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SUFFICIENT SETS FOR BOUNDEDNESS OF L-INDEX IN DIRECTION FOR ENTIRE FUNCTIONS

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The conditions on a set A providing the equality $N_{\mathbf{b}}(F,L)=\max\{N(g_{z^0},l_{z^0}) : z^0\in A\}$, where $l_{z^0}(t)\equiv L(z^0+t\mathbf{b})$, are obtained. And sufficient conditions of boundedness L-index of solutions of some partial differential equations are obtained.

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Установлены условия на множество A, обеспечивающие справедливость равенства $N_{\mathbf{b}}(F,L)=\max\{N(g_{z^0},l_{z^0}) : z^0\in A\}$, где $l_{z^0}(t)\equiv L(z^0+t\mathbf{b})$. А также получены достаточные условия ограниченности L-индекса для решений некоторых дифференциальных уравнений в частных производных.

1. Introduction. Let L be a positive continuous function on $\mathbb{C}^n$. An entire function $F(z)$, $z\in\mathbb{C}^n$, is called (see [1]) a function of *bounded L-index in a direction* $\mathbf{b}\in\mathbb{C}^n$, if there exists $m_0\in\mathbb{Z}_+$ such that for $m\in\mathbb{Z}_+$ and every $z\in\mathbb{C}^n$ the following inequality performs

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\right|\le\max\left\{|F(z)|,\ \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right| : 1\le k\le m_0\right\}\right\},\qquad(1)$$

where $\frac{\partial F(z)}{\partial\mathbf{b}}=\sum_{j=1}^n\frac{\partial F(z)}{\partial z_j}b_j=\langle\mathbf{grad}\ F,\overline{\mathbf{b}}\rangle$, $\frac{\partial^k F(z)}{\partial\mathbf{b}^k}=\frac{\partial}{\partial\mathbf{b}}\big(\frac{\partial^{k-1}F(z)}{\partial\mathbf{b}^{k-1}}\big)$, $k\ge 2$.

The least of such $m_0=m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}\in\mathbb{C}^n$ *of the function* $F(z)$ and is denoted by $N_{\mathbf{b}}(F,L):=m_0$.

In the case of $n=1$ and $L(z)=l(z)$, $z\in\mathbb{C}$, we receive a definition of a function of bounded l-index, and in the case $L(z)\equiv 1$ we get a definition of a function of bounded index. Accordingly, under the $N(g,l)$ we will understand l-index of an entire function $g(z), z\in\mathbb{C}$. In the paper [1] we proved the following statement.

Lemma 1. *If $F(z)$ is a function of bounded L-index in direction $\mathbf{b}\in\mathbb{C}^n$ and j_0 is such that $b_{j_0}\ne 0$, then $N_{\mathbf{b}}(F,L)=\max\{N(g^{\mathbf{b}}_{z^0},l^{\mathbf{b}}_{z^0}) : z^0\in\mathbb{C}^n, z^0_{j_0}=0\}$ and if there $\|\mathbf{b}\|\ne 0$ then $N_{\mathbf{b}}(F,L)=\max\{N(g^{\mathbf{b}}_{z^0},l^{\mathbf{b}}_{z^0}) : z^0\in\mathbb{C}^n, \|z^0\|=0\}$, where $\|z^0\|=\sum_{j=1}^n z^0_j$, $g^{\mathbf{b}}_{z^0}(t)=F(z^0+t\mathbf{b})$, $l^{\mathbf{b}}_{z^0}(t)=L(z^0+t\mathbf{b})$.*

In [2] we proved that $\exists F(z)$, $z\in\mathbb{C}^2$, an entire function and $\mathbf{b}\in\mathbb{C}^2$ such that $N_{\mathbf{b}}(F,L)=+\infty$ and $(\forall\ z^0\in\mathbb{C}^2):\ N(g^{\mathbf{b}}_{z^0},l^{\mathbf{b}}_{z^0})<+\infty$, where $L\equiv 1$. In connection with this statement we posed the following natural *question* in the mentioned article: under what minimum

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requirements on a set A the equality $N_{\mathbf{b}}(F, L) = \max\{N(g^{\mathbf{b}}_{z^0}, l^{\mathbf{b}}_{z^0}) : z^0 \in A\}$, where $l^{\mathbf{b}}_{z^0}(t) \equiv L(z^0 + t\mathbf{b})$ will be carried out?

