УДК 519.512

I. V. Krasikova

A NOTE ON NARROW OPERATORS IN L_∞

I. V. Krasikova. *A note on narrow operators in L_∞*, Mat. Stud. **31** (2009), 102–106.

A usual technique used in various investigations on narrow operators does not work in the case when the domain space is L_∞. We show that the example of a narrow projection in an r.i. function space E on $[0, 1]$ with an absolutely continuous norm onto a subspace $E_1 \subset E$ isometric to E constructed by A. Plichko and M. Popov extends to the case of $E = L_\infty$. We also prove that a sum of two narrow operators in L_∞ need not be narrow, answering to a question of O. Maslyuchenko, V. Mykhaylyuk and M. Popov.

И. В. Красикова. *Замечание об узких операторах в L_∞* // Мат. Студiї. – 2009. – Т.31, №1. – С.102–106.

Обычная техника, используемая в различных исследованиях об узких операторах, не работает, когда операторы заданы на пространстве L_∞. В заметке доказано, что пример узкой проекции в любом перестановочно-инвариантном пространстве E на $[0, 1]$ с абсолютно непрерывной нормой на подпространство $E_1 \subset E$, изометричное пространству E, построенный А. Пличко и М. Поповым, распространяется на случай $E = L_\infty$. Доказано также, что сумма узких операторов в L_∞ не обязана быть узким оператором. Это — ответ на вопрос А. Маслюченко, В. Михайлюка и М. Попова.

1. Introduction. The notation and terminology on Banach spaces are standard and will follow that of [6], [7], and on vector lattices we follow mainly [2]. By B_E we denote the closed unit ball of a Banach space E; the symbols $\mathcal{L}(X, Y)$ and $\mathcal{L}(X)$ stand for the spaces of all linear bounded operators acting from X to Y and from X to X respectively. By L_p we denote the space $L_p(\lambda)$ with the Lebesgue measure λ on the Borel σ-algebra $\mathcal{B}$ on $[0, 1]$.

For convenience of the notation, throughout the paper we consider operators on L_∞ only, but nevertheless all results automatically can be extended to operators on $L_\infty(\mu)$ with any finite non-atomic measure μ.

First narrow operators were introduced and considered acting from a symmetric (in other terminology, rearrangement invariant, r.i. in short) function space E to a Banach space (or, more general, to an F-space) X by Plichko and Popov in [9]. Although the assumption of absolute continuity of the norm in E is not used in the definition, it essentially have been used in all the investigations on narrow operators for a long time. So, there were no result on narrow operators acting from L_∞ before [8].

In their general direction, narrow operators were considered as a generalization of compact operators, still having some of their properties (see [1], [9]). Nevertheless, a compact (actually, rank-one) operator $T \in \mathcal{L}(L_\infty)$ need not be narrow (see [8]). Let us represent an example

2000 *Mathematics Subject Classification:* 46B20, 47B38.

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from [8]. Let $\overline{\mathcal{B}}$ be the Boolean algebra of the Borel subsets of $[0,1]$ equals up to measure null sets, and $\mathcal{U}$ be any ultrafilter on $\overline{\mathcal{B}}$. Then the linear functional $f_{\mathcal{U}}\colon E \to \mathbb{R}$ defined by

$$f_{\mathcal{U}}(x) = \lim_{A\in\mathcal{U}} \frac{1}{\mu(A)} \int\limits_A x d\mu$$

is obviously bounded and not narrow. Indeed, for each $x \in L_\infty$ of the form $x = \chi(A) - \chi(B)$ where $[0,1] = A \sqcup B$ one has $f_{\mathcal{U}}(x) = \pm 1$ depending of whether $A \in \mathcal{U}$ or $B \in \mathcal{U}$. The reason why $f_{\mathcal{U}}$ is not narrow is that it is not order-to-norn continuous. A positive result on narrowness of operators from L_∞ is also obtained in [8]: every AM-compact order-to-norm continuous linear operator $T\colon L_\infty(\mu) \to X$ is narrow for any Banach space X.

The most interesting phenomena concerning narrow operators is that if an r.i. function space E on $[0,1]$ has un unconditional basis (e.g., the spaces L_p with $1 < p < \infty$) then each operator $T \in \mathcal{L}(E)$ is a sum of two narrow operators and, on the other hand, the sum of two narrow operators from $\mathcal{L}(L_1)$ is narrow [9]. In a recent paper [8] O. Maslyuchenko, V. Mykhaylyuk and M. Popov explained this phenomena by showing that the sum of two regular (i.e. a difference of two positive operators) narrow operators in E is narrow for a large class of vector lattices E containing L_p with $1 \le p < \infty$. Moreover, they proved that the set $N_r(E)$ of all narrow regular operators in E is a band in the lattice $L_r(E)$ of all regular operators in E. Since all operators from $\mathcal{L}(L_1)$ are regular, this implies the above fact on the sum of two narrow operators in L_1. On the other hand, examples of narrow operators in L_p with $1 < p < \infty$ with non-narrow sum involve non-regular operators.

