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TRACE PROPERTIES IN NORMED SPACES ESTABLISHED BY USING OF MIXED DERIVATIVES

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In this paper, trace properties of functions in weighted function spaces established by free $n+1 \leq |\Sigma| \leq 2^n$ mixed (non-mixed) derivatives defined in an n-dimensional domain are studied. We estimate the $L_p(\Gamma_s)$ norm of the derivatives of the function defined on an s-dimensional surface via the weighted $L_p(G)$ norm of these functions. In order to prove thıs theorem, we use a special form of the integral representation for differentiable functions.

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В статье изучены свойства следов функций в весовых пространствах. С помощью определённых на n-мерной области $n+1 \leq |\Sigma| \leq 2^n$ свободных смешанных производных оцениваются $L_p(\Gamma_s)$-нормы производных функций, которые определены на s-мерной поверхности, через весовую $L_p(G)$-норму этих функций. Для доказательства использовано специальное интегральное представление дифференцируемой функции.

Let us give basic definitions and notation to be used in this paper. Let $n \geq 2$ be a natural number, $\mathbb{E}_n$ be a real arithmetical space consisting of the sets of n real numbers. If $x = (x_1, \ldots, x_n)$ is an element of $\mathbb{E}_n$, then $\operatorname{supp} x$ is a set of $j \in \{1, \ldots, n\}$ such that $x_j \neq 0$.

Let $\mathbb{Z}_n^+$ be the subspace of the space $\mathbb{E}_n$ such that $\alpha \in \mathbb{Z}_n^+$ if $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1, \ldots, \alpha_n$ are integer numbers and $\alpha_1 \geq 0, \ldots, \alpha_n \geq 0$. An element $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $\mathbb{Z}_n^+$ is called a multiindex and $|\alpha| = \alpha_1 + \ldots + \alpha_n$ is its length.

Let $e_n = \{1, 2, ..., n\}$ be a natural number set, $\Sigma \neq \varnothing$ be a subset of e_n, $\Sigma^* \subseteq \Sigma$. A special case $\Sigma^* = \varnothing$ or $\Sigma^* = \Sigma$ can be considered. By definition, we put $\Sigma_0^* := \Sigma^* \cup \{0\}$.

Thus, the number of all possible subsets (including also $\varnothing$ and $\Sigma \setminus \Sigma^*$) of the set $\Sigma \setminus \Sigma^*$ is $N = 2^{|\Sigma \setminus \Sigma^*|}$ (we denote the power of any set E (number of elements) by $|E|$; for example, if $E = e_n$ then $|E| = |e_n| = n$). For convenience, let us list all possible subsets of $\Sigma \setminus \Sigma^*$ as $e^1, e^2, ..., e^N$.

Now we consider the set of vectors $\{m^{i,k} = (m_1^{i,k}, ..., m_n^{i,k}) \in \mathbb{Z}_n^+ : i \in \Sigma_0^*, k \in \{1, \ldots, N\}\}$, such that their coordinates satisfy the conditions:

$$m_j^{0,k} > 0 \quad \text{if} \quad j \in e^k, \qquad m_j^{0,k} \geq 0 \quad \text{if} \quad j \in \Sigma \setminus e^k, \qquad m_j^{0,k} = 0 \quad \text{if} \quad j \in e_n \setminus \Sigma,$$

$$m_j^{i,k} > 0 \quad \text{if} \quad j \in \{i\} \cup e^k, \quad m_j^{i,k} \geq 0 \quad \text{if} \quad j \in \Sigma \setminus \left(\{i\} \cup e^k\right), \quad m_j^{i,k} = 0 \quad \text{if} \quad j \in e_n \setminus \Sigma,$$

that is,

1) $e^k \cup \{i\} \subseteq \operatorname{supp} m^{i,k} \subseteq \Sigma$, $i \in \Sigma^*, k \in \{1, \ldots, N\}$;

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2) $e^k \subseteq \operatorname{supp} m^{0,k} \subseteq \Sigma$, $k \in \{1,\dots,N\}$.

