)$, where $\nu \in EX,\ \varphi \in C(Y)$. One can check that E is a covariant functor in the category **Comp**.

It can be shown that the functor E is weakly normal and can be completed to a monad.

The next lemma was proved by A. Stan'ko and J. Camargo, and we outline its proof below for the sake of completeness.

Lemma 1. *Consider any $A \subset C(X)$ which contains all constant functions and a functional $\mu_0 \colon A \to \mathbb{R}$ which is non-expanding on A. Then there exists a non-expanding functional $\mu \colon C(X) \to \mathbb{R}$ such that $\mu|_A = \mu_0$.*

Proof. It is sufficient to prove that μ_0 can be extended on one more function $\varphi \in C(X)\backslash A$. According to the definition of a non-expanding functional, the choice of $\mu_0(\varphi)$ must satisfy the inequality $\mu_0(\psi) - d(\psi,\varphi) \le \mu_0(\varphi) \le \mu_0(\psi) + d(\psi,\varphi)$ for any $\psi \in A$. Denote by I_ψ the closed interval $[\mu_0(\psi) - d(\psi,\varphi), \mu_0(\psi) + d(\psi,\varphi)]$. Now, for arbitrary $\psi_1, \psi_2 \in A$ we have $\mu_0(\psi_1) - d(\psi_1,\varphi) \le \mu_0(\psi_2) + d(\psi_1,\psi_2) - d(\psi_1,\varphi) \le \mu_0(\psi_2) + d(\psi_2,\varphi)$. That's why, according to the axiom of completeness, $\bigcap_{\psi\in A} I_\psi \neq \varnothing$. It is sufficient to choose $\mu_0(\varphi)$ from the set $\bigcap_{\psi\in A} I_\psi$. The lemma is proved. □

3. In this section we prove the finite openness of functor E, then show that it is not bicommutative and not open.

First consider the following technical lemma.

Lemma 2. *Let X, Y be finite Hausdorff spaces, $f \colon X \to Y$ a continuous surjective function. Then for any $\phi \in C(Y)$ and any $\varphi \in C(X)$ there exists a function $\psi \in C(Y)$ such that $d(\phi,\psi) + d(\psi \circ f, \varphi) = d(\phi \circ f, \varphi)$, and $d(\varphi, \psi \circ f) = \inf_{\mu\in C(Y)} d(\varphi, \mu \circ f)$.*

Proof. Denote $b = \inf_{\mu\in C(Y)} d(\varphi, \mu\circ f)$. Let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$. We make the following denotations: $X_i = \{x_{k_{i-1}+1}, \dots, x_{k_i}\}$, where $f(X_i) = y_i$ and $k_0 = 0$; $\varphi(x_j) = \varphi^i_j$ for $x_j \in X_i$, supposing all x_j are numbered the way that $\varphi^i_j \ge \varphi^i_l$ if $j \ge l$; $\psi(y_i) = \psi_i$; $\phi(y_i) = \phi_i$. Let $d(\phi \circ f, \varphi) = c > b$ (in case $c = b$ we can put $\psi = \phi$). Note that $b = \max_{1\le i\le m} \frac{\varphi^i_{k_j} - \varphi^i_{k_{j-1}+1}}{2}$. So any function $\psi \in C(Y)$ with the property $d(\psi \circ f, \varphi) = b$ must satisfy the condition $\varphi^j_{k_j} - b \le \psi_j \le \varphi^j_{k_{j-1}+1} + b$, where $j = \overline{1,m}$. Choose ψ_j the following way: for any $j \in \{1, \dots, m\}$ if $\phi_j \in [\varphi^j_{k_j} - b, \varphi^j_{k_{j-1}+1} + b]$ then put $\psi_j = \phi_j$, in the other case, choose that end of the segment $[\varphi^j_{k_j} - b, \varphi^j_{k_{j-1}+1} + b]$, which is closer to ϕ_j. The obtained function $\psi \in CY$ satisfies the necessary conditions: $d(\psi \circ f, \varphi) = b$, and it's easy to see that $d(\psi \circ f, \varphi) + d(\phi \circ f, \psi \circ f) = d(\phi \circ f, \varphi)$. □

Now we deal with the notion of E-convexity, which will be useful in the following discussion.