Concerning the space L_∞, it is known that the set $N_r(L_\infty)$ of all narrow regular operators in L_∞ is not a band in the lattice $L_r(L_\infty)$ of all regular operators in L_∞ [8]. The authors asked a question, whether a sum of two narrow operators (regular, or not necessary regular) in L_∞ is narrow? In Section 3 we answer this question for not necessary regular operators. Section 2 is devoted to an example of a "very" non-compact narrow operator in L_∞, a narrow projection to an infinite dimensional subspace of L_∞.

2. An example of a narrow projection in L_∞ onto a subspace isometric to L_∞. Let (Ω, Σ, μ) be a measure space with a σ-finite non-atomic measure μ and E be a rearrangement invariant Banach function space [1] (or, more general, Köthe function F-space) on (Ω, Σ, μ). Recall that a continuous linear operator $T \in \mathcal{L}(E, X)$ from E to a Banach space (or, more general, F-space) X is called *narrow* if for every $A \in \Sigma$ and every $\varepsilon > 0$ there is an $x \in E$ such that $x^2 = \chi(A)$, $\int_\Omega x d\mu = 0$ and $\|Tx\| < \varepsilon$.

As it was shown in [9], in every symmetric space E with an absolutely continuous norm on $[0,1]$ there exists a narrow projection onto a subspace isometric to E. We show that the same is true for L_∞. Moreover, the example is the same, the conditional expectation operator with respect to a suitable sub-σ-algebra. But the technique used in [9] failed in this case, because it essentially is based on absolute continuity of the norm. Actually, we prove more.

Let E be a Köthe function F-space on $[0,1]$. According to [8], a continuous linear operator $T \in \mathcal{L}(E, X)$ from E to an F-space X is called *strictly narrow* if for every $A \in \mathcal{B}$ there is an $x \in \ker T$ such that $x^2 = \chi(A)$ and $\int_{[0,1]} x d\lambda = 0$. Obviously, every strictly narrow operator is narrow.

Theorem 1. *There exists a sub-σ-algebra $\mathcal{F}$ of the Borel σ-algebra $\mathcal{B}$ on* $[0,1]$ *such that for every $p \in [1, \infty]$ the conditional expectation operator E with respect to $\mathcal{F}$ is a strictly narrow projection of L_p onto a subspace isometric to L_p.*

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r.i., in short

In our proof we consider the same sub-σ-algebra as in [9, p. 57] and, as in the proof of the corresponding statement from [9], for convenience we consider the space $L_p[0,1]^2$ instead of L_p. It will be the same by the Carathéodory theorem [5, p. 121]. First we need the following lemma.

Lemma 1. *Let $A \subseteq [0,1]^2$ be any measurable subset. Then there is a measurable function $\varphi_A\colon\ [0,1] \to [0,1]$ such that*

$$\lambda\Big(A_x \cap \big[0, \varphi_A(x)\big]\Big) = \frac{\lambda(A_x)}{2} \tag{1}$$

for almost all $x \in [0,1]$ where $A_x = \big\{y \in [0,1]\colon\ (x,y) \in A\big\}$.

Proof of Lemma 1. Using Fubini's theorem and standard arguments, one can show that since $A \subseteq [0,1]^2$ is measurable, the set A_x is measurable for almost all $x \in [0,1]$ (in the sequel, for simplicity of the notation we consider these values of x only). Now set

$$M_x = \Big\{y \in [0,1]\colon\ \lambda\Big(A_x \cap \big[0,y\big]\Big) = \frac{\lambda(A_x)}{2}\Big\}.$$

Standard arguments show that M_x is a closed nonempty set for almost all values of x. Then we finally set $\varphi_A(x) = \max M_x$. $\square$

Proof of Theorem 1. Consider the following sub-σ-algebra $\mathcal{F} = \mathcal{B} \times \big\{[0,1]\big\}$ of the Borel σ-algebra $\mathcal{B} \times \mathcal{B}$ on $[0,1]^2$. Then the conditional expectation operator $P\colon L_p[0,1]^2 \to L_p[0,1]^2$ acting as $Px = E(x/\mathcal{F})$ at an elements $x \in L_p[0,1]^2$ is the integration over the second variable

$$Px = \int\limits_{[0,1]} x(s,t)\, d\lambda(t).$$

It is well known and that P is a projection of norm 1 in L_p with any $p \in [1,\infty]$ (for $1 \le p < \infty$ see [4, p. 123], and the case $p = \infty$ is obvious). It is evident that the range of P is the subspace of all functions $x \in L_p[0,1]^2$ depending only of the first variable which is isometric to L_p. Thus, it remains to prove that P is strictly narrow. Let $A \subseteq [0,1]^2$ be a measurable set. Using Lemma 1, choose a function $\varphi_A\colon\ [0,1] \to [0,1]$ such that (1) holds. Then we set

$$z(x,y) = \begin{cases} 1, & \text{if } (x,y) \in A \quad \text{and} \quad y \le \varphi_A(x); \\ -1, & \text{if } (x,y) \in A \quad \text{and} \quad y > \varphi_A(x); \\ 0, & \text{if } (x,y) \notin A \end{cases}$$