Let $G \subset \mathbb{E}_n$ be a domain. If $g = g(x)$, $x \in G$, is a measurable positive (weight) function, $1 < p < +\infty$, then we say that $L_p(G;g)$ is a linear normed space of measurable functions $f : G \to \mathbb{R}$ such that

$$\|f\|_{L_p(G;g)} \equiv \|gf\|_{L_p(G)} = \Big(\int\limits_G (g(x)|f(x)|\,)^p \; dx\Big)^{1/p} < +\infty.$$

For the vector $m = (m_1, ..., m_n) \in \mathbb{Z}_n^+$ the notation of $D^m f\,(x_1, x_2, ..., x_n)$ stands for

$$D^m f(x_1, ..., x_n) := D_1^{m_1} ... D_n^{m_n} f(x_1, ..., x_n) = \frac{\partial^{m_1}}{\partial x_1^{m_1}} ... \frac{\partial^{m_n}}{\partial x_n^{m_n}} f(x_1, ..., x_n).$$

Let $D^m f$ be Sobolev's generalized derivative of the function f. By definition, we set

$$L_p^{\langle m\rangle}(G;g) = \{f : \; D^m f \in L_p(G,g)\}.$$

Let $g_{i,k}$ $(i \in \Sigma_0^*,\ k \in \{1,\dots,N\})$ be some weight functions. By definition, let

$$\bigcap_{i\in\Sigma_0^*} \bigcap_{k=1}^{N} L_p^{\langle m^{i,k}\rangle}(G; g_{i,k}) \tag{1}$$

be the closure of the set of infinite differentiable functions in $\mathbb{E}_n$ with respect to the norm

$$\|f\|_{\bigcap\limits_{i\in\Sigma_0^*} \bigcap\limits_{k=1}^{N} L_p^{\langle m^{i,k}\rangle}(G;g_{i,k})} = \sum_{i\in\Sigma_0^*} \sum_{k=1}^{N} \|f\|_{L_p^{\langle m^{i,k}\rangle}(G;g_{i,k})}. \tag{2}$$

If we consider (2) with $\Sigma^* = \varnothing$, then we have a weighted space $\bigcap_{k=1}^{|\Sigma|} L_p^{\langle m^k\rangle}\,(G; g_k)$ in (1). This space was earlier investigated in [4]. If in a special case we take $g = 1$ and the vector m^k $(k = 1,\dots,|\Sigma|)$ as the projection from zero vector to the coordinate axes and coordinate surfaces of any vector $\overline{r} = (r_1, ..., r_n)$, then we obtain the well known Nikolski's space $S_p^{\overline{r}}W(\Omega)$. General information about this space can be found [1].

Let us consider (1) when $\Sigma^* = \Sigma$. Then the space (1) turns into $\bigcap_{i\in\Sigma^*} L_p^{\langle m^i\rangle}\,(G; g_i)$ space. This space was investigated by A. D. Dzabrailov ([5]) in the weighted case, and by V.P. II'in [7] in the not weighted case.

Let $H = (H_1,\dots,H_n)$ be a vector with nonnegative coordinates such that $H_i = 0$ if $i \in e_n \setminus \Sigma$, and $H_i > 0$ if $i \in \Sigma$. We consider functions $\varphi_1,\dots,\varphi_n$ such that $\varphi_i \equiv 1$ for every $i \in \{1,\dots,n\} \setminus \Sigma$, for every $i \in \Sigma$ the function φ_i defined on the set $G \times (0, H_i]$ and for each $x \in G$ $\varphi_i(x;\eta_i), \eta_i \in (0, H_i]$, be a function increasing and differentiable (with respect to the parameter η_i), and $\lim_{\eta_i\to 0^+} \varphi_i(x;\eta_i) = 0$. By definition, we put $\varphi = \varphi(x;\eta) = (\varphi_1(x,\eta_1),\dots,\varphi_n(x,\eta_n))$.

Now we consider the vector $\delta = (\delta_1, ..., \delta_n)$ such that supp$\delta \subseteq \Sigma$, and suppose that $\delta_i = +1$ or $\delta_i = -1$ $(i \in \Sigma)$. The number of δ vectors is $2^{|\Sigma|}$. Let us denote the vectors in different numbers by δ^j $(j \in \{1, ..., 2^{|\Sigma|}\})$.

We denote by $R_\delta = R_\delta(\varphi;H)$ the set

$$\bigcup_{0<\eta_j\le H_j, j\in\Sigma} \left\{ y^\Sigma : \quad c_i \le \frac{y_i\delta_i}{\varphi_i(x;\eta_i)} \le c_i^*; \quad c_i, c_i^* > 0 \quad \text{are constants}, \quad i \in \Sigma \right\},$$

where $y^\Sigma \in \mathbb{E}_n$, $y_i^\Sigma = 0$ if $i \in e_n \setminus \Sigma$.

