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# THE BEST POSSIBLE DESCRIPTION OF EXEPTIONAL SET IN BOREL'S RELATION FOR MULTIPLE DIRICHLET SERIES

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The best possible in some sense description of the exceptional set in Borel's relation between the logarithms of the maximum modulus and of the maximal term of the multiple Dirichlet series is obtained.

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Получено неулучшаемое в некотором смысле описание исключительного множества в соотношении Бореля между логарифмами максимума модуля и максимального члена целого кратного ряда Дирихле.

**1. Conjecture and main result.** Let $\Lambda = (\lambda_n)$ be a fixed sequence such that $\lambda_n = (\lambda_{n_1}^{(1)}, \ldots, \lambda_{n_p}^{(p)})$ for $n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p$ and $0 \le \lambda_k^{(j)} \uparrow +\infty$ $(k \to +\infty)$ for all $1 \le j \le p$. By $H^p(\Lambda)$ we denote the class of entire Dirichlet series (absolutely convergent in $\mathbb{C}^p,\ p \ge 1$)

$$F(z) = \sum_{\|n\|=0}^{+\infty} F_n e^{\langle z, \lambda_n \rangle}, \quad z \in \mathbb{C}^p, \tag{1}$$

such that $\sharp\{n \in \mathbb{Z}_+^p : \ F_n \ne 0\} = +\infty$, where $\langle a, b\rangle = a_1 b_1 + \ldots a_p b_p$ and $\|b\| = b_1 + \ldots + b_p$ for $a = (a_1, a_2, \ldots, a_p) \in \mathbb{C}^p, b = (b_1, b_2, \ldots, b_p) \in \mathbb{C}^p$. Let $\mathbb{R}_+ = (0, +\infty)$.

For $\Lambda = (\lambda_n)$ and $x \in \mathbb{R}^p$ we define the following counting functions

$$n(t, x, F) = \sum_{\langle x, \lambda_n \rangle \le t, F_n \ne 0} 1, \qquad n_j(t) = \sum_{\lambda_k^{(j)} \le t} 1$$

and denote $\widetilde{n}(t, x, F) := n(t|x|, x, F)$, $n_0(t, F) = n(t, \mathbf{e}_0, F) = \sum_{\|\lambda_n\| \le t} 1$, where $\mathbf{e}_0 = (1, \ldots, 1)$.
For $F \in H^p(\Lambda)$ and $x \in \mathbb{R}^p$ we denote

$$M(x, F) = \sup\{|F(x + iy)| : \ y \in \mathbb{R}^p\}, \quad \mu(x, F) = \max\{|F_n| e^{\langle x, \lambda_n \rangle} : \ n \in \mathbb{Z}_+^p\}.$$

It follows from [2] (see also [3]) that if a sequence $\Lambda = (\lambda_n)$ satisfies for all $1 \le j \le p$ the condition

$$\sum_{k=1}^{+\infty} \frac{1}{k \lambda_k^{(j)}} < +\infty, \tag{2}$$

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then for any entire function $F \in H^p(\Lambda)$ there exists a set $E \subset \mathbb{R}^p$ such that

$$(\forall\ \varepsilon > 0):\quad \int\limits_E \frac{dx}{|x|^{p-1+\varepsilon}} < +\infty \tag{3}$$

and for any real cone $K$ in $\mathbb{R}^p$ with the vertex at the point $O$ such that

$$\overline{K}\setminus\{O\} \subset \gamma(F) := \Big\{x \in \mathbb{R}^p:\ \lim_{t\to+\infty} \frac{\ln \mu(tx, F)}{t} = +\infty\Big\},$$

one has

$$\ln M(x, F) = (1 + o(1)) \ln \mu(x, F) \tag{4}$$

as $|x| \to +\infty$ $(x \in K \setminus E)$. In [3] it is proved that in the case $p = 2$ relation (3) can be replaced by the relation

$$\int\limits_{E\cap K_1} \frac{dx}{|x|} < +\infty,$$

where $K_1$ is an arbitrary real cone in $\mathbb{R}^2$ with the vertex at the point $O$ such that for $p = 2$

$$\overline{K}_1 \setminus \{O\} \subset \gamma_1(F) := \{x \in \gamma_+(F):\ \int^{+\infty} t^{-2} \ln \widetilde{n}(t, x, F)dt < +\infty\},$$
$$\gamma_+(F) := \{x \in \mathbb{R}^p:\ (\forall\ t > 0)[\widetilde{n}(t, x, F) < +\infty]\}.$$