In this paper we prove propositions which give a partial answer for this question. The answer is partial in the sense that it is not known that the obtained set is the least from those which the mentioned equality is valid for.

2. Structure of a set sufficient for boundedness of L-index in direction.

Lemma 2. *Let $\mathbf{b} \in \mathbb{C}^n$ be a fixed direction, A_0 be an arbitrary set in $\mathbb{C}^n$ such that $\{z + t\mathbf{b} : t \in \mathbb{C}, z \in A_0\} = \mathbb{C}^n$. An entire function $F(z)$, $z \in \mathbb{C}^n$, is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exists $M > 0$ such that for each $z^0 \in A_0$ the function $g^{\mathbf{b}}_{z^0}(t) = F(z^0 + t\mathbf{b})$ is of bounded $l^{\mathbf{b}}_{z^0}$-index $N(g^{\mathbf{b}}_{z^0}, l^{\mathbf{b}}_{z^0}) \le M < +\infty$ as a function of $t \in \mathbb{C}$ ($l^{\mathbf{b}}_{z^0}(t) \equiv L(z^0 + t\mathbf{b})$). Thus, $N_{\mathbf{b}}(F, L) = \max\{N(g^{\mathbf{b}}_{z^0}, l^{\mathbf{b}}_{z^0}) : z^0 \in A_0\}$.*

Proof. According to Theorem 1 of [1], a function $F(z)$, $z \in \mathbb{C}^n$ is a function of bounded L-index in a direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exists $M > 0$ such that for every $z^0 \in \mathbb{C}^n$ the function $g^{\mathbf{b}}_{z^0}(t) = F(z^0 + t\mathbf{b})$ is of bounded $l^{\mathbf{b}}_{z^0}$-index $N(g^{\mathbf{b}}_{z^0}, l^{\mathbf{b}}_{z^0}) \le M < +\infty$, as a function of $t \in \mathbb{C}$ ($l^{\mathbf{b}}_{z^0}(t) \equiv L(z^0 + t\mathbf{b})$). But for every $z^0 + t\mathbf{b}$, in view of the definition of the set A_0, there will be a point $\widetilde{z}^0 \in A_0$ and $\widetilde{t} \in \mathbb{C}$, of $z^0 + t\mathbf{b} = \widetilde{z}^0 + \widetilde{t}\mathbf{b}$.

That is why the condition that $g^{\mathbf{b}}_{z^0}(t)$ is of bounded $l^{\mathbf{b}}_{z^0}$-index for all $z^0 \in \mathbb{C}$ is equivalent to the condition that $g^{\mathbf{b}}_{\widetilde{z}^0}(t)$ is of bounded $l^{\mathbf{b}}_{\widetilde{z}^0}$-index for all $\widetilde{z}^0 \in A_0$. □

Remark 1. An arbitrary hyperplane $A_0 = \{\widetilde{z} \in \mathbb{C}^n : \langle \widetilde{z}, c\rangle = 1\}$, where $\langle c, \mathbf{b}\rangle \neq 0$, satisfies the conditions of Lemma 2.

We prove that for all $w \in \mathbb{C}^n$ there exist $\widetilde{z} \in A_0$ and $\widetilde{t} \in \mathbb{C}$ such that $w = \widetilde{z} + \widetilde{t}\mathbf{b}$. It is clear that there exists $z \in \mathbb{C}^n$, $t \in \mathbb{C}$ with $w = z + t\mathbf{b}$. Choose $\widetilde{z} = z + \dfrac{1 - \langle z, c\rangle}{\langle \mathbf{b}, c\rangle}\mathbf{b}$ and $\widetilde{t} = \dfrac{\langle z, c\rangle + t\langle \mathbf{b}, c\rangle - 1}{\langle \mathbf{b}, c\rangle}$. We obtain $\widetilde{z} + \widetilde{t}\mathbf{b} = z + \dfrac{1 - \langle z, c\rangle}{\langle \mathbf{b}, c\rangle}\mathbf{b} + \dfrac{\langle z, c\rangle + t\langle \mathbf{b}, c\rangle - 1}{\langle \mathbf{b}, c\rangle}\mathbf{b} = z + t\mathbf{b}$.