The set $x + R_\delta(\varphi, H)$ is called a "φ-semi horn" with vertex at a point x. In the special case when $\Sigma^* = \Sigma = e_n$, a "φ-semi horn" turns into a "φ-horn" (see O.V. Besov [1]).

We say that a domain G satisfies the *condition of "φ-semi horn" according to the variable* $x_j, j \in \Sigma$, if there exists a finite number of the open sets $G_1, \ldots, G_\gamma$ and vectors $\delta^1, \ldots, \delta^\gamma$, respectively, such that for every $i \in \{1, \ldots, \gamma\}$ we have $G_j \subset G$ and

$$x + R_{\delta^j}(\varphi, H) \subset \overline{G}$$

for every $x \in \overline{G}_j$. In this case we write $G \in A(\varphi; H; \Sigma)$.

Suppose that $G \in A(\varphi; H; \Sigma)$ and the set of the open sets $G_1, \ldots, G_\gamma$ and the vectors $\delta^1, \ldots, \delta^\gamma$ are such that for every $i \in \{1, \ldots, \gamma\}$ we have $G_j \subset G$ and $x + R_{\delta^j}(\varphi, H) \subset \overline{G}$ for every $x \in \overline{G}_j$. We say that a function $b = b(x) > 0$, $x \in G$, satisfies condition (B) if for every number $j \in \{1, \ldots, \gamma\}$ we have

$$c_j \leq \frac{b(x)}{b(x+y)} \leq C_j^*, \quad x \in G_j, \quad y \in R_{\delta^j},$$

where $c_i, c_i^* > 0$ are constants.

For convenience, further, we deal with $\Sigma = e_n$ and $\Sigma^* \subseteq e_n$.

Now consider the s-dimensional $(1 \leq s \leq n-1)$ surface Γ_s, which is defined by the equations

$$x_1 = x_1, \ \ x_2 = x_2, \ \ \ldots \ \ x_s = x_s, \ \ x_{s+1} = \psi_{s+1}(x_1, \ldots, x_s), \ \ \ldots \ \ x_n = \psi_n(x_1, ..., x_s), \quad (3)$$

where the function $\psi_i(x^*)$ $(x^* = (x_1, ..., x_s)$, $i = s+1, ..., n)$ is defined on the domain Ω_s of space the $\mathbb{E}_s$ and has bounded partial derivatives of first order. We denote the surfaces having such properties by $A^{1)}$.

The surface $\Gamma_s + z^{**}$ is an s-dimensional surface and is connected to the surface Γ_s with the following equations

$$x_1 = x_1, \ \ x_2 = x_2, \ \ \ldots \ \ x_s = x_s, \ \ x_{s+1} = \psi_{s+1}(x^*) + z_{s+1}, \ \ \ldots \ \ x_n = \psi_n(x^*) + z_n,$$

where $\psi_i(x^*)$ $(i = s+1, ..., n)$ is the same function as in the equations of surface Γ_s in statement (3), $z^{**} = (0, ..., 0, z_{s+1}, ..., z_n)$. Let us assume that if $\Gamma_s \subset \partial G$ then $z^{**} = (0, ..., 0, z_{s+1}, ..., z_n)$ is chosen such that $\Gamma_s + z^{**} \subset G$.

If $\Gamma_s \subset G$ then for all $\lambda \in \mathbb{Z}_n^+$

$$\Big(\int_{\Gamma_s} |D^\lambda f|_{\Gamma_s}|^p d\Gamma_s\Big)^{1/p} \leq \Big(\int_{\Omega_s} |D^\lambda f(P(x^*))|^p\, |F|^p dx^*\Big)^{1/p} \leq C\Big(\int_{\Omega_s} |D^\lambda f(P(x^*))|^p dx^*\Big)^{1/p}. \quad (4)$$

Here F is the Jacobian matrix of the map $P(x^*) = (x_1, ..., x_s, \psi_{s+1}(x_1, ..., x_s), ..., \psi_n(x_1, ..., x_s))$.