**Conjecture 1.** *Let $p \geq 2$. In order that for every function $F \in H^p(\Lambda)$ relation (4) hold as $|x| \to +\infty$ $(x \in K_1 \setminus E)$, where $K_1$ is an arbitrary real cone in $\mathbb{R}^p$ with the vertex at the point $O$ such that $\overline{K}_1 \setminus \{O\} \subset \gamma(F)$ and the set $E \subset \mathbb{R}^p$ such that*

$$\tau_p(E \cap \gamma(F)) := \int\limits_{E\cap\gamma(F)} \frac{dx}{|x|^{p-1}} < +\infty, \tag{5}$$

*it is necessary and sufficient that*

$$\int^{+\infty} \frac{\ln n_0(t)}{t^2} dt < +\infty. \tag{6}$$

The result cited above implies that Conjecture 1 is true in the case $p = 2$ by replacing $\gamma(F)$ on $\gamma_1(F)$.

In the present paper we establish that in the general case $(p > 2)$ the conjecture is true by replacing $\gamma(F)$ on $\mathbb{R}^p_+$.

It is easy to see that $(\forall\ j,\ 1 \leq j \leq p)$:

$$n_j(t|x|/x_j) \leq \widetilde{n}(t, x, F) = \widetilde{n}(t, x/|x|, F) \leq n_1(t|x|/x_1) n_2(t|x|/x_2) \cdots n_p(t|x|/x_p) \tag{7}$$

for $x \in \mathbb{R}^p_+$, hence $n_j(t) \leq n_0(t, F) \leq n_1(t)n_2(t)\cdots n_p(t)$. It is known ([1]) that a sequence $(\lambda^{(j)}_n)$, $1 \leq j \leq p$, satisfies condition (2) if and only if $\int^{+\infty} t^{-2} \ln n_j(t, F)dt < +\infty$, and thus a sequence $\Lambda = (\lambda_n)$ satisfies condition (2) for all $1 \leq j \leq p$ iff condition (6) holds. From condition (2) and inequality (7) it follows also that $\mathbb{R}^p_+ \subset \gamma_1(F)$. From the other hand ([5, 6]) $x \in \gamma(F) \Leftrightarrow \sup\{\langle x, \lambda_n\rangle:\ F_n \neq 0, n \in \mathbb{Z}^p_+\} = +\infty$, in particular $\gamma_1(F) \subset \gamma(F)$ as well.

The following theorem is true.

**Theorem 1.** *Let $F \in H^p(\Lambda)$, $p \geq 2$. If a sequence $\Lambda = (\lambda_n)$ satisfies condition (6) then relation (4) is valid as $|x| \to +\infty$ $(x \in K \setminus E)$, where $K$ is an arbitrary real cone in $\mathbb{R}^p_+$ with the vertex at the point $O$ such that $\overline{K} \setminus \{O\} \subset \mathbb{R}^p_+$ and the set $E \subset \mathbb{R}^p$ such that $\tau_p(E \cap K) < +\infty$.*

*Proof.* Without lost of generality suppose that for $F \in H^p(\Lambda)$ at the form (1) $F(0) = 1$ and $F_n \geq 0$ $(n \geq 0)$. For fixed $x_0$ consider the function $g(t) = g_0(t) := \ln F(tx_0)$, $t > 0$, and the measure $\nu$ on $\mathbb{R}^p_+$ such that for every bounded set $E \subset \mathbb{R}^p_+$

$$\nu(E) = \sum_{\lambda_n \in E} F_n \delta_n(E),$$

where measures $\delta_n(E) = 1$ $(\lambda_n \in E)$ and $\delta_n(E) = 0$ $(\lambda_n \notin E)$ for all $n \in \mathbb{Z}^p_+$. Thus

$$F(x) = \int_{\mathbb{R}^p_+} e^{\langle x,y\rangle} \nu(dy), \quad x \in \mathbb{R}^p_+. \tag{8}$$