Lemma 3. *Let $\overline{A} = \mathbb{C}^n$, that is, A is a dense set in $\mathbb{C}^n$. An entire function $F(z)$, $z \in \mathbb{C}^n$, is a function of bounded L-index in a direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exists $M > 0$ such that for all $z^0 \in A$ the function $g^{\mathbf{b}}_{z^0}(t) = F(z^0 + t\mathbf{b})$ is of bounded $l^{\mathbf{b}}_{z^0}$-index $N(g^{\mathbf{b}}_{z^0}, l^{\mathbf{b}}_{z^0}) \le M < +\infty$, as a function of $t \in \mathbb{C}$ ($l^{\mathbf{b}}_{z^0}(t) \equiv L(z^0 + t\mathbf{b})$). Thus, $N_{\mathbf{b}}(F, L) = \max\{N(g^{\mathbf{b}}_{z^0}, l^{\mathbf{b}}_{z^0}) : z^0 \in A\}$.*

Proof. Necessity arises from Lemma 2 of [1] (in this lemma a proper condition is true for all $z^0 \in \mathbb{C}^n$.

Sufficiency. As $\overline{A} = \mathbb{C}^n$, for all $z^0 \in \mathbb{C}^n$ there is $\{z^{(m)}\}$, such that $z^{(m)} \longrightarrow z^0$ as $m \longrightarrow +\infty$. But $F(z + t\mathbf{b})$ is of bounded l_z-index for all $z \in \overline{A}$ as a function of t ($l_z \equiv l^{\mathbf{b}}_z$). That is why, in view of the definition of bounded l_z-index we have $\dfrac{|g^{(p)}_z(t)|}{p!l^p_z(t)} \le \max\left\{\dfrac{|g^{(k)}_z(t)|}{k!l^k_z(t)} : 0 \le k \le M\right\}$ with $g_z(t) \equiv g^{\mathbf{b}}_z(t)$. Then $\dfrac{|g^{(p)}_{z^{(m)}}(t)|}{p!l^p_{z^{(m)}}(t)} \le \max\left\{\dfrac{|g^{(k)}_{z^{(m)}}(t)|}{k!l^k_{z^{(m)}}(t)} : 0 \le k \le M\right\}$ for every $m \ge 1$.

In other words,

$$\frac{1}{p!L^p(z^{(m)} + t\mathbf{b})}\left|\frac{\partial^p F(z^{(m)} + t)}{\partial \mathbf{b}^p}\right| \le \max\left\{\frac{1}{k!L^k(z^{(m)} + t\mathbf{b})}\left|\frac{\partial^k F(z^{(m)} + t\mathbf{b})}{\partial \mathbf{b}^k}\right| : 0 \le k \le M\right\}.$$

But F is an entire function, and L is a continuous positive function, that is why in the obtained expression the limiting transition is possible as $m \longrightarrow +\infty$ ($z^{(m)} \longrightarrow z$). Hence the function $F(z+t\mathbf{b})$ is of bounded $L(z+t\mathbf{b})$-index as a function of t at each $z\in\mathbb{C}^n$. Remaining for a complete proof we refer to the sufficiency in Theorem 1 of [1]. □

From Lemma 3 and Remark 1 imply following corollary.

Corollary 1. *Let $\mathbf{b}\in\mathbb{C}^n$ be a fixed direction, A_0 is an arbitrary set in $\mathbb{C}^n$ such that $\overline{A}_0=\langle\widetilde{z},c\rangle=1$, where $\langle c,\mathbf{b}\rangle\neq 0$. An entire function $F(z)$, $z\in\mathbb{C}^n$, is a function of bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n$ if and only if there exists $M>0$ such that for all $z^0\in A_0$ the function $g^{\mathbf{b}}_{z^0}(t)=F(z^0+t\mathbf{b})$ is of bounded $l^{\mathbf{b}}_{z^0}$-index $N(g^{\mathbf{b}}_{z^0},l^{\mathbf{b}}_{z^0})\le M<+\infty$, as a function of $t\in\mathbb{C}$ ($l^{\mathbf{b}}_{z^0}(t)\equiv L(z^0+t\mathbf{b})$). Thus, $N_{\mathbf{b}}(F,L)=\max\{N(g^{\mathbf{b}}_{z^0},l^{\mathbf{b}}_{z^0}):z^0\in A_0\}$.*