If the surface Γ_s does not belong to G $(\Gamma_s \not\subset G)$ then by the definition $\Gamma_s \subset \partial G$ and

$$\Big(\int_{\Gamma_s} \big|D^\lambda f\big|_{\Gamma_s}\big|^p d\Gamma_s\Big)^{1/p} = \lim_{|z^{**}| \to 0} \Big(\int_{\Gamma_s + z^{**}} \big|D^\lambda f\big|_{\Gamma_s + z^{**}}\big|^p d\left(\Gamma_s + z^{**}\right)\Big)^{1/p},$$

where $P\left(x^*\right) + z^{**} = (x_1, \ldots, x_s, \psi_{s+1}(x^*) + z_{s+1}, \ldots, \psi_n(x^*) + z_n)$.

Let the vectors $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\{m^{i,k} = (m_1^{i,k}, ..., m_n^{i,k}) : \ i \in \Sigma_0^*, k \in \{1, 2, ..., N\}\}$ have non-negative integer coordinates and also satisfy the following conditions:

i) $e^k \subseteq \operatorname{supp} m^{0,k} \subseteq e_n, \{i\} \cup e^k \subseteq \operatorname{supp} m^{i,k} \subseteq e_n (i \in \Sigma^*, k \in \{1, 2, ..., N\})$;

ii) $\lambda_j < m_j^{0,k}$ if $j \in e^k$, $\lambda_j \geq m_j^{0,k}$ if $j \in e_n \backslash e^k$;

iii) $\lambda_j < m_j^{i,k}$ if $j \in \{i\} \cup e^k$, $\lambda_j \geq m_j^{i,k}$ if $j \in e_n \setminus (\{i\} \cup e^k)$ $(i \in \Sigma^*, k \in \{1, \ldots, N\})$.

We put $\gamma_j = 0$ if $j \in \{1, ..., s\}$, $\gamma_j = \frac{1}{p}$ if $j \in \{s+1, ..., n\}$. Now assume that $H = (H_1, \ldots, H_n) \in \mathbb{E}_n$ is a vector with the components $H_j = h_j > 0$ if $j \in e_n \setminus \Sigma^*$, and $H_j = T > 0$ if $j \in \Sigma^*$.

Theorem. *Suppose that*
1) differentiable functions $\tau_j = \tau_j(\eta_j), \eta_j \in (0, H_j], (j \in \{1, \ldots, n\})$ *are such that* $\tau_j'(\eta_j) > 0$, $\lim\limits_{\eta_j \to 0+} \tau_j(\eta_j) = 0$, *and*

$$\int_0^{h_j} [\tau_j(v_j)]^{m_j^{i,k} - \lambda_j - \gamma_j} \frac{d\tau_j(v_j)}{\tau_j(v_j)} < \infty \quad (i \in \Sigma_0^*, \quad k \in \{1, \ldots, N\}, \quad j \in e^k),$$

$$\int_0^{T} \prod_{j \in \Sigma^*} [\tau_j(t)]^{m_j^{i,k} - \lambda_j - \gamma_j} \frac{d\tau_i(t)}{\tau_i(t)} < \infty \quad (i \in \Sigma_0^*, \ k \in \{1, \ldots, N\});$$

2) $G \in A(\varphi; H; e_n)$, *where* $\varphi = (b_1\tau_1, ..., b_n\tau_n)$, *and functions* b_j $(j \in \{1, \ldots, n\})$ *satisfy condition (B);*

3) $f \in \bigcap\limits_{i \in \Sigma_0^*} \bigcap\limits_{k=1}^{N} L_p^{\langle m^{i,k} \rangle}(G; g_{ik})$, $1 < p < \infty$, *where* $g_{ik}(x) := \prod\limits_{j=1}^{n} [b_j(x)]^{m_j^{i,k} - \lambda_j - \gamma_j}$, $x \in G$, $(i \in \Sigma_0^*$, $k \in \{1, ..., N\})$.