Obviously, $z^2 = \chi(A)$. Besides,

$$\int\limits_{[0,1]^2} z(x,y)\, d\lambda(x)\, d\lambda(y) \ =$$

$$= \lambda\{(x,y) \in A\colon y \le \varphi_A(x)\} \ - \ \lambda\{(x,y) \in A\colon y > \varphi_A(x)\}$$

and

$$\lambda\{(x,y) \in A\colon y \le \varphi_A(x)\} = \int\limits_{[0,1]} \lambda\Big(A_x \cap \big[0, \varphi_A(x)\big]\Big) d\lambda(x) =$$

$$\int_{[0,1]} \frac{\lambda(A_x)}{2} d\lambda(x) = \frac{1}{2} \int_{[0,1]} \lambda(A_x) d\lambda(x) = \frac{1}{2}\lambda(A).$$

The above equalities imply that $\lambda\{(x,y) \in A \colon y > \varphi_A(x)\} = \lambda(A)/2$, i.e.

$$\int_{[0,1]^2} z(x,y) d\lambda(x) d\lambda(y) = 0.$$

Finally, we obtain

$$Pz(x,y) = \int_0^1 z(x,y')d\lambda(y') = \int_{[0,\varphi_A(x)]} z(x,y')d\lambda(y') + \int_{[\varphi_A(x),1]} z(x,y')d\lambda(y') =$$

$$= \int_{[0,\varphi_A(x)]\cap A_x} d\lambda(y') - \int_{[\varphi_A(x),1]\cap A_x} d\lambda(y') =$$

$$= \lambda([0,\varphi_A(x)] \cap A_x) - \lambda([\varphi_A(x),1] \cap A_x) = \frac{1}{2}\lambda(A_x) - \frac{1}{2}\lambda(A_x) = 0.$$

□

3. A sum of two narrow operators in L_∞ need not be narrow.

Lemma 2. *For each $p \in [1,\infty)$ the identity embedding $J_p \colon L_\infty \to L_p$ is a sum of two narrow operators $T, S \colon L_\infty \to L_p$.*

Proof. Suppose first that $p > 1$. According to [9], the identity map $I_p \colon L_p \to L_p$ is a sum of two narrow operators $P, Q \in \mathcal{L}(L_p)$. We set $T = P \circ J_p$ and $S = Q \circ J_p$. Now we prove that T and S are narrow operators. Indeed, given any measurable set $A \subseteq [0,1]$ and $\varepsilon > 0$, choose an $x \in L_p$ so that $x^2 = \chi(A)$, $\int_{[0,1]} x d\lambda = 0$ and $\|Px\| < \varepsilon$. Since $x \in L_\infty$, we have that $\|Tx\| = \|Px\| < \varepsilon$, thus, T is narrow. Likewise, S is narrow. Besides,

$$T + S = P \circ J_p + Q \circ J_p = (P+Q) \circ J_p = J_p.$$

Now let $p = 1$. Remark that $J_1 = J \circ J_2$ where $J \colon L_2 \to L_1$ is the identity embedding. Let $J_2 = T + S$ where $T, S \colon L_\infty \to L_2$ narrow operators. By [9], $J \circ T$ and $J \circ S$ are narrow operators from L_∞ to L_1. Thus,

$$J_1 = J \circ (T + S) = J \circ T + J \circ S$$

is the desired expansion. □

Theorem 2. *A sum of two narrow operators in L_∞ need not be narrow.*

Proof. Since by the Mazur theorem, $C[0,1]$ is universal space for all separable Banach spaces [3] and $C[0,1]$ is isometrically embedded into L_∞, the last space is universal as well. Let $U \colon L_2 \to L_\infty$ be any isomorphic embedding and $T, S \in \mathcal{L}(L_\infty, L_2)$ be any narrow operators such that $T + S = J_2$ where $J_2 \colon L_\infty \to L_2$ be the identity embedding (see Lemma 2).

According to [9], the operators $U \circ T$ and $U \circ S$ are narrow members of $\mathcal{L}(L_\infty)$. We show that their sum V is not narrow. Indeed,

$$V = U \circ T + U \circ S = U \circ J_2,$$

whence for each $x \in L_\infty$ with $x^2 = \chi_{[0,1]}$ one has

$$\|Vx\| \geq \|U^{-1}\|^{-1} \|J_2 x\| = \|U^{-1}\|^{-1} \int\limits_\Omega x^2 d\mu = \|U^{-1}\|^{-1},$$

and thus V is not narrow. $\square$

It is still unknown whether a sum of two regular narrow operators in L_∞ is narrow. Besides, we do not know whether the condition $\int_0^1 x d\lambda = 0$ in the definition of a narrow operator is essential for operators acting from L_∞. Remark that in an r.i. function space E on $[0, 1]$ with an absolutely continuous norm this condition is unnecessary [9].

I express many thanks to prof. M.M. Popov for valuable remarks.

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Department of Mathematics
Zaporizhzhya State University
str. Zhukovs'koho 2
Zaporizhzhya, Ukraine
yudp@mail.ru

Received 16.10.08
Revised 04.12.08