3. **Bounded sets.** Let L be a positive continuous function on $\mathbb{C}^n$. For $\eta>0$, $z\in\mathbb{C}^n$, $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{0\}$ and a positive continuous function $L:\ \mathbb{C}^n\to\mathbb{R}_+$ we define

$$\lambda_1^{\mathbf{b}}(z,t_0,\eta)=\inf\Big\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}:|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\Big\},$$

$\lambda_1^{\mathbf{b}}(z,\eta)=\inf\{\lambda_1^{\mathbf{b}}(z,t_0,\eta):t_0\in\mathbb{C}\}$, $\lambda_1^{\mathbf{b}}(\eta)=\inf\{\lambda_1^{\mathbf{b}}(z,\eta):z\in\mathbb{C}^n\}$, and also

$$\lambda_2^{\mathbf{b}}(z,t_0,\eta)=\sup\Big\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}:|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\Big\},$$

$\lambda_2^{\mathbf{b}}(z,\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,t_0,\eta):t_0\in\mathbb{C}\}$, $\lambda_2^{\mathbf{b}}(\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,\eta):z\in\mathbb{C}^n\}$.
By $Q^n_{\mathbf{b}}$ we denote the class of functions L which for all $\eta\ge 0$ satisfy the condition

$$0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty.$$

We prove propositions, which are certain multidimensional analogues of some theorems obtained by M. Sheremeta, M. Bordulyak and S. Shah (see [3, p. 33] Theorem 2.2, and also [4], [5]). At first, we prove some auxiliary statements. In particular, the following is true.

Theorem 1. *Let G be an arbitrary bounded domain in $\mathbb{C}^n$. If $L(z)$ is a positive continuous function and $F(z)$ is an entire function, where $z\in\mathbb{C}^n$, then for a function $F(z)$ and for any direction $\mathbf{b}\in\mathbb{C}^n$ there exists $m_0=m_0(\mathbf{b})$ such that for each $m\in\mathbb{Z}_+$ and for all $z\in G$ inequality (1) is true.*

Proof. Let $\overline{G}$ be the closure of the domain G and $z^0\in\overline{G}$ be a fixed point. We denote $g_{z^0}(t)\equiv F(z^0+t\overline{\mathbf{b}})$, $t\in\mathbb{C}$, $D_{z^0}=\overline{G}\cap\{z^0+t\mathbf{b}:t\in\mathbb{C}\}$, $d=\operatorname{diam}\overline{G}$. Then on the bounded set D_{z^0} the entire function $g_{z^0}(t)$ has a finite number of zeros or identically equals to zero. In the last case, in view of the uniqueness theorem, this function identically equals zero for all $t\in\mathbb{C}$. It is clear that inequality (1) is implemented on the set D_{z^0}.

Regarding to finite number of zeros, for simplification of the proof we will consider the case when this function has only one zero on D_{z^0}. It is easily seen from the next proof how similar arguments can be generalized for the case of finite number of zeros.

Consequently, let $g_{z^0}(t)$ have one zero at a point a_{z^0} on D_{z^0}. Then there is a derivative $g_{z^0}^{(q_{z^0})}$ such that $g_{z^0}^{(q_{z^0})}(a_{z^0})\neq 0$. So $|g_{z^0}^{(q_{z^0})}(t)|\ge h_{z^0}>0$ in some circle $\overline{K_{z^0}}=\{t\in\mathbb{C}:|t-a_{z^0}|\le d_{z^0}<\frac{d}{\sqrt{n}|\mathbf{b}|}\}$. We note that all these constants are uniformly bounded. There exist positive constants q, h such that $q_{z^0}\le q$, $h_{z^0}\ge h$, for all $z^0\in\mathbb{C}^n$. Suppose that this is not true. Then there is a convergent sequence z^0_p, $p\in\mathbb{N}$ and corresponding convergent sequences $q_{z^0_p}$ and $a_{z^0_p}$ such that $z^0_p\longrightarrow z^*\in\overline{G}$, $a_{z^0_p}\longrightarrow a^*\in D_{z^*}$, but $q_{z^0}\longrightarrow+\infty$. Then there exists a finite point a^*, at which the entire function $g_{z^*}(t)$ would have a zero of infinity multiplicity, that is impossible.