For any set $B \subset EX$ let $\sup B$ and $\inf B$ be the functionals defined by the formulae $\sup B(\varphi) = \sup\{\mu(\varphi) \mid \mu \in B\}$ and $\inf B(\varphi) = \inf\{\mu(\varphi) \mid \mu \in B\}$ for all $\varphi \in C(X)$. Then we have $\sup B$, $\inf B \in EX$. Indeed, suppose that, for example, $\sup B$ is expanding. This means that $|\sup B(\phi) - \sup B(\psi)| > d(\phi,\psi)$ for some $\phi, \psi \in C(X)$. Consider, for example, the case when $\sup B(\phi) < \sup B(\psi) - d(\phi,\psi)$, from where we get that there's a functional $\mu \in B$ such that $\sup B(\phi) < \mu(\psi) - d(\phi,\psi)$, and hence $\mu(\phi) < \mu(\psi) - d(\phi,\psi)$, which is a contradiction because $\mu \in EX$. A contradiction can be obtained from the inequality $\sup B(\psi) + d(\phi,\psi) < \sup B(\phi)$ the same way. That is why $\sup B \in EX$. Also, similar arguments give us $\inf B \in EX$.

From the previous arguments we obtain the following

Proposition 1. *For any compact Hausdorff space X the set EX is a compact sublattice of $\prod_{\varphi\in C(X)}[\min\varphi, \max\varphi]$.*

Hence, the maps $\sup, \inf\colon \exp EX \to EX$ are continuous (see [7]).

The set $B \subset EX$ is called *E-convex* if for any $\mu \in EX$ with $\inf B \le \mu \le \sup B$ we have $\mu \in B$. Assume $f\colon X \to Y$ is a continuous map. It is easy to check that the set of the form $E(f)^{-1}(\nu)$ is E-convex for any $\nu \in EY$. Given a sequence $\{B_n\}_{n\in\mathbb{N}}$ of E-convex subsets of EX which converges to some $B \subset EX$, a question arises whether B is also E-convex.

Lemma 3. *Let $\{B_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of EX converging to $B \subset EX$ in $\exp EX$ and assume that all B_n are E-convex. Then B is E-convex as well.*

Proof. Consider any $\mu \in EX$ such that $\inf B \le \mu \le \sup B$. For any $n \in \mathbb{N}$ define the functional ν_n the following way:

$$\nu_n = \sup\{\inf\{\mu, \sup B_n\}, \inf B_n\}.$$

Due to the continuity of the maps sup and inf we have $\mu = \lim_{n\to\infty} \nu_n$. Also, $\inf B_n \le \nu_n \le \sup B_n$ for all $n \in \mathbb{N}$, so $\nu_n \in B_n$, and we get $\mu \in \lim_{n\to\infty} B_n$. The lemma is proved. □

Theorem 1. *Functor E is finitely open.*

Proof. Let X and Y be two finite Hausdorff spaces, $f\colon X \to Y$ be a surjective map (obviously continuous and open). We want to show that $E(f)$ is open. Assume the opposite: let $E(f)$ be not open, i.e. there are a functional $\mu_0 \in EX$, a sequence of functionals $\{\nu_n\}_{n\in\mathbb{N}} \subset EY$ converging to $\nu_0 = Ef(\mu_0)$ and a neighborhood W of μ_0 such that $(Ef)^{-1}(\nu_n) \cap W = \varnothing$ for each $n \in \mathbb{N}$. Since EX is compact, we can suppose that the sequence consisting of $B_n = (Ef)^{-1}(\nu_n),\ n \in \mathbb{N}$, converges to $B \subset (Ef)^{-1}(\nu_0)$ in $\exp EX$. Then $\mu_0 \notin B$. Hence, recalling Lemma 3, we see that there is a function $\varphi \in C(X)$ such that $\mu_0(\varphi) > \sup B(\varphi)$ or $\mu_0(\varphi) < \inf B(\varphi)$. We will consider the first case. Denote $a = \sup B(\varphi)$, $\varepsilon = \mu_0(\varphi) - a$. Then $U = \{\mu \in EX \mid \mu(\varphi) > a\}$ is the open neighborhood of μ_0. Denote $U_1 = \{\mu \in EX \mid \mu(\varphi) \ge a + \frac{\varepsilon}{2}\}$. We have $U \cap B = \varnothing$ because for any $\mu \in B$ $\inf B \le \mu \le \sup B$. Thus the set $(Ef)(U_1)$ cannot cover any neighborhood of ν_0, otherwise the intersection of U and B would not be empty. Denote $K = \{\psi \in C(Y) \mid d(\varphi, \psi\circ f) = \inf_{\phi\in C(Y)} d(\varphi, \phi\circ f)\}$. Then K is bounded and closed in the space $C(Y)$, which is homeomorphic to $\mathbb{R}^m$, where $m = |Y|$. Thus K is compact, which gives us the following fact: for any $\delta > 0$ there are functions $\psi_1, \dots, \psi_k \in C(Y)$ such that $K \subset \bigcup_{i=1}^k \{\phi \mid d(\phi, \psi_i) < \delta\}$. We can choose positive numbers δ and ε_1 such that $\varepsilon_1 + 2\delta = \frac{\varepsilon}{2}$. Consider the neighborhood $V = O(\nu_0; \psi_1, \dots, \psi_k; \varepsilon_1)$ of ν_0. Take any $\nu \in V$. Put $\overline{\mu}(\psi\circ f) = \nu(f)$ for any $\psi \in C(Y)$. The functional $\overline{\mu}\colon \{\psi\circ f \mid \psi \in C(Y)\} \to \mathbb{R}$ is non-expanding, because so is ν. Our task now is to define the value $\overline{\mu}(\varphi)$ the way that $\overline{\mu}(\varphi) \ge a + \frac{\varepsilon}{2}$.

