УДК 517.927

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ON THE BOUNDARY VALUE PROBLEM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH DISCRETE-CONTINUOUS DISTRIBUTION OF PARAMETERS

V. Mazurenko, M. Stasyuk, R. Tatsiy. *On the boundary value problem for a system of ordinary differential equations with discrete-continuous distribution of parameters*, Mat. Stud. **31** (2009), 65–74.

We establish necessary and sufficient conditions for the existence of solutions of the boundary value problem for a system of differential equations with discrete-continuous distribution of parameters. Representations for solutions in integral form and in the Schmidt form are obtained. Besides, the structure of the Green matrix and its properties are investigated.

В. Мазуренко, М. Стасюк, Р. Таций. *О граничной задаче для системы обыкновенных дифференциальных уравнений с дискретно-непрерывным распределением параметров* // Мат. Студії. – 2009. – Т.31, №1. – C.65–74.

Установлены необходимые и достаточные условия существования решений граничной задачи для системы дифференциальных уравнений с дискретно-непрерывным распределением параметров. Получены представления решений в интегральной форме и в форме Шмидта. Исследована структура матрицы Грина и ее свойства.

Introduction. Investigations of the varied physical phenomena, which take into account natural unity of discrete and continuous, result to the necessity of creation of adequate mathematical models. Many of them are described by differential and quasi-differential (q.-d.) equations with generalized functions in the coefficients and in the right-hand side part (see [1, 2]). Boundary value problems for differential equations with the divisions in coefficients are successfully studied by mathematicians and mechanics for a long time. This subject got the substantial shove for development due to fundamental works of M. G. Krein (see [3, p. 648]) in relation to differential equations of the second order, which describe free vibrations of a string, the mass of which assumes except for continuous yet and the point division. Before introducting of the concept of δ-functions, points of singularities appeared in problems in the form of specific conditions of conjugate for the solution and its derivative in points, which in view of the modern theory, belong to the singular support of the coefficients of the equation. Such researches mostly had partial character, because they concern the equation of a concrete kind.

In papers [4, 5] the authors established existence and uniqueness of a solution and studied spectral properties of a wide class of correct (in research of which the problem of multiplication of functionals does not arise up) boundary value problems for q.-d. equations of any finite order. In [6] the authors obtained an analogue of Fredholm's alternative for a system

2000 *Mathematics Subject Classification:* 34B05, 34L05, 34A37.
Keywords: boundary value problem, system of ordinary differential equations, discrete-continuous distribution of parameter
doi:10.30970/ms.31.1.65-74

of the first order, and in [7] the Green matrix for a system of q.-d. equations of fourth order is constructed. Some results from this paper can be generalized to systems of equations of higher orders.

1. Preliminaries. By $\mathbb{C}^{p\times q}$ we denote the linear space of complex matrices with p rows and q columns (s.c. $p\times q$-matrices), and by $\overline{\mathcal{D}}\,(I)$ (I stands for the interval of $\mathbb{R}$) the space of continuous functions $I\to\mathbb{C}^p$ with a compact support. The conjugate space to $\overline{\mathcal{D}}\,(I)$ is the space $\overline{\mathcal{D}}'(I)$ of the vector distributions.

Let $AC_p[a,b]$, $L_p[a,b]$ and $L^2_p[a,b]$ be the spaces of matrix functions $F\colon[a,b]\to\mathbb{C}^{p\times p}$ with absolutely continuous, Lebesgue integrable and Lebesgue module square-integrable on the interval $[a,b]$ elements $f_{ij}(x)$, respectively, and let $BV^+_p[a,b]$ be the space of function matrices with the elements $f_{ij}(x)$, where $f_{ij}(x)$ are right continuous on the interval $[a,b)$ scalar functions of bounded variation such that $f_{ij}(b-0)=f_{ij}(b)$. Respectively, spaces of p-vector functions are denoted with upper line.

Let 0 be zero element (matrix, vector or number), E_p an identity matrix of order p, τ the symbol of transposition, $\Delta F(x)=F(x)-F(x-0)$ the jump of the function $F\in BV^+_p[a,b]$ at the point $x\in[a,b]$, (f,φ) the value of the functional f at the function $\varphi(x)$.

2. Statement of a problem. We consider the quasi-differential expression of any odd or even order m with matrix coefficients:

$$l_m[y]=(-1)^l\left\{iB(x)\big[B(x)y^{(l)}\big]^{(m-2l)}\right\}^{(l)}+\sum_{r,s=0}^{l}(-1)^{l-s}\left(B_{rs}(x)y^{(l-r)}\right)^{(l-s)},\tag{1}$$

where $y\colon[a,b]\to\mathbb{C}^p$, $l,m\in\mathbb{N}$, $l\le\left[\frac{m}{2}\right]$, i is the imaginary unit. Furthermore, for $m=2n$ we mean (formally) $l=n$, $B\equiv0$, so $l_{2n}[y]=\sum_{r,s=0}^{n}(-1)^{n-s}\left(B_{rs}(x)y^{(n-r)}\right)^{(n-s)}$.