Now we prove that $\sup\{q_{z^0} : z^0 \in \overline{G}\} \equiv q < +\infty$, where q_{z^0} is the multiplicity of all zeros of the function $g^{\mathbf{b}}_{z^0}(t) = F(z^0 + t\mathbf{b})$ in $\overline{K} = \{t \in \mathbb{C} : |t| \le \frac{d}{\sqrt{n}|\mathbf{b}|}\}$. To the contrary, in view of Montel's theorem, we suppose that $q = +\infty$. But $\overline{G}$ is a compact. Then there exists a sequence $z^0_j \longrightarrow z^0 \in \overline{G}$ and a sequence of zeros $a_j = a_{z_j} \longrightarrow a \in \overline{G}$ of multiplicity $q_j = q_{z^0_j} \longrightarrow +\infty$ such that the corresponding sequence $g_j(t) \equiv g_{z_j}(t)$ uniformly converges on $\overline{K}$ to an analytic function $g(t)$. Therefore, the point a is a zero of infinite multiplicity for $g(t)$, that is impossible. Thus, we have such estimates $q_{z^0} \le q$, $h_{z^0} \ge h$ for all $z^0 \in \overline{G}$.

Let $L_* = \max\{L(z) : z \in \overline{G}\}$ and $\mu = \min\{|F(z)| : z \in \overline{G} \setminus \bigcup_{z^0\in\mathbb{C}^n} K_{z^0}\}$.
Then for all $z \in \overline{G}$ the following inequality is true

$$\max\left\{|F(z)|, \frac{1}{q_z!L^{q_z}(z)}\left|\frac{\partial^{q_z}F(z)}{\partial\mathbf{b}^{q_z}}\right|\right\} = \max\left\{|F(z)|, \frac{|g^{q_z}_z(0)|}{q_z!L^{q_z}(z)}\right\} \ge$$
$$\ge \min\left\{\mu, \frac{h_z}{q_z!L^{q_z}_*}\right\} \ge \min\left\{\mu, \frac{h}{q!L^q_*}\right\} = T > 0.$$

We choose $\alpha > |\mathbf{b}|\sqrt{n}$ and consider the set $\overline{G^*} = \overline{G} \cup \bigcup_{z\in\overline{G}} \left\{w \in \mathbb{C}^n : |w - z| \le \frac{\alpha}{L(z)}\right\}$.

We denote $M = \max\{|F(z)| : z \in \overline{G^*}\}$. According to the Cauchy inequality for all $z \in \overline{G}$ the following inequality holds

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\right| = \frac{|g^{(m)}_z(0)|}{m!L^m(z)} \le \left(\frac{|\mathbf{b}|\sqrt{n}L(z)}{\alpha}\right)^m \frac{1}{L^m(z)} \max\left\{|g_z(\theta)| : |\theta| = \frac{\alpha}{|\mathbf{b}|\sqrt{n}L(z)}\right\} =$$
$$= \left(\frac{|\mathbf{b}|\sqrt{n}}{\alpha}\right)^m \max\left\{|F(z+\theta\mathbf{b})| : |z+\theta\mathbf{b} - z| = \frac{\alpha}{L(z)}\right\} \le$$
$$\le \left(\frac{|\mathbf{b}|\sqrt{n}}{\alpha}\right)^m \max\left\{|F(w)| : |w - z| = \frac{\alpha}{L(z)}\right\} \le M\left(\frac{|\mathbf{b}|\sqrt{n}}{\alpha}\right)^m.$$

Thus, for every $z \in \overline{G}$ we obtain

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\right| \le M\left(\frac{|\mathbf{b}|\sqrt{n}}{\alpha}\right)^m = \frac{M(|\mathbf{b}|\sqrt{n})^m}{T\alpha^m}T \le$$
$$\le \frac{M|\mathbf{b}\sqrt{n}|^m}{T\alpha^m}\max\left\{|F(z)|, \frac{1}{q_z!L^{q_z}(z)}\left|\frac{\partial^{q_z}F(z)}{\partial\mathbf{b}^{q_z}}\right|\right\} \le$$
$$\le \frac{M(|\mathbf{b}|\sqrt{n})^m}{T\alpha^m}\max\left\{\frac{1}{p!L^p(z)}\left|\frac{\partial^p F(z)}{\partial\mathbf{b}^p}\right| : 0 \le p \le q\right\}.$$