Similar to [6, p.127] we prove that Markov's inequality for functions $F$ of the form (8) implies

$$F(tx_0) \leq \frac{C}{C-1} \int_{\{y\colon \langle x_0,y\rangle \leq c g_0'(t)\}} e^{t\langle x_0,y\rangle} \nu(dy) \quad (\forall\ x_0 \in \mathbb{R}^p_+,\ C > 1,\ t > 0). \tag{9}$$

In fact, for fixed $t > 0$ and $x_0 \in \mathbb{R}^p_+$ consider probabilistic space $(\mathbb{R}^p_+, \mathcal{A}, \mathbb{P}(dy))$, where $\mathbb{P}(dy) = \dfrac{e^{t\langle y,x_0\rangle}}{F(tx_0)} \nu(dy)$ probabilistic measure on the countably additive algebra $\mathcal{A}$ of $\nu$-measurable subsets of $\mathbb{R}^p_+$, and random variable $\xi(y) = \langle y, x_0\rangle$. Since mean $\mathbf{M}\xi = g_0'(t)$, we have Markov's inequality $\mathbb{P}\{y\colon \xi > a\} \leq \mathbf{M}\xi/a$, $a > 0$, as $a = C\mathbf{M}\xi$, $C > 1$, implies that

$$\int_{\{y\colon \xi > C\mathbf{M}\xi\}} e^{t\xi} \nu(dy) \leq F(tx_0)/C.$$

Hence

$$F(tx_0) \leq \int_{\{y\colon \xi \leq C\mathbf{M}\xi\}} e^{t\xi} \nu(dy) + F(tx_0)/C,$$

which implies inequality (9).

From inequality (9) for $C = 2$, $t = |x|$ and $x_0 = x/|x|$ we obtain

$$F(x) \leq 2 \sum_{\langle x_0,\lambda_n\rangle \leq 2g_0'(t)} F_n e^{\langle x,\lambda_n\rangle} \leq$$

$$\leq 2\mu(x,F) n(2g_0'(t), x_0, F) \leq 2\mu(x,F) \sup\{n(2g_0'(t), \sigma, F)\colon |\sigma| = 1, \sigma \in \overline{K}\}. \tag{10}$$

Let $x^* := \inf\{\inf\{x_j\colon x = (x_1, \ldots, x_j, \ldots, x_p), |x| = 1, x \in \overline{K}\}\colon 1 \leq j \leq p\}$. Since $x^* > 0$, we have for $|\sigma| = 1, \sigma \in \overline{K}$, $t > 0$

$$n(t, \sigma, F) = \sum_{\langle \sigma,\lambda_n\rangle \leq t} 1 \leq \sum_{x^*\|\lambda_n\| \leq t} 1 = n_0(t/x^*),$$

and thus

$$\sup\{n(2g'(t),\sigma,F)\colon |\sigma|=1,\sigma\in\overline{K}\}\le n_0(2g'(t)/x^*).$$

Hence, using inequality (10) for $x=tx_0$ we obtain

$$F(x)\le 2\mu(x,F)n_0(2g_0'(t)/x^*).$$

It is proved in [6] (Proposition 5') that for every real cone $K$ with the vertex at the point $O$ such that $\overline{K}\setminus\{O\}\subset\gamma(F)$

$$\lim_{|x|\to+\infty,\ x\in K}\frac{\ln F(x)}{|x|}=+\infty.$$

Since variance $\mathbf{D}\xi=g_0''(t)$, we have $g_0''(t)\ge 0$ for all $t>0$ and fixed $x_0$. Thus the function $g_0(t)$ is convex on $(0,+\infty)$ for fixed $x_0$, $g_0(0)=0$, and we have

$$g_0'(t)\ge\frac{g_0(t)}{t}=\frac{\ln F(tx_0)}{t}\ge\inf\left\{\frac{\ln F(t\sigma)}{t}\colon |\sigma|=1,\sigma\in K\right\}\to+\infty\quad(t\to+\infty).$$

Hence

$$g_0'(t)\to+\infty,\quad t=|x|\to+\infty\quad(x=tx_0,\ x\in K).$$

Condition (6) implies that ([6], p.130–131) there exists a nonnegative continuous increasing function $\psi(u)$ such that

$$\int_0^{+\infty}\frac{du}{\psi(u)}<+\infty\quad\text{and}\quad\ln n_0(t)=o(\psi^{-1}(t))\quad(t\to+\infty),$$

where the function $\psi^{-1}(t)$ is inverse to the function $\psi(t)$.