Q.-d. expressions of type (1) with sufficiently smooth and Lebesgue integrable $(p\times p)$-matrix coefficients were investigated by various authors. In fact, [8] contains an elementary theory and a spectral analysis of differential operators generated by q.-d. expressions of type (1) in the case when $m=2n$, $p=1$, $B_{rs}\equiv0$ $(r\ne s)$. The case when in (1) m is any finite number, $l=n$, $B_{rs}\equiv0$ for all r,s such that $r\ne s+\{0,\pm1\}$, for $p=1$ is considered in [9], and for $p>1$ is considered in [10]. More generally, q.-d. expressions are considered in [11, 12]. A brief review of other papers in this direction of domestic and foreign authors can be found in [8].

In this paper we weaken requirements on coefficients of q.-d. expression (1). We assume that the following holds:

(A) if $m=2n$, then $B^{-1}_{00}(x)$ is bounded and measurable for $x\in[a,b]$ matrix, $B_{r0},B_{0s}\in L^2_p[a,b]$, $B_{rs}=A'_{rs}$, $A_{rs}\in BV^+_p[a,b]$ $\forall\,r,s\in\{1,\dots,n\}$; in the other case $B^{-1}(x)$ is bounded and measurable for $x\in[a,b]$, and $B_{00},B_{r0},B_{0s}\in L_p[a,b]$, $B_{rs}=A'_{rs}$, $A_{rs}\in BV^+_p[a,b]$ $\forall\,r,s\in\{1,\dots,n\}$ if $l=n$, and $B_{rs}=A'_{rs}$, $A_{rs}\in BV^+_p[a,b]$ $\forall\,r,s\in\{0,\dots,l\}$ if $l\ne n$;

(B) $B^*_{j0}=B_{0j}$, $A^*_{rs}=A_{sr}$, where $j\in\in\{0,\dots,n\}$, $r,s\in\{1,\dots,n\}$ if $l=n$, and $A_{rs}=A^*_{sr}$, where $r,s\in\{1,\dots,n\}$ if $l\ne n$. Moreover, if $m=2n+1$, then $B^*=B$.

Note that the existence and the uniqueness of solutions of initial problems for q.-d. expressions with matrix coefficients and also the elements of linear theory of corresponding q.-d. equations are given in [13].

On a finite interval $[a,b]$ we consider the two-point boundary value problem

$$l_m[y] - \lambda\omega(x)y = \sum_{j=0}^{m-l-1} (-1)^{j+1} f_j^{(j+1)}(x), \tag{2}$$

$$U_k(y) \equiv \sum_{\nu=1}^{m} \left[P_{k\nu} y^{[\nu-1]}(a) + Q_{k\nu} y^{[\nu-1]}(b)\right] = 0, \quad k \in \{1,\dots,m\}; \tag{3}$$

where λ is the spectral parameter, $\omega(x)$ and $f_j(x)$ satisfy the condition

(C) $\omega = \sigma'$, $\sigma \in BV_p^+[a,b]$, $\sigma^* = \sigma$, $f_j \in \overline{BV}_p^+[a,b]$, $j \in \{0,\dots,m{-}l{-}1\}$,

and $U_k(y)$ are linearly independent boundary value forms with given numerical matrices $P_{k\nu}, Q_{k\nu}$ $(k,\nu \in \{1,\dots,m\})$ and quasi-derivatives $y^{[k]}(x)$ $(k \in \{0,\dots,m\})$, that are defined by the expressions (for $m = 2n$ here it is necessary to assume $l = n$, $B \equiv 0$, $f_n \equiv 0$):

$$\begin{aligned}
&y^{[k]} = y^{(k)},\ k \in \{0,\dots,l{-}1\}; \quad y^{[l+k-1]} = -\tfrac{1+i}{\sqrt{2}} B(x) y^{(l+k-1)},\ k \in \{1,\dots,m{-}2n\};\\
&y^{[l+k]} = -\left(y^{[l+k-1]}\right)' + f'_{m-l-k},\ k \in \{1,\dots,2(n{-}l)\};\\
&y^{[m-l]} = -\tfrac{1+i}{\sqrt{2}} B(x)\left(y^{[m-l-1]}\right)' + \sum_{r=0}^{l} B_{r0}(x) y^{(l-r)} + f'_l;\\
&y^{[m-l+k]} = -\left(y^{[m-l+k-1]}\right)' + \sum_{r=0}^{l} B_{rk}(x) y^{(l-r)} + f'_{l-k},\ k \in \{1,\dots,l\}.
\end{aligned} \tag{4}$$

Definition 1. By a *solution of boundary value Problem* (2), (3) we understand a function $y \in \overline{AC}_p[a,b]$ such that $U_k(y) = 0$, $k \in \{1,\dots,m\}$ and the following identity holds

$$\left(y^{[m]} - \lambda\omega y,\ \varphi\right) = 0 \quad \forall\varphi \in \overline{\mathcal{D}}(I), \quad I \supseteq [a,b].$$

If $f_j(x) \equiv const$ $(j \in \{0,\dots,m{-}l{-}1\})$, then Problem (2), (3) becomes

$$l_m[y] = \lambda\omega(x)y, \quad U_k(y) = 0,\ k \in \{1,\dots,m\}. \tag{5}$$

Values of λ for which Problem (5) has nontrivial solutions, are eigenvalues for this problem, and these solutions are respective eigenvectors.