But $\frac{M(|\mathbf{b}|\sqrt{n})^m}{t\alpha^m} \longrightarrow 0$ as $m \longrightarrow +\infty$, then we choose $m^* \in \mathbb{N}$ so that $\frac{M(|\mathbf{b}|\sqrt{n})^m}{t\alpha^m} \le 1$ for all $m \ge m^*$. We obtain (1) at $m_0 = n^*$, that was necessary to prove. □

In [1] the following criterion of boundedness of L-index in direction is obtained.

Lemma 4 ([1])**.** *Let $L \in Q^n_{\mathbf{b}}$. An entire function $F(z), z \in \mathbb{C}^n$, is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that $(\forall\ z \in \mathbb{C}^n)$ one has* $\left|\frac{1}{L^{p+1}(z)}\frac{\partial^{p+1}F(z)}{\partial\mathbf{b}^{p+1}}\right| \le C\max\left\{\left|\frac{1}{L^k(z)}\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right| :\ 0 \le k \le p\right\}.$

According to Theorem 1, we obtain the following consequence from this criterion.

Theorem 2. *Let $L \in Q^n_{\mathbf{b}}$, G be a bounded domain in $\mathbb{C}^n$. An entire function $F(z), z \in \mathbb{C}^n$, is a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for all $z \in \mathbb{C}^n \setminus G$ the following inequality is true*

$$\left|\frac{1}{L^{p+1}(z)}\frac{\partial^{p+1}F(z)}{\partial \mathbf{b}^{p+1}}\right| \le C \max\left\{\left|\frac{1}{L^k(z)}\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : \ 0 \le k \le p\right\}.$$

In [1] we proved the following statement.

Lemma 5 ([1]). *An entire function $F(z)$, $z \in \mathbb{C}^n$, is a function of bounded L-index in a direction $\mathbf{b} \in \mathbb{C}^n$ if and only if there exists a constant $M > 0$ such that for all $z^0 \in \mathbb{C}^n$ the function $g^{\mathbf{b}}_{z^0}(t) = F(z^0 + t\mathbf{b})$ is a function of bounded $l^{\mathbf{b}}_{z^0}$-index $N(g^{\mathbf{b}}_{z^0}, l^{\mathbf{b}}_{z^0}) \le M < +\infty$, as a function of $t \in \mathbb{C}$ ($l^{\mathbf{b}}_{z^0}(t) \equiv L(z^0 + t\mathbf{b})$) and $N_{\mathbf{b}}(F, L) = \max\{N(g^{\mathbf{b}}_{z^0}, l^{\mathbf{b}}_{z^0}) : z^0 \in \mathbb{C}^n\}$.*

Remark that in [3, p. 132] the following statement is formulated and proved.

Lemma 6 ([3], Th.7.4, p. 132). *If $f(z)$, $z \in \mathbb{C}$, is an entire transcendental function, which satisfies the equation*

$$w^{(p)} + a_1 w^{(p-1)} + \dots + a_p w = 0 \tag{2}$$

with constant coefficients then f is of bounded index and

$$N(f) \le \min\left\{k \ge p-1 : \frac{|a_1|}{k+1} + \frac{|a_2|}{k(k+1)} + ... + \frac{|a_p|}{(k-p+2)\dots(k+1)} \le 1\right\}.$$

Below we formulate and prove an analogue of Lemma 6 for entire functions of bounded index in direction.

Theorem 3. *Let $F(z)$, $z \in \mathbb{C}^n$, be an entire transcendental function, which satisfies the equation*

$$\frac{\partial^p F(z)}{\partial \mathbf{b}^p} + a_1 \frac{\partial^{p-1} F(z)}{\partial \mathbf{b}^{p-1}} + \dots + a_p F(z) = 0 \tag{3}$$

with constant coefficients. Then $F(z)$ is of bounded index in an arbitrary direction $b \in \mathbb{C}^n$ and

$$N_{\mathbf{b}}(F) \le \min\left\{k \ge p-1 : \frac{|a_1|}{k+1} + \frac{|a_2|}{k(k+1)} + ... + \frac{|a_p|}{(k-p+2)\dots(k+1)} \le 1\right\}.$$