We denote $E(x_0)=\{x=tx_0\colon t>0,\frac{2}{x^*}g_0'(t)>\psi(g_0(t))\}$ for the fixed $x_0\in K$ and

$$E=\bigcup_{|x_0|=1,\ x_0\in K}E(x_0).$$

Since $g_0'(t)\to+\infty$ $(t=|x|\to+\infty,\ x=tx_0,\ x\in K)$, we obtain the following asymptotic estimates as $t=|x|\to+\infty$ $(x\in\overline{K}\setminus E,\ x=tx_0)$

$$\ln F(x)\le\ln 2+\ln\mu(x,F)+\ln n_0\Big(\frac{2}{x^*}g_0'(t)\Big)=\ln\mu(x,F)+o\Big(\psi^{-1}\Big(\frac{2}{x^*}g_0'(t)\Big)\Big)\le$$
$$\le\ln\mu(x,F)+o(\psi^{-1}(\psi(\ln F(x))))=\ln\mu(x,F)+o(\ln F(x)).$$

Using the Cauchy inequality $(\mu(x,F)\le F(x))$, hence we obtain relation (4) as $|x|\to+\infty$ $(x\in K\setminus E)$.

Let $S_1=\{x\in K\colon |x|=1\}$. Finally, we obtain the following estimate for the set $E$

$$\tau_p(E\cap K)=\iint\limits_{E\cap K}\frac{dx}{|x|^{p-1}}=\int\limits_{S_1}\Big(\int\limits_{E(x_0)}dt\Big)ds\le\frac{2}{x^*}\int\limits_{S_1}\Big(\int\limits_{E(x_0)}\frac{g_0'(t)}{\psi(g_0(t))}dt\Big)ds\le$$
$$\le\frac{2}{x^*}\int\limits_{S_1}\Big(\int\limits_{g_0(0)}^{g_0(+\infty)}\frac{du}{\psi(u)}\Big)ds\le\frac{2}{x^*}\Big(\frac{\sqrt{\pi}}{2}\Big)^{p-2}\frac{\Gamma(\frac{p-1}{2})\ldots\Gamma(1)}{\Gamma(\frac{p}{2})\ldots\Gamma(\frac{3}{2})}\int\limits_0^{+\infty}\frac{du}{\psi(u)}<+\infty.$$

This completes the proof of Theorem 1. □

From Theorem 1 and proof of Corollary 1 ([2], p.48) we deduce the following

**Corollary 1.** *Let $p \geq 2$. Conjecture 1 is true by replacing $\gamma(F)$ on $\mathbb{R}^p_+$.*

*Proof.* By Theorem 1 we immediately get sufficiency of the condition (6) .
Necessity of the condition (6). In proof of Corollary 1 ([2], p.48) in order that condition (6) be false it is proved that there exist a function $F \in H^p(\Lambda)$, a constant $h > 0$ such that for all $x \in E := \{x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p_+ \colon x_1 \geq x_1^0, x_1 \geq \max\{x_2, x_3, \dots, x_p\}\}$

$$\ln M(x, F) \geq (1 + h) \ln \mu(x, F). \tag{11}$$

Finally for the set $E$ we obtain

$$\tau_p(E \cap \mathbb{R}^p_+) = \int\limits_E \frac{dx}{|x|^{p-1}} \geq p^{-\frac{p-1}{2}} \int\limits_E \frac{dx}{x_1^{p-1}} \geq \int_{x_1^0}^{+\infty} \frac{dx_1}{x_1^{p-1}} \int_0^{x_1} dx_2 \int_0^{x_1} dx_3 \cdots \int_0^{x_1} dx_p = +\infty.$$

□

**2. Description (5) of the exceptional set is the best possible.**

The following proposition shows that description (5) of the exceptional set in relation (4) is the best possible in some sense.