3. Method of research. The main technical idea in the paper consists of the replacement of a boundary value problem for a system of equations of a high order with any equivalent problem for the system of equations of the first order. The advantage of such a method is obvious, it is the presence of first derivatives only in the system, and it enables to reduce the analysis of singularities to the question on a location of jumps of some functions which derivatives generated by these singularities.

Lemma 1. *Let conditions* (A)–(C) *and the following ratio hold*

$$\begin{aligned}
&\sum_{j=1}^{l}\left[P_{kj}P^*_{\nu,m-j+1} - P_{k,m-j+1}P^*_{\nu j}\right] + i\sum_{j=l+1}^{n}(-1)^{j-l}\left[P_{kj}P^*_{\nu,m-j+1} + P_{k,m-j+1}P^*_{\nu j}\right] +\\
&+ i(-1)^{n-l+1}P_{k,n+1}P^*_{\nu,n+1} = \sum_{j=1}^{l}\left[Q_{kj}Q^*_{\nu,m-j+1} - Q_{k,m-j+1}Q^*_{\nu j}\right] +\\
&+ i\textstyle\sum_{j=l+1}^{n}(-1)^{j-l}\left[Q_{kj}Q^*_{\nu,m-j+1} + Q_{k,m-j+1}Q^*_{\nu j}\right] + i(-1)^{n-l+1}Q_{k,n+1}Q^*_{\nu,n+1}
\end{aligned} \tag{6}$$

(if $m = 2n$ then in (6) *it is necessary to assume that $l=n$, $P_{k,n+1}=Q_{k,n+1}=0$, $k \in \{1,\dots,2n\}$). Then Problem* (2), (3) *is equivalent to the following problem:*

$$\mathcal{J}\mathcal{Y}' = [\mathcal{B}'(x) + \lambda\mathcal{A}'(x)]\mathcal{Y} + \mathcal{F}'(x), \quad a \le x \le b, \tag{7}$$

$$\mathcal{Y}(a) = \mathcal{M}v,\ \mathcal{Y}(b) = \mathcal{N}v, \quad \mathcal{M}^*\mathcal{J}\mathcal{M} = \mathcal{N}^*\mathcal{J}\mathcal{N}, \tag{8}$$

where $\mathcal{J},\mathcal{M},\mathcal{N} \in \mathbb{C}^{mp\times mp}$, $\mathcal{Y}\colon [a,b] \to \mathbb{C}^{mp}$ is an unknown vector, $\mathcal{B},\mathcal{A} \in BV^+_{mp}[a,b]$, $\mathcal{F} \in \overline{BV}^+_{mp}[a,b]$, $v \in \mathbb{C}^{mp}$ a parameter, $\operatorname{rank}(\mathcal{M}\,|\,\mathcal{N}) = mp$, moreover, $\mathcal{J}^* = -\mathcal{J}$, $\mathcal{J}^*\mathcal{J} = E$, $\mathcal{B}^* = \mathcal{B}$, $\mathcal{A}^* = \mathcal{A}$.

Proof. System (2) reduces to the form (7) by means of the vector $Y = \big(y, y^{[1]}, \dots, y^{[m-1]}\big)^\tau$ that consists of quasi-derivatives (4). Thus, the vector

$$\mathcal{F} = \Big(-f_0, \dots, -f_{l-1}, \tfrac{1+i}{\sqrt{2}}\int B^{-1}df_l, -if_{l+1}, \dots, (-1)^{n-l}if_n, \dots, if_{m-l-1}, \underbrace{0,\dots,0}_{l}\Big)^\tau$$

and matrices $\mathcal{J}$, $\mathcal{A}(x)$, $\mathcal{B}'(x)$ of order mp have the block structure (we describe non-zero $p\times p$-blocks only $\widehat{J}_{k\nu}$, $\widehat{A}_{k\nu}$, $\widehat{B}_{k\nu}$ $k,\nu \in \{1,\dots,m\}$ of each of the matrices) as follows:

$$\widehat{J}_{k,m-k+1} = \begin{cases} -E_p, & k \in \{1,\dots,l\};\\ (-1)^{m-l-k}iE_p, & k \in \{l+1,\dots,m-l\};\\ E_p, & k \in \{m-l+1,\dots,m\}; \end{cases} \qquad \widehat{A}_{11} = \sigma;$$

for even m

$$\widehat{B}_{k\nu} = \begin{cases} B_{0,n-k+1}B_{00}^{-1}B_{n-\nu+1,0} - B_{n-\nu+1,n-k+1}, & k,\nu \in \{1,\dots,n\};\\ -B_{0,n-k+1}B_{00}^{-1}, & k \in \{1,\dots,n\},\ \nu = n+1;\\ -B_{00}^{-1}B_{n-\nu+1,0}, & k = n+1,\ \nu \in \{1,\dots,n\};\\ -B_{00}^{-1}, & k = \nu = n+1;\\ E_p, & k \in \{2,\dots,m\},\ k \ne n+1,\ \nu = m-k+2; \end{cases}$$