Proof. Let $g^{\mathbf{b}}_{z^0}(t) \equiv F(z^0 + t\mathbf{b})$, $z^0 \in \mathbb{C}^n$, $t \in \mathbb{C}$. At every fixed z^0 function $g(t) \equiv g^{\mathbf{b}}_{z^0}(t)$, as function of variable t, transform equation (3) in equation (2) relatively to the variable t, because $g^{(p)}(t) = \frac{\partial^p F(z^0+t\mathbf{b})}{\partial \mathbf{b}^p}$. Applying Lemma 6, we obtain that $g^{\mathbf{b}}_{z^0}(t)$ is a function of bounded index and its index is equal to

$$N(g^{\mathbf{b}}_{z^0}) \le \min\left\{k \ge p-1 : \tfrac{|a_1|}{k+1} + \tfrac{|a_2|}{k(k+1)} + ... + \tfrac{|a_p|}{(k-p+2)...(k+1)} \le 1\right\}.$$

But the value in the right-hand-side of the inequality does not depend on $z^0 \in \mathbb{C}^n$. Then indices of functions $g^{\mathbf{b}}_{z^0}(t)$ are uniformly bounded. Therefore, in view of Lemma 5, for function $g^{\mathbf{b}}_{z^0}(t)$ we obtain the desired conclusion. □

Let $\mathcal{K}$ be the class of positive on $[0, +\infty)$, continuous differentiable on $(0, +\infty)$ functions l such that $l'(x) = o(l^2(x))$ as $x \to +\infty$. We denote by $\widetilde{\mathcal{K}}$ the class of functions $L(z)$, $z \in \mathbb{C}^n$, such that $L(z) = l(|z|)$, where $l \in \mathcal{K}$.

In addition, we denote $l^*(r) = \int_0^r l(r)dr$, $M(r, f) = \max\{|f(t)| : |t| = r\}$, where $t \in \mathbb{C}$. In [4] the following two lemmas are formulated and proved.

Lemma 7 ([4]). *Let $l \in \mathcal{K}$, $f(t)$ be an entire function, $t \in \mathbb{C}$. If there are numbers $p \in \mathbb{Z}_+$, $C > 0$, such that for all $t \in \mathbb{C}, |t| \geq R$, one has $\frac{|f^{(p+1)}(t)|}{l^{p+1}(|t|)} \leq C \max\left\{\frac{|f^{(k)}(t)|}{l^k(|t|)} : 0 \leq k \leq p\right\}$, then*

$$\varlimsup_{r\to+\infty} \frac{\ln M(r,f)}{l^*(r)} \leq \max\{1, C\}.$$

Lemma 8 ([4]). *Let $l \in \mathcal{K}$, $f(t)$ be an entire function, $t \in \mathbb{C}$. If there are numbers $p \in \mathbb{Z}_+$, $C > 0$ such that $(\forall\, t \in \mathbb{C}, |t| \geq R)$: $\frac{|f^{(p+1)}(t)|}{(p+1)!l^{(p+1)!}(|t|)} \leq \max\left\{\frac{|f^{(k)}(t)|}{k!l^k(|t|)} : 0 \leq k \leq p\right\}$, then*

$$\varlimsup_{r\to+\infty} \frac{\ln M(r,f)}{l^*(r)} \leq (p+1)\max\{1, C\}.$$

Theorem 4. *Let $L \in \widetilde{\mathcal{K}} \cap Q^n_{\mathbf{b}}$. For all $z \in \mathbb{C}^n$ $|z| \geq R$, entire functions $g_0, g_1, \ldots, g_p$ and h satisfy the following conditions: 1) $|g_j(z)| \leq m_j l^j(|z|) g_0(z)$, $(1 \leq j \leq p)$; 2) $\left|\frac{\partial g_j(z)}{\partial \mathbf{b}}\right| < M_j L^{j+1}(z) g_0(z)$, $(0 \leq j \leq p)$; 3) $\left|\frac{\partial h(z)}{\partial \mathbf{b}}\right| \leq ML(z)|h(z)|$, where m_j, M_j, M are nonnegative constants. If an entire function $F(z)$, $z \in \mathbb{C