If $D^\lambda f|_{\Gamma_s} \in L_p(\Gamma_s)$ *then the following inequality holds*

$$\|D^\lambda f|_{\Gamma_s}\|_{L_p(\Gamma_s)} \leq C \sum_{i \in \Sigma_0^*} \sum_{k=1}^{N} W_{iks}(h, T) \, ||g_{ik} D^{m^{i,k}} f||_{L_p(G)}.$$

Here $C > 0$ *is a constant independent of* f, *and for* $i = 0$ *we have*

$$W_{0ks}(h; T) = \prod_{j \in (e_n \setminus \Sigma^*) \setminus e^k \cap \{s+1, \ldots, n\}} \left[\tau_j(h_j)\right]^{m_j^{0,k} - \lambda_j - \frac{1}{p}} \prod_{j \in \Sigma^* \cap \{s+1, \ldots, n\}} [\tau_j(T)]^{m_j^{0,k} - \lambda_j - \frac{1}{p}} \times$$

$$\times \prod_{j \in (e_n \setminus \Sigma) \setminus e^k \cap \{1, \ldots, s\}} \left[\tau_j(h_j)\right]^{m_j^{0,k} - \lambda_j} \prod_{j \in \Sigma^* \cap \{1, \ldots, s\}} [\tau_j(T)]^{m_j^{0,k} - \lambda_j} \prod_{j \in e^k \cap \{s+1, \ldots, n\}} \times$$

$$\times \int_0^{h_j} \left[\tau_j(v_j)\right]^{m_j^{0,k} - \lambda_j - \frac{1}{p}} \frac{d\tau_j(v_j)}{\tau_j(v_j)} \prod_{j \in e^k \cap \{1, \ldots, s\}} \int_0^{h_j} \left[\tau_j(v_j)\right]^{m_j^{0,k} - \lambda_j} \frac{d\tau_j(v_j)}{\tau_j(v_j)},$$

and for $i \neq 0$

$$W_{iks}(h; T) = \prod_{j \in (e_n \setminus \Sigma^*) \setminus e^k \cap \{s+1, \ldots, n\}} \left[\tau_j(h_j)\right]^{m_j^{i,k} - \lambda_j - \frac{1}{p}} \prod_{j \in (e_n \setminus \Sigma^*) \setminus e^k \cap \{1, \ldots, s\}} \left[\tau_j(h_j)\right]^{m_j^{i,k} - \lambda_j} \times$$

$$\times \int_0^{T} \prod_{j \in \Sigma^* \cap \{s+1, \ldots, n\}} [\tau_j(t)]^{m_j^{i,k} - \lambda_j - \frac{1}{p}} \prod_{j \in \Sigma^* \cap \{1, \ldots, s\}} [\tau_j(t)]^{m_j^{i,k} - \lambda_j} \frac{d\tau_i(t)}{\tau_i(t)} \times$$

$$\times \prod_{j \in e^k \cap \{s+1, \ldots, n\}} \int_0^{h_j} \left[\tau_j(v_j)\right]^{m_j^{i,k} - \lambda_j - \frac{1}{p}} \frac{d\tau_j(v_j)}{\tau_j(v_j)} \prod_{j \in e^k \cap \{1, \ldots, s\}} \int_0^{h_j} \left[\tau_j(v_j)\right]^{m_j^{i,k} - \lambda_j} \frac{d\tau_j(v_j)}{\tau_j(v_j)}.$$

Remark. Special cases of this theorem were examined in papers [2], [4], [6], [9].

Proof. We write the integral in the form (see [8])

$$D^{\lambda} f(x) = \sum_{k=1}^{N} C_{0k}(h;T) \prod_{j\in e^k} \int_0^{h_j} [\tau_j(v_j)]^{m_j^{0,k}-\lambda_j} \frac{d\tau_j(v_j)}{\tau_j(v_j)} \times$$

$$\times \frac{1}{\text{mes}\ (R_\delta \cap E_n)_{0k}} \int_{E_n} \prod_{j\in e_n} [b_j(x+y)]^{m_j^{0,k}-\lambda_j} \ D^{m^{0,k}} f(x+y)\ \phi_{0k\delta}\ dy +$$

$$+ \sum_{i\in\Sigma^*} \sum_{k=1}^{N} C_{ik}(h) \int_0^T \prod_{j\in\Sigma^*} [\tau_j(t)]^{m_j^{i,k}-\lambda_j} \frac{d\tau_i(t)}{\tau_i(t)} \prod_{j\in e^k} \int_0^{h_j} [\tau_j(v_j)]^{m_j^{i,k}-\lambda_j} \times$$

$$\times \frac{d\tau_j(v_j)}{\tau_j(v_j)} \frac{1}{\text{mes}\ (R_\delta \cap E_n)_{ik}} \int_{E_n} \prod_{j\in e_n} [b_j(x+y)]^{m_j^{i,k}-\lambda_j} \ D^{m^{i,k}} f(x+y)\ \phi_{ik\delta} dy. \tag{5}$$