for odd m

$$\widehat{B}_{k\nu} = \begin{cases} -B_{l-\nu+1,l-k+1}, & k,\nu \in \{1,\dots,l\};\\ \frac{(1-i)}{\sqrt{2}}B_{0,l-k+1}B^{-1}, & k \in \{1,\dots,l\},\ \nu = l+1;\\ \frac{(1+i)}{\sqrt{2}}B_{l-\nu+1,0}, & k = l+1,\ \nu \in \{1,\dots,l\};\\ -B^{-1}B_{00}B^{-1}, & k = \nu = l+1;\\ -\frac{(1+i)}{\sqrt{2}}B^{-1}, & k = l+1,\ \nu = m-l+1;\\ -\frac{(1-i)}{\sqrt{2}}B^{-1}, & k = m-l+1,\ \nu = l+1;\\ E_p, & k \in \{2,\dots,m\},\ k \ne l+1, m-l+1,\ \nu = m-k+2. \end{cases}$$

Therefore, conditions (A) and (C) imply that $\mathcal{B},\mathcal{A} \in BV^+_{mp}[a,b]$, $\mathcal{F} \in \overline{BV}^+_{mp}[a,b]$. It is easy to see that $\mathcal{J}^* = -\mathcal{J}$, $\mathcal{J}^*\mathcal{J} = E$ and, by condition (B), equalities $\mathcal{B}^* = \mathcal{B}$, $\mathcal{A}^* = \mathcal{A}$ are valid.

Conditions (3) are reduced to the form (8) by means of matrices $\mathcal{M} = \mathcal{J}\mathcal{P}^*, \mathcal{N} = -\mathcal{J}\mathcal{Q}^*$, where $\mathcal{P}$, $\mathcal{Q}$ are block matrices of order mp that consist of coefficients $P_{k\nu}$, $Q_{k\nu}$ of the boundary value forms $U_k(y)$. Thus, we have

$$\mathcal{M}^*\mathcal{J}\mathcal{M} - \mathcal{N}^*\mathcal{J}\mathcal{N} = \mathcal{P}\mathcal{J}^*\mathcal{J}\mathcal{J}\mathcal{P}^* - \mathcal{Q}\mathcal{J}^*\mathcal{J}\mathcal{J}\mathcal{Q}^* = \mathcal{P}\mathcal{J}\mathcal{P}^* - \mathcal{Q}\mathcal{J}\mathcal{Q}^* = 0$$

Here we use equality (6), which, taking into account the structure of the matrix $\mathcal{J}$, is equivalent to the condition $\mathcal{P}\mathcal{J}\mathcal{P}^* = \mathcal{Q}\mathcal{J}\mathcal{Q}^*$. □

Remark 1. The correctness of the definition of a solution (see [14]) of Problem (2), (3) follows from conditions (A), (C) and the equations

$$\Big\{\mathcal{J}\big[\Delta\mathcal{B}(x) + \lambda\Delta\mathcal{A}(x)\big]\Big\}^2 = 0, \quad \big[\Delta\mathcal{B}(x) + \lambda\Delta\mathcal{A}(x)\big]\mathcal{J}\Delta\mathcal{F}(x) = 0, \tag{9}$$

which (as it is easy to verify) are true for any $\lambda \in \mathbb{C}$, $x \in [a, b]$.

4. Main results. From now on, we assume that conditions (A)-(C) and (6) hold.

Theorem 1. *Let $\sigma(x)$ be a non-decreasing with respect to $x \in [a, b]$ and non-constant matrix. Thus, Problem (5) has, at most, countable quantity of real eigenvalues λ_k, $k \in \{1, 2, \ldots\}$ that have no finite accumulation point. Eigenvectors $y(x, \lambda_k)$, $k \in \{1, 2, \ldots\}$, which correspond to distinct eigenvalues, are σ-orthogonal in the sense that*

$$\int_a^b y^*(x, \lambda_k) d\sigma(x) y(x, \lambda_\nu) = 0, \quad \lambda_k \neq \lambda_\nu. \tag{10}$$

Proof. Let $y(x, \lambda_k)$ be an eigenvector of boundary value Problem (5) which corresponds to an eigenvalue λ_k. By Lemma 1 we have that

$$\mathcal{J}\mathcal{Y}_k' = [\mathcal{B}'(x) + \lambda_k\mathcal{A}'(x)]\mathcal{Y}_k, \tag{11}$$

$$\mathcal{Y}_k(a) = \mathcal{M}v_k,\ \mathcal{Y}_k(b) = \mathcal{N}v_k, \quad \mathcal{M}^*\mathcal{J}\mathcal{M} = \mathcal{N}^*\mathcal{J}\mathcal{N}, \tag{12}$$

where $\mathcal{Y}_k(x) = \Big(y(x, \lambda_k), y^{[1]}(x, \lambda_k), \ldots, y^{[m-1]}(x, \lambda_k)\Big)^\tau$. By correctness conditions (9), the products $\mathcal{Y}_k^*\mathcal{J}\mathcal{Y}_\nu'$ and $\mathcal{Y}_k^{*\prime}\mathcal{J}\mathcal{Y}_\nu$ exist in the sense of the generalized functions theory and, moreover, the following equation holds