According to Lemma 2, for any function $\xi \in C(Y)$ we can find $\psi \in K$ such that $d(\xi\circ f, \psi\circ f) + d(\varphi, \psi\circ f) = d(\xi\circ f, \varphi)$. For this function ψ there is the number $i_0 \in \{1, \dots, k\}$ such that $d(\psi, \psi_{i_0}) < \delta$. Then $\nu(\xi) + d(\xi\circ f, \varphi) = \nu(\xi) + d(\xi\circ f, \psi\circ f) + d(\psi\circ f, \varphi) \ge \nu(\xi) + d(\xi\circ f, \psi_{i_0}\circ f) + d(\psi_{i_0}\circ f, \varphi) - 2\delta \ge \nu(\psi_{i_0}) + d(\psi_{i_0}\circ f, \varphi) - 2\delta \ge \nu_0(\psi_{i_0}) - \varepsilon_1 + d(\psi_{i_0}\circ f, \varphi) - 2\delta = \mu_0(\psi_{i_0}\circ f) - \varepsilon_1 + d(\psi_{i_0}\circ f, \varphi) - 2\delta \ge \mu_0(\varphi) - \varepsilon_1 - 2\delta = a + \varepsilon - \varepsilon_1 - 2\delta$. Recalling that $\varepsilon_1 + 2\delta = \frac{\varepsilon}{2}$, we eventually get $\nu(\xi) + d(\xi\circ f, \varphi) \ge a + \frac{\varepsilon}{2}$ for any $\xi \in C(Y)$. Choose $\overline{\mu}(\varphi) = \inf_{\xi\in C(Y)}\{\nu(\xi) + d(\xi\circ f, \varphi)\}$, which gives us $\overline{\mu}(\varphi) \ge a + \frac{\varepsilon}{2}$. Then, evidently, $\overline{\mu}(\varphi) \ge \nu(\xi) - d(\xi\circ f, \varphi)$ for all $\xi \in C(Y)$, and therefore $\overline{\mu}$ is non-expanding on $\{\psi\circ f \mid \psi \in C(Y)\} \cup \{\varphi\}$. Extend $\overline{\mu}$ to a non-expanding functional $\mu\colon C(X) \to \mathbb{R}$. Then, due to the construction, $\mu \in U_1$ and $Ef(\mu) = \nu$. Hence $V \subset (Ef)(U_1)$, which is a contradiction, because we have found the open neighborhood V

of ν_0 contained in $(Ef)(U_1)$. So, our assumption that $E(f)$ is not open, turned out to be incorrect. The theorem is proved. □

Theorem 2. *Let $f\colon X \to Y$ be continuous, and $E(f)$ be open. Then f is open.*