Here

$$C_{0k}(h;T) = (-1)^{|m^{0.k}-\lambda|}\, 2^{|e^k|} \prod_{j\in(e_n\setminus\Sigma^*)\setminus e^k} [\tau_j(h_j)]^{m_j^{0,k}-\lambda_j} \prod_{j\in\Sigma^*} [\tau_j(T)]^{m_j^{0,k}-\lambda_j},$$

$$C_{ik}(h) = (-1)^{|m^{i.k}-\lambda|}\, 2^{1+|e^k|} \prod_{j\in(e_n\setminus\Sigma^*)\setminus e^k} [\tau_j(h_j)]^{m_j^{i,k}-\lambda_j},$$

$$\text{mes}\ (R_\delta \cap E_n)_{0k} = \prod_{j\in e_n} b_j(x) \prod_{j\in(e_n\setminus\Sigma^*)\setminus e^k} \tau_j(h_j) \prod_{j\in\Sigma^*} \tau_j(T) \prod_{j\in e^k} \tau_j(v_j)\ ,$$

$$\text{mes}\ (R_\delta \cap E_n)_{ik} = \prod_{j\in e_n} b_j(x) \prod_{j\in(e_n\setminus\Sigma^*)\setminus e^k} \tau_j(h_j) \prod_{j\in\Sigma^*} \tau_j(T) \prod_{j\in e^k} \tau_j(v_j)$$

($i \in \Sigma^*$, $k = 1, ..., N$) and $\phi_{0k\delta}$, $\phi_{ik\delta}$ are known kernel functions.

For convenience, we rewrite (5) as follows

$$D^{\lambda} f(x) = \sum_{k=1}^{N} Q_{0k}(x;h;T) \int_{E_n} F_{0k}(x+y)\phi_{ok\delta} dy + \sum_{i\in\Sigma^*} \sum_{k=1}^{N} Q_{ik}(x;h;T) \int_{E_n} F_{ik}(x+y)\phi_{ik\delta} dy, \tag{6}$$

where

$$Q_{0k}(x;h;T) = (-1)^{|m^{0,k}-\lambda|} 2^{|e^k|} \prod_{j\in e^k} \int_0^{\eta_j} [\tau_j(v_j)]^{m_j^{0,k}-\lambda_j-\gamma_j} \frac{d\tau_j(v_j)}{\tau_j(v_j)} \times$$

$$\times \frac{\prod_{j\in(e_n\setminus\Sigma^*)\setminus e^k} \left[\tau_j(h_j)\right]^{m_j^{0,k}-\lambda_j-\gamma_j} \prod_{j\in\Sigma^*} [\tau_j(T)]^{m_j^{0,k}-\lambda_j-\gamma_j}}{[\text{mes}\,(R_\delta \cap E_n)_{0k}]^{1-\gamma_j}},$$

$$Q_{ik}(x;h;T) = (-1)^{|m^{0,k}-\lambda|}\ 2^{1+|e^k|} \int_0^T \prod_{j\in\Sigma^*} [\tau_j(t)]^{m_j^{i,k}-\lambda_j-\gamma_j} \frac{d\tau_i(t)}{\tau_i(t)} \times$$

$$\times\int_0^h \prod_{j\in e^k}\left[\tau_j(v_j)\right]^{m_j^{i,k}-\lambda_j-\gamma_j}\frac{d\tau_j(v_j)}{\tau_j(v_j)}.\frac{\prod_{j\in(e_n\backslash\Sigma^*)\backslash e^k}\left[\tau_j(h_j)\right]^{m_j^{i,k}-\lambda_j-\gamma_j}}{\left[\text{mes}\,(R_\delta\cap E_n)_{ik}\right]^{1-\gamma_j}},$$

$$F_{ik}=D^{m^{i.k}}f\prod_{j\in e_n}[b_j]^{m_j^{i,k}-\lambda_j-\gamma_j}. \tag{7}$$