$$\begin{aligned}\big(\mathcal{Y}_k^*\mathcal{J}\mathcal{Y}_\nu\big)' &= \mathcal{Y}_k^*\mathcal{J}\mathcal{Y}_\nu' + \mathcal{Y}_k^{*\prime}\mathcal{J}\mathcal{Y}_\nu = \mathcal{Y}_k^*\big(\mathcal{J}\mathcal{Y}_\nu'\big) - \big(\mathcal{J}\mathcal{Y}_k'\big)^*\mathcal{Y}_\nu = \\ &= \mathcal{Y}_k^*[\mathcal{B}' + \lambda_\nu\mathcal{A}']\mathcal{Y}_\nu - \mathcal{Y}_k^*[\mathcal{B}^{*\prime} + \overline{\lambda}_k\mathcal{A}^{*\prime}]\mathcal{Y}_\nu = (\lambda_\nu - \overline{\lambda}_k)\mathcal{Y}_k^*\mathcal{A}^{*\prime}\mathcal{Y}_\nu,\end{aligned}$$

thus,

$$\mathcal{Y}_k^*(b)\mathcal{J}\mathcal{Y}_\nu(b) - \mathcal{Y}_k^*(a)\mathcal{J}\mathcal{Y}_\nu(a) = (\lambda_\nu - \overline{\lambda}_k)\int_a^b \mathcal{Y}_k^*(x) d\mathcal{A}(x)\mathcal{Y}_\nu(x).$$

Here equalities (11) and the properties of matrices $\mathcal{J}$, $\mathcal{A}(x)$, $\mathcal{B}(x)$ are used. Taking into account conditions (12), the structure of the matrix $\mathcal{A}(x)$ and of the eigenvector $\mathcal{Y}_k(x)$, we obtain

$$(\lambda_\nu - \overline{\lambda}_k)\int_a^b y^*(x, \lambda_k) d\sigma(x) y(x, \lambda_\nu) = 0. \tag{13}$$

Let $\lambda_\nu = \lambda_k$. Since $\sigma(x)$ is not decreasing with respect to $x \in [a, b]$ and is a non-constant matrix, for any eigenvector $y(x, \lambda_k)$ one has that $\int_a^b y^*(x, \lambda_k) d\sigma(x) y(x, \lambda_k) > 0$. Hence, (13) yields $\lambda_k = \overline{\lambda}_k$, that is, all eigenvalues of Problem (5) are real.

For $\lambda_\nu \neq \lambda_k = \overline{\lambda}_k$ condition (13) implies the σ-orthogonality (10) of eigenvectors $y(x,\lambda_k)$ and $y(x,\lambda_\nu)$. The eigenvalues of the problem are zeros of the characteristic determinant $\Delta(\lambda) = \det\big[U_k(Y_\nu)\big]_{k,\nu=1}^{m}$, where $Y_1(x,\lambda),\dots,Y_m(x,\lambda)$ is the fundamental system of solutions of the operator equation (see [8, p. 110])

$$l_m[Y] - \lambda\omega(x)Y = 0, \quad Y:[a,b]\to\mathbb{C}^{p\times p}, \tag{14}$$

In view of a theorem from [15], these solutions are entire functions dependent on the parameter λ, therefore, $\Delta(\lambda)$ is also an entire function. It has just been shown that this function has no non-real zero, thus, it does not vanish identically. Thus, the set of zeros of this function has no finite accumulation point. □

Remark 2. The order of any eigenvalue λ of boundary value Problem (5) coincides with its order as a root of the characteristic equation $\Delta(\lambda) = 0$.

Remark 3. The eigenvectors, which correspond to the same eigenvalue, can be also chosen to be orthogonal, applying the process of orthogonality (see, [3, p. 298]).

Remark 4. If $\sigma(x)$ is the staircase matrix-function with a finite set of points of discontinuity, that is, the matrix $\omega = \sigma'$ has a singularity of impulse type, then Problem (5) has a finite set of eigenvalues and its corresponding eigenvectors.

Theorem 2. *If λ is not an eigenvalue of Problem* (5), *then non-homogeneous Problem* (2), (3) *has a unique solution $y \in \overline{AC}_p[a,b]$, that is, represented as*

$$y(x) = \sum_{j=0}^{m-l-1}\int_a^b \frac{\partial^j G(x,t,\lambda)}{\partial t^j}df_j(t). \tag{15}$$

Proof. We will construct solutions of the problems

$$l_m[y] - \lambda\omega(x)y = (-1)^{j+1}f_j^{(j+1)}(x),\ U_k(y) = 0, \quad k\in\{1,\dots,m\} \tag{16}$$

for $j\in\{0,\dots,m-l-1\}$ and summing them.