Proof. Assume f is not open. The latter means that there exist a point x_0 in X, a net $\{y_\alpha\}_{\alpha\in A}$ converging to $y_0 = f(x_0)$ in Y and an open neighborhood V of x_0 in X such that $f^{-1}(y_\alpha) \cap V = \varnothing$ for any $\alpha \in A$. Choose a continuous function $\varphi\colon X \to [0,1]$ such that $\varphi(x_0) = 1$, $\varphi(X\backslash V) = 0$. Put $C = \{x \in X|\ \varphi(x) \geq \frac{1}{2}\}$. The set C is closed in X and $C \cap X\backslash V = \varnothing$. Also we have $\delta_{x_0}(\varphi) = \varphi(x_0) = 1$, $E(f)(\delta_{x_0}) = \delta_{y_0}$, and the net $\{\delta_{y_\alpha}\}_{\alpha\in A}$ converges to δ_{y_0}. Choose the open neighborhood $W = \{\nu \in EX|\ \nu(\varphi) > \frac{1}{2}\}$ of δ_{x_0} in EX. We will show that $(Ef)^{-1}(\delta_{y_\alpha}) \cap W = \varnothing$ for every $\alpha \in A$. Indeed, take any $\alpha \in A$ and any $\mu \in (Ef)^{-1}(\delta_{y_\alpha})$. Choose a continuous function $\psi\colon Y \to [0,\frac{1}{2}]$ such that $\psi(y_\alpha) = 0$, $\psi(f(C)) = \frac{1}{2}$. The distance between $\psi\circ f$ and φ is $\frac{1}{2}$, because for $x \in X$ with $\varphi(x) \geq \frac{1}{2}$ we have $\psi(f(x)) = \frac{1}{2}$, and $\psi(f(x)) \in [0,\frac{1}{2}]$ when $\varphi(x) < \frac{1}{2}$ for some $x \in X$, therefore, the absolute value of their difference cannot be greater than $\frac{1}{2}$. Then $|\mu(\varphi) - \mu(\psi\circ f)| \leq d(\psi\circ f, \varphi) = \frac{1}{2}$, but on the other side $|\mu(\varphi) - \mu(\psi\circ f)| = |\mu(\varphi) - \delta_{y_\alpha}(\psi)| = |\mu(\varphi)|$, which means that $|\mu(\varphi)| \leq \frac{1}{2}$ and therefore $\mu \notin W$. Thus, we have shown that there are the functional $\delta_{x_0} \in EX$, the net of functionals δ_{y_α} in EY converging to $\delta_{y_0} = Ef(\delta_{x_0})$ and the open neighborhood W of δ_{x_0} such that for any $\alpha \in A$ $(Ef)^{-1}(\delta_{y_\alpha}) \cap W = \varnothing$, which means that $E(f)$ is not open. We obtain a contradiction, which finishes the proof of the theorem. □

It is known that the subfunctors P, O of E are bicommutative. The fact about E is that it does not possess this property.

Let $D = \{0,1\}$ be equipped with the discrete topology.

Theorem 3. *The diagram $E(\pi^2(D))$ is not bicommutative.*

Proof. Let $A \subset C(D)$ be a subspace of $C(D)$ such that $A = \{f \in C(D)|f(0) = 0$ *or* $f(1) = 0\} \cup \{c_D|c \in \mathbb{R}\}$. Define $\overline{\lambda}\colon A \to \mathbb{R}$ as follows: $\overline{\lambda}(c_D) = c$, $\overline{\lambda}(f) = 0$ if $f(0) = 0$ or $f(1) = 0$. It is easy to see that $\overline{\lambda}$ is non-expanding on A. According to Lemma 1, we can extend it to a non-expanding functional $\lambda\colon C(D) \to \mathbb{R}$ such that $\lambda|_A = \overline{\lambda}$. Take now any function $g \in C(D)$. It is defined by its values on 0 and 1: $g\colon 0 \mapsto a$, $g\colon 1 \mapsto b$, so we will simply denote such functions as vectors: $g = (a,b)$. Due to the fact that λ is non-expanding we have that for any $f_1 = (0,x)$ and $f_2 = (y,0)$ the inequalities $-d(g,f_1) \leq \lambda(g) \leq d(g,f_1)$, $-d(g,f_2) \leq \lambda(g) \leq d(g,f_2)$ hold, from where we get

$$|\lambda(g)| \leq \max\{|a|, |b-x|\} \tag{1}$$

$$|\lambda(g)| \leq \max\{|a-y|, |b|\} \tag{2}$$

Denote $S = \{f\circ pr_1|f \in C(D)\}$. Now consider functions $\varphi,\psi\colon D^2 \to \mathbb{R}$ defined as follows: $\varphi\colon (0,0) \mapsto -2, (0,1) \mapsto 2, (1,0) \mapsto 0, (1,1) \mapsto 0$, $\psi\colon (0,0) \mapsto 0, (0,1) \mapsto 0, (1,0) \mapsto -2, (1,1) \mapsto 2$. Consider the functionals $\overline{\nu}$, $\overline{\mu}$ such that $\overline{\nu}(\varphi) = 2$, $\overline{\mu}(\psi) = -2$ and for any $f \in C(D)$ $\overline{\nu}(f\circ pr_1) = \overline{\mu}(f\circ pr_1) = \lambda(f)$. We will show that $\overline{\nu}$ and $\overline{\mu}$ are non-expanding on $S\cup\{\varphi\}$ and $S\cup\{\psi\}$ respectively. First consider the functional $\overline{\nu}$, the proof for $\overline{\mu}$ is analogous. So, for any $f \in C(D), f = (a,b)$ we have $|\overline{\nu}(\varphi) - \overline{\nu}(f\circ pr_1)| = |\overline{\nu}(\varphi) - \lambda(f)| = |2 - \lambda(f)|$. Applying the inequality (1) from above, one can get $|2 - \lambda(f)| \leq \max\{|a|, |b-y|\} + 2$ for any $y \in \mathbb{R}$, and when $y = b$ the latter turns into $|2 - \lambda(f)| \leq |a| + 2$. On the other hand, $d(f\circ pr_1, \varphi) = \max\{|a+2|, |a-2|, |b|\}$, and therefore $|2 - \lambda(f)| \leq |a| + 2 \leq$