Let $G=\bigcup_{q=1}^{M}U_q$ and $U_q+R_{\delta^q}\subset G$. Thus, subset U_q satisfies the condition of "φ-semi horn". For the subset $U_q+R_{\delta^q}$ we consider the auxiliary function $F_q=F\left(D^\lambda f\,;U_q+R_{\delta^q}\right)$. This function coincides with the function D^λ in the region $\{U_q+R_{\delta^q}\}\cap\{\Gamma_s+z^{**}\}$.
Then

$$F_q=F\left(D^\lambda f;U_q+R_{\delta^q}\right)=Q_{0k}\left[P(x^*)+z^{**};h\,;T\right]\int_{E_n}\widetilde{F}_{0kq}\left[P(x^*)+z^{**}+y\right]\phi_{0k\delta^q}dy+$$

$$+\sum_{k=1}^{N}\sum_{i\in\Sigma^*}Q_{ik}\left[P(x^*)+z^{**};h;T\right]\int_{E_n}\widetilde{F}_{ikq}\left[P(x^*)+z^{**}+y\right]\phi_{ik\delta^q}dy. \tag{8}$$

where $\widetilde{F}_{ikq}=\chi(U_q+R_{\delta^q})F_{ikq}$, $\chi(U_q+R_{\delta^q})$ is the characteristic function of $U_q+R_{\delta^q}$.

If in statement (6) we put $\delta=\delta^q$ and compare with (8), then it is easy to see that the functions $D^\lambda f$ and F_q coincide at the intersection of the surface Γ_s+z^{**} and the subset $U_q+R_{\delta^q}$.

If we consider inequalities (4), then

$$\left\|\,D^\lambda\,f\mid_{\Gamma_s+z^{**}}\right\|_{L_p(\Gamma_s+z^{**})}\le C\left\|\,D^\lambda f\left(P\left(\cdot\right)+z^{**}\right)\right\|_{L_p(E_s)}\le\sum_{i\in\Sigma_0^*}\sum_{k=1}^{N}\sum_{q=1}^{M}\|F_{ikq}\|_{L_p(\mathbb{E}_s)}.$$

Here at first we evaluate $\|F_{ikq}\|_{L_p(\mathbb{E}_s)}$ and set $\delta^q=(1,...,1)$ for convenience.

Denote the vector $y=\left(y_1,...,y_s,y_{s+1},...,y_n\right)$ by $y=(y',y'')$, where $y'=(y_1,...,y_s)$ and $y''=(y_{s+1},...,y_n)$. Let us denote the set of all points y', whose coordinates satisfy the inequality $x_k\le y_k\le x_k+b_k(P(x^*)+z^{**})$, $\tau_k(\eta_k)$ $(k=1,...,s)$, by $\Omega_s^*\ (...)$, and the set of all points y'', whose coordinates satisfy the inequality

$$\psi_k\left(x^*\right)+z_k\le y_k\le z_k+\psi_k\left(x^*\right)+b_k\left(P\left(x^*\right)+z^{**}\right)\tau_k\left(\eta_k\right),k\in\{s+1,\dots,n\},$$

by $\Omega_{n-s}^{**}\ (...)$. Here

$$\text{mes}\,\Omega_s^*(...)=\prod_{j\in\{1,...,s\}}b_j(P(x^*)+z^{**})\tau_j(\eta_j),\text{mes}\,\Omega_{n-s}^{**}(...)=\prod_{j\in\{s+1,...,n\}}b_j(P(x^*)+z^{**})\tau_j(\eta_j).$$

Since the functions b_j and the see G satisfy condition (B) and have the finite property of the kernel function,

$$\left|\widetilde{F}_{ikq}\right|\le C\prod_{j\in(e_n\backslash\Sigma^*)\backslash e^k}\left[\tau_j(h_j)\right]^{m_j^{i,k}-\lambda_j-\gamma_j}\int_0^T\prod_{j\in\Sigma^*}\left[\tau_j(t)\right]^{m_j^{i,k}-\lambda_j-\gamma_j}\frac{d\tau_i(t)}{\tau_i(t)}\prod_{j\in e^k}\int_0^{h_j}\left[\tau_j(v_j)\right]^{m_j^{i,k}-\lambda_j-\gamma_j}\times$$

$$\times\frac{d\tau_j(v_j)}{\tau_j(v_j)}\,\frac{1}{\left\{\text{mes}\,\Omega_s^*\,(...)\,\text{mes}\,\Omega_{n-s}^{**}\,(...)\right\}^{1-\gamma_j}}\int_{\Omega_s^*(...)}dy'\int_{\Omega_{n-s}^*(...)}\left|F_{ikq}(y';y'')\right|\,dy''. \tag{9}$$