Let the function $\mathcal{K}\colon [a,b]\times[a,b]\to\mathbb{C}^{p\times p}$ of a variable x be a solution of operators equation (14) such that $\mathcal{K}_x^{[k]}(x,t)\Big|_{x=t} = 0, k\in\{0,\dots,m-2\}$, $\mathcal{K}_x^{[m-1]}(x,t)\Big|_{x=t} = E_p$, $t\in[a,b]$. Then the general solution of system (16) is of the form

$$y_j(x) = \begin{cases} \sum\limits_{r=1}^{m}\mathcal{K}^{*\{r-1\}*}(x,a,\lambda)c_r^j + \int_a^x \mathcal{K}^{*\{j\}*}(x,t,\lambda)df_j(t), \ j\neq l; \\ \sum\limits_{r=1}^{m}\mathcal{K}^{*\{r-1\}*}(x,a,\lambda)c_r^j + \frac{1-i}{\sqrt{2}}\int_a^x \mathcal{K}^{*\{l\}*}(x,t,\lambda)B^{-1}(t)df_l(t), \ j = l; \end{cases} \tag{17}$$

symbol $\{\cdot\}$ means the quasi-derivative in the sense of equation conjugated to (14) [13, p. 50]. Taking into account the expressions for the quasi-derivatives

$$\begin{aligned} &Y^{\{k\}} = Y^{(k)},\ k\in\{0,\dots,l-1\};\quad Y^{\{l+k-1\}} = \tfrac{1-i}{\sqrt{2}}B^*(x)Y^{(l+k-1)};\ k\in\{1,\dots,m-2n\}\\ &Y^{\{l+k\}} = \left(Y^{\{l+k-1\}}\right)',\ k\in\{1,\dots,2(n-l)\};\\ &Y^{\{m-l\}} = \tfrac{1-i}{\sqrt{2}}B^*(x)\left(Y^{\{m-l-1\}}\right)' - \sum_{s=0}^{l}B_{0s}^*(x)Y^{(l-s)};\\ &Y^{\{m-l+k\}} = -\left(Y^{\{m-l+k-1\}}\right)' - \sum_{s=0}^{l}B_{ks}^*(x)Y^{(l-s)},\ k\in\{1,\dots,l\} \end{aligned} \tag{18}$$

we rewrite (17) in the form

$$y_j(x) = \sum_{r=1}^{m} \mathcal{K}^{*\{r-1\}*}(x,a,\lambda)c_r^j + \int_a^x \mathcal{K}^{*<j>*}(x,t,\lambda)df_j(t); \tag{19}$$

here the symbol $<\cdot>$ means the ordinary derivative $(\cdot)$ for $j \in \{0,\dots,l\}$ and the quasi-derivative $\{\cdot\}$ for $j \in \{l{+}1,\dots,m{-}l{-}1\}$.

The solution (19) contains m unknown numerical vectors. For their searching, it is necessary to satisfy boundary value conditions (3). Thus, we get the system of matrix equations

$$\sum_{r=1}^{m} U_k\Big(\mathcal{K}^{*\{r-1\}*}(x,a,\lambda)\Big)c_r^j + \sum_{\nu=1}^{m} Q_{k\nu}\int_a^b \mathcal{K}^{[\nu-1]*<j>*}(b,t,\lambda)df_j(t) = 0.$$

Since, by the assumptions of the theorem, λ is not an eigenvalue, the determinant of this system is not equal to zero. Thus, vectors c_r^j, $r \in \{1,\dots,m\}$ are searched immediatelly from the system.

By $V_{kr}(\lambda)$ we denote the matrix of order p, consisting of the algebraic complements to the elements of the matrix $U_k\Big(\mathcal{K}^{*\{r-1\}*}(x,a,\lambda)\Big)$, $k,r \in \{1,\dots,m\}$ in the determinant $\Delta(\lambda)$. Let $W_{kr}(\lambda) = -V_{kr}(\lambda)^{\tau}$. Then for $r \in \{1,\dots,m\}$ we have that

$$c_r^j = \sum_{k,\nu=1}^{m} \frac{W_{kr}(\lambda)Q_{k\nu}}{\Delta(\lambda)} \int_a^b \mathcal{K}^{[\nu-1]*<j>*}(b,t,\lambda)df_j(t).$$

We put these values to formula (17). We call the expression

$$G_j(x,t,\lambda) = \begin{cases} \sum\limits_{r,k,\nu=1}^{m} \mathcal{K}^{*\{r-1\}*}(x,a,\lambda)\dfrac{W_{kr}(\lambda)Q_{k\nu}}{\Delta(\lambda)}\mathcal{K}^{[\nu-1]*<j>*}(b,t,\lambda), \ x<t; \\ \sum\limits_{r,k,\nu=1}^{m} \mathcal{K}^{*\{r-1\}*}(x,a,\lambda)\dfrac{W_{kr}(\lambda)Q_{k\nu}}{\Delta(\lambda)}\mathcal{K}^{[\nu-1]*<j>*}(b,t,\lambda) + \mathcal{K}^{*<j>*}(x,t,\lambda), \ x \ge t \end{cases}$$

the *Green matrix* of boundary value Problem (16). Taking into account the structure of this matrix, we, obviously, obtain the following its property:

$$G_j(x,t,\lambda) = \frac{\partial^j G_0(x,t,\lambda)}{\partial t^j} \equiv \frac{\partial^j G(x,t,\lambda)}{\partial t^j} \quad \forall j \in \{0,\dots,l\}. \tag{20}$$