$\max\{|a+2|, |a-2|, |b|\} = d(f\circ pr_1, \varphi)$, which means that $\overline{\nu}$ is non-expanding on $S\cup\{\varphi\}$. Similarly, using the inequality (2) instead of (1), one can verify that $\overline{\mu}$ satisfies the definition of a non-expanding functional on $S\cup\{\psi\}$. Thus, in accordance with Lemma 1, we can extend $\overline{\nu}$, $\overline{\mu}$ to non-expanding functionals $\nu, \mu\colon C(D^2) \to \mathbb{R}$. Now note that $|\nu(\varphi) - \mu(\psi)| = 4$, whereas $d(\varphi\circ pr_{12}, \psi\circ pr_{13}) = 2$. In other words, there is no non-expanding functional $\Theta \in ED^4$ such that $E(pr_{12})(\Theta) = \nu$ and $E(pr_{13})(\Theta) = \mu$, thus the diagram

$$\begin{array}{ccc} ED^4 & \xrightarrow{E(pr_{13})} & ED^2 \\ {\scriptstyle E(pr_{12})}\downarrow & & \downarrow{\scriptstyle E(pr_1)} \\ ED^2 & \xrightarrow[E(pr_1)]{} & ED \end{array}$$

is not bicommutative. The theorem is proved. □

Corollary 1. *Functor E is not bicommutative.*

Say that the diagram

$$\begin{array}{ccc} X_1' & \xrightarrow{g_1} & X_2' \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ X_3' & \xrightarrow[g_2]{} & X_4' \end{array}$$

is a *retract* of the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ X_3 & \xrightarrow[f_2]{} & X_4 \end{array}$$

if for each $i = \overline{1,4}$ the space X_i' is a retract of X_i and the diagrams

$$\begin{array}{ccc} X_j & \xrightarrow{f_j} & X_{j+1} \\ {\scriptstyle r_j}\downarrow\uparrow{\scriptstyle i_j} & & {\scriptstyle r_{j+1}}\downarrow\uparrow{\scriptstyle i_{j+1}} \\ X_j' & \xrightarrow{g_j} & X_{j+1}' \end{array}$$

and

$$\begin{array}{ccc} X_j & \xrightarrow{p_j} & X_{j+2} \\ {\scriptstyle r_j}\downarrow\uparrow{\scriptstyle i_j} & & {\scriptstyle r_{j+2}}\downarrow\uparrow{\scriptstyle i_{j+2}} \\ X_j' & \xrightarrow{q_j} & X_{j+2}' \end{array}$$

where $j = \overline{1,2}$, are commutative. By $r_j\colon X_j \to X_j'$ we denote the respective retractions, and by $i_j\colon X_j' \to X_j$ the imbeddings, $j = \overline{1,4}$.

Suppose now that a diagram is bicommutative and consider its retract. Then, using the notation of the previous definition, for arbitrary $z \in X_3'$ and $y \in X_2'$ such that $g_2(z) = q_2(y)$ we have $f_2(i_3(z)) = p_2(i_2(y))$. Hence, there is $x \in X_1$ with the property $p_1(x) = i_3(z)$ and

$f_1(y) = i_2(y)$. Then, evidently, the element $r_1(x) \in X_1'$ satisfies the conditions $q_1(r_1(x)) = z$ and $g_1(r_1(x)) = y$.

Therefore, we obtain the following

Lemma 4. *A retract of a bicommutative diagram is also bicommutative.*

Theorem 4. *Functor E is not open.*

Proof. Suppose the opposite. Then, using the same arguments as in [6, Proposition 2.10.2], one can show that the diagram $E(\pi^2(D^\omega))$ is bicommutative. Diagram $\pi^2(D)$ could be viewed as the retract of $\pi^2(D^\omega)$, and therefore $E(\pi^2(D))$ is the retract of $E(\pi^2(D^\omega))$. Using Lemma 3.3 we obtain the bicommutativity of $E(\pi^2(D))$, which is a contradiction with Theorem 3. Hence, E is not open. □

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