If we use the Hölder inequality for the last integral of inequality (9), then we obtain

$$\int\limits_{\Omega^*_{n-s}(...)} |F_{ikq}(y';y'')| \; dy'' \le C \, \|F_{ikq}(y';.)\|_{L_p(E_{n-s})} \left\{\operatorname{mes}\Omega^{**}_{n-s}(...)\right\}^{1-\frac{1}{p}}.$$

Hence,

$$\left|\widetilde{F}_{ikq}\right| \le C \prod_{j\in(e_n\setminus\Sigma^*)\setminus e^k} \left[\tau_j(h_j)\right]^{m_j^{i,k}-\lambda_j-\gamma_j} \int\limits_0^T \prod_{j\in\Sigma^*} \left[\tau_j(t)\right]^{m_j^{i,k}-\lambda_j-\gamma_j} \frac{d\tau_i(t)}{\tau_i(t)} \times$$

$$\times \int\limits_0^h \prod_{j\in e^k} \left[\tau_j(v_j)\right]^{m_j^{i,k}-\lambda_j-\gamma_j} \frac{d\tau_j(v_j)}{\tau_j(v_j)} \cdot \frac{1}{\operatorname{mes}\Omega^*_s(...)} \int\limits_{\Omega^*_s(...)} \|F_{kq}(y';.)\|_{L_p(E_{n-s})} \; dy'. \qquad (10)$$

In addition, using the Hölder inequality for the last integral of inequality (10), we have

$$\int\limits_{\Omega^*_s(...)} \left\|\widetilde{F}_{ikq}(y';.)\right\|_{L_p(E_{n-s})} dy' \le C \left\{\operatorname{mes}\Omega^*_s(...)\right\}^{1-\frac{1}{p}} \left(\int\limits_{\Omega^*_s(...)} \left\|\widetilde{F}_{ikq}(y';.)\right\|^p dy' \right)^{1/p}.$$

Then

$$\left\|\widetilde{F}_{ikq}\right\|_{L_p(E_s)} \le C W_{iks}(h) \left\{ \int\limits_{E_s} \frac{dx'}{\operatorname{mes}\Omega^*_s(...)} \int\limits_{\Omega^*_s(...)} \left\|\widetilde{F}_{ikq}(y';.)\right\|^p_{L_p(E_{n-s})} dy' \right\}^{1/p} \le$$

$$\le C W_{iks}(h;T) \; \|F_{ikq}\|_{L_p(E_n)}.$$

Similarly,

$$\left\|\widetilde{F}_{0k\;q}\right\|_{L_p(E_s)} \le C W_{0ks}\,(h;T) \, \|F_{0kq}\|_{L_p(E_n)}. \qquad (11)$$

From inequalities of (10) and (11)

$$\left\|D^\lambda f|_{\Gamma_s+z^{**}}\right\|_{L_p(\Gamma_s+z^{**})} \le C \sum_{i\in\Sigma^*_0} \sum_{k=1}^N \sum_{q=1}^M \|F_{ikq}\|_{L_p(E_n)} \le C \sum_{i\in\Sigma^*_0} \sum_{k=1}^N W_{iks}(h;T) \sum_{q=1}^M \|F_{ikq}\|_{L_p(E_n)} \le$$

$$\le C \sum_{i\in\Sigma^*_0} \sum_{k=1}^N W_{iks}(h;T) \left\| \prod_{j\in e^n} [b_j]^{m_j^{i,k}-\lambda_j-\gamma_j} D^{m^{i,k}} f \right\|_{L_p(G)}. \qquad (12)$$

The constants $C>0$, f, H_j and T_0 are independent. Passing in inequality (12) to the limit as $|z^{**}|\to 0$, we obtain the following estimate

$$\left\|D^\lambda f|_{\Gamma_s}\right\|_{L_p(G)} \le \lim_{|z^{**}|\to 0} \left\|D^\lambda f|_{\Gamma_s+z^{**}}\right\|_{L_p(\Gamma_s+z^{**})} \le C \sum_{i\in\Sigma^*_0} \sum_{k=1}^N W_{iks}(h;T) ||g_{i,k} D^{m^{i,k}} f||_{L_p(G)}.$$

This completes the proof of the theorem. □

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