By (18), we have $\mathcal{K}^{*\{l+k\}*}(x,t,\lambda) = \frac{\partial^k \mathcal{K}^{*\{l\}*}(x,t,\lambda)}{\partial t^k}$, $k \in \{1,\dots,m{-}2l{-}1\}$, thus, property (20) is also true for $j \in \{l{+}1,\dots,m{-}l{-}1\}$, that completes the proof. □

Remark 5. One could prove this statement similar to that in the proof of Theorem A from [6] whence the solution of boundary value Problem (7), (8), with the condition that λ is not an eigenvalue, is of the form

$$\mathcal{Y}(x) = \int_a^b \mathcal{R}(x,t,\lambda)d\mathcal{F}(t), \tag{21}$$

where the solution kernel

$$\mathcal{R}(x,t,\lambda) = \begin{cases} \Phi(x,a,\lambda)\mathcal{M}[\Phi(b,a,\lambda)\mathcal{M}-\mathcal{N}]^{-1}\Phi(b,t,\lambda)\mathcal{J}, & x<t, \\ \Phi(x,a,\lambda)\mathcal{M}[\Phi(b,a,\lambda)\mathcal{M}-\mathcal{N}]^{-1}\Phi(b,t,\lambda)\mathcal{J} - \Phi(x,t,\lambda)\mathcal{J}, & x \ge t \end{cases}$$

is Hermitian in the sense that $\mathcal{R}(x,t,\lambda)=\mathcal{R}^*(t,x,\overline{\lambda})$ if $x\neq t$. But in order to obtain a concrete form for the solution, the way we have choose, seems to be more constructive. Nevertheless, comparing forms (16) and (21), it is easy to realize that first $(p\times mp)$-block row of the kernel $\mathcal{R}(x,t,\lambda)$ has the following form:

$$\Big(-G,...,-\frac{\partial^{l-1}G}{\partial t^{l-1}},\frac{1-i}{\sqrt{2}}\frac{\partial^{l}G}{\partial t^{l}}B^{-1},i\frac{\partial^{l+1}G}{\partial t^{l+1}},...,(-1)^{n-l+1}i\frac{\partial^{n}G}{\partial t^{n}},...,-i\frac{\partial^{m-l-1}G}{\partial t^{m-l-1}},\underbrace{0,...,0}_{l}\Big).$$

Theorem 3. *The Green matrix $G_j(x,t,\lambda)$ has the following properties:*

1) *$G_j(x,t,\lambda)$ on each of the intervals $[a,t)$ and $(t,b]$ with respect to the variable x is a solution of boundary value Problem* (5)*;*
2) *$G_j(x,t,\lambda)$ and its quasi-derivatives with respect to the variable x, up to the $(m-l-1)$-th order inclusive, are joint continuous functions by totality of variables, absolutely continuous with respect to each variables and entire functions of the parameter λ;*
3) *$G_j(x,t,\lambda)$ on the diagonal $x=t$ satisfies:*

$$G_j^{[q]}(t+0,t,\lambda)-G_j^{[q]}(t-0,t,\lambda)=0,\ q\in\{0,\dots,m-l-1\};$$

$$G_j^{[m-l-1+q]}(t+0,t,\lambda)-G_j^{[m-l-1+q]}(t-0,t,\lambda)=$$

$$=\sum_{r,k,\nu=1}^{m}\sum_{s=0}^{l-1}\Delta A_{l-s,q}(t)\mathcal{K}^{*\{r-1\}*[s]}(t,a,\lambda)\frac{W_{kr}(\lambda)Q_{k\nu}}{\Delta(\lambda)}K^{[\nu-1]*<j>*}(b,t,\lambda)+\Theta_{j+q,l},$$

$$q\in\{1,\dots,l-1\},\ \Theta_{k\nu}=\begin{cases}0,\ k\neq\nu;\\ E_p,\ k=\nu;\end{cases}$$

$$G_j^{[m-1]}(t+0,t,\lambda)-G_j^{[m-1]}(t-0,t,\lambda)=$$

$$=\sum_{r,k,\nu=1}^{m}\Big\{\sum_{s=0}^{l-1}\Delta A_{l-s,l}(t)\mathcal{K}^{*\{r-1\}*[s]}(t,a,\lambda)-\lambda\Delta\sigma(t)\mathcal{K}^{*\{r-1\}*}(t,a,\lambda)\Big\}\times$$

$$\times\frac{W_{km}(\lambda)Q_{k\nu}}{\Delta(\lambda)}K^{[\nu-1]*<j>*}(b,t,\lambda)+\Theta_{j0};$$

4) *$G_j(x,t,\lambda)=\dfrac{\partial^j G_0(x,t,\lambda)}{\partial t^j}$;*
5) *$G_j(x,t,\lambda)=G_j^*(t,x,\overline{\lambda})$ for $x\neq t$;*