**Proposition.** *Let $p \geq 2$. For every $\varepsilon > 0$ there exist a sequence $\Lambda = (\lambda_n)$ satisfying condition (6), a function $F \in H^p(\Lambda)$, a locally measurable set $E \subset \mathbb{R}^p_+$ and a constant $h > 0$ such that for all $x \in E$ inequality (11) hold and $\tau_{p-\varepsilon}(E \cap \mathbb{R}^p_+) = +\infty$.*

*Proof.* The proof is similar to that of Theorem 2 in [3]. It is known ([3], Theorem 3) that for every $\varepsilon > 0$ there exist a sequence $\lambda^{(1)} = (\lambda_k^{(1)})$ satisfying for $j = 1$ condition (2), a function $f_1 \in H^1(\lambda^{(1)})$, sequences $c_k \to +0$ $(k \to +\infty)$, $\sum_{k=1}^{+\infty} c_k = +\infty$, and $y_k \uparrow +\infty$ $(1 \leq k \to +\infty)$ and a constant $h_1 > 0$ such that for all $k \geq 1$ and $x \in I_k := [y_k; y_k + c_k y_k^{-\varepsilon}]$ inequality (11) with $F = f_1$ hold. For $j$, $2 \leq j \leq p$, consider arbitrary functions $f_j \in H^1(\lambda^{(j)})$, such that $\mu(t, f_j) \leq \mu(t, f)$ $(t > 0)$, where $\lambda^{(j)} = (\lambda_k^{(j)})$ is an arbitrary sequence (in particular, satisfy condition (2)). It is obvious that such a function $f_i$ exists. Then the function $F(x_1, x_2, \dots, x_p) = \prod\limits_{j=1}^{p} f_j(x_j) \in H^p(\Lambda)$ and for $x \in E := \big\{x = (x_1, \dots, x_p) \colon x_1 \in \bigcup\limits_{k=1}^{+\infty} I_k,\ x_1 \geq \max\{x_2, x_3, \dots, x_p\}\big\}$ by the Cauchy inequality we have

$$\ln F(x) = \sum_{j=1}^{p} \ln f_j(x_j) \geq (1 + h_1) \ln \mu(x_1, f_1) + \sum_{j=2}^{p} \ln \mu(x_j, f_j) \geq$$

$$\geq (1 + \frac{h_1}{p}) \sum_{j=1}^{p} \ln \mu(x_j, f_j) = (1 + h) \ln \mu(x, F), \quad h = \frac{h_1}{p}.$$

Finally we note that

$$\tau_{p-\varepsilon}(E \cap \mathbb{R}^p_+) = \int_E \frac{dx}{|x|^{p-1-\varepsilon}} \geq \sum_{k=1}^{+\infty} \int_{I_k} dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_1} \frac{dx_p}{|x|^{p-1-\varepsilon}} \geq$$

$$\geq p^{-(p-1-\varepsilon)/2}\sum_{k=1}^{+\infty}\int_{I_k}dx_1\int_0^{y_k}dx_2\cdots\int_0^{y_k}\frac{dx_p}{x_1^{p-1-\varepsilon}}\geq p^{-(p-1-\varepsilon)/2}\sum_{k=1}^{+\infty}\frac{c_ky_k^{p-1-\varepsilon}}{(y_k+c_ky_k^{-\varepsilon})^{p-1-\varepsilon}}=+\infty.$$

□

**Conjecture 2.** *Let $\ell(x)$ be an arbitrary positive measurable on $\mathbb{R}_+^p$ function such that $\ell(x)\to+\infty$ $(|x|\to+\infty, x\in\mathbb{R}_+^p)$. There exist a sequence $\Lambda=(\lambda_n)$ satisfying condition (6), a function $F\in H^p(\Lambda)$, a locally measurable set $E\subset\mathbb{R}_+^p$ and a constant $h>0$ such that for all $x\in E$ inequality (11) hold and*

$$\int\limits_{E\cap\mathbb{R}_+^p}\frac{\ell(x)dx}{|x|^{p-1}}=+\infty.$$

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