Proof. The property 5) follows from the hermicity of a resolvent $\mathcal{R}(x,t,\lambda)$. Property 4) has been proved above. Properties (14) follow from the structure of the Green matrix and the conjugate matrix. Indeed, the matrix function $\mathcal{K}^{*\{r\}*}(x,t,\lambda)$, $r\in\{0,\dots,m-1\}$ and $\mathcal{K}^{*[q]}(x,t,\lambda)$, $q\in\{0,\dots,m-1\}$ form normal for $x{=}t$ fundamental systems of solutions of these equations. Therefore, by results of [13], this functions along with the quasi-derivatives in variables x and t, respectively, up to the order $m-l-1$ inclusive, belong to the space $AC_p[a,b]$ for fast other variables, and quasi-derivatives of higher orders are elements of the space $BV_p^+[a,b]$. Furthermore, for any $r\in\{0,\dots,m-1\}$

$$\mathcal{K}^{*\{r\}*[q]}(x,t,\lambda)=\begin{cases}E_p,\ r+q=m-1;\\ 0,\ r+q\neq m-1,\end{cases}\quad \Delta_x\mathcal{K}^{*\{r\}*[q]}(x,t,\lambda)=0,\ q\in\{0,m-l-1\},$$

$$\Delta_x \mathcal{K}^{*\{r\}*[m-l-1+q]}(x,t,\lambda) = \sum_{s=0}^{l-1} \Delta A_{l-s,q}(x) K^{*\{r\}*[s]}(x,t,\lambda),\ q\in\{1, l-1\},$$

$$\Delta_x \mathcal{K}^{*\{r\}*[m-1]}(x,t,\lambda) = \sum_{s=0}^{l-1} \Delta A_{l-s,l}(x) \mathcal{K}^{*\{r\}*(s)}(x,t,\lambda) - \lambda \Delta\sigma(x) \mathcal{K}^{*\{r\}*}(x,t,\lambda).$$

□

Theorem 4. *Let $\lambda = \lambda_k$, $k \in \{1, 2, \dots\}$ be a r_k-multiple eigenvalue of Problem (5). Then non-homogeneous Problem (2), (3) has solutions if and only if the following r conditions hold:*

$$\sum_{j=0}^{l} \int_a^b y^{(j)*}(t,\lambda_\nu) df_j(t) - \sum_{j=l+1}^{m-l-1} (-1)^{j-l} \int_a^b y^{[j]*}(t,\lambda_\nu) df_j(t) = 0,\ \nu\in\{k,\dots,k+r_k-1\};$$

thus, the solutions are represented in the form

$$y(x) = \lambda_k \sum_{\nu=1,\lambda_\nu\neq\lambda_k}^{\infty} \frac{y(x,\lambda_\nu)}{\lambda_\nu(\lambda_k - \lambda_\nu)} \Bigg(\sum_{j=0}^{l} \int_a^b y^{(j)*}(t,\lambda_\nu) df_j(t) - \sum_{j=l+1}^{m-l-1} (-1)^{j-l} i \int_a^b y^{[j]*}(t,\lambda_\nu) df_j(t) \Bigg) +$$
$$+ \sum_{\nu=k}^{k+r_k-1} c_\nu y(x,\lambda_\nu) + \sum_{j=0}^{m-l-1} \int_a^b \frac{\partial^j G(x,t,0)}{\partial t^j} df_j(t),$$

where c_ν are any constants and the series in the right-hand side converge absolutely and uniformly on $x \in [a,b]$.

For the case when zero is not an eigenvalue of Problem (5), the claim of the theorem follows from Theorem 2 of [6] on the basis of Lemma 1 and Property 5) of Green's matrix $G(x,t,0)$.

In the other case there exists a value λ_0 of the parameter λ which is not an eigenvalue. Thus, system (5) can be written as

$$l_m^0[y] - \lambda^- \omega(x) y = 0, \tag{22}$$

where $l_m^0[y] = l_m[y] - \lambda_0 \omega(x) y$, $\lambda^- = \lambda - \lambda_0$. Clearly, $\lambda^- = 0$ is not an eigenvalue of Problem (22), (3), thus, we can apply the previous arguments. We notice also that eigenvectors of this problem coincide with eigenvectors of Problem (5), (3) with the only difference, that now $y(x,\lambda_k)$ corresponds to the "displaced" eigenvalue $\lambda_k^- = \lambda_k - \lambda_0$.

Remark 6. By Theorem 1 [6], the solution of (15) can be represented also by the formula of Schmidt (in the form of absolutely and uniformly on $x \in [a,b]$ converging series by eigenvectors)

$$y(x) = \lambda_k \sum_{k=1}^{\infty} \frac{y(x,\lambda_\nu)}{\lambda_\nu(\lambda - \lambda_\nu)} \Bigg(\sum_{j=0}^{l} \int_a^b y^{(j)*}(t,\lambda_\nu) df_j(t) -$$
$$- \sum_{j=l+1}^{m-l-1} (-1)^{j-l} i \int_a^b y^{[j]*}(t,\lambda_\nu) df_j(t) \Bigg) + \sum_{j=0}^{m-l-1} \int_a^b \frac{\partial^j G(x,t,0)}{\partial t^j} df_j(t),$$

This formula has on advantage before Fredholm formula (15), because we have an explicit form of a solution as a meromorphic function of parameter λ.

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Received 21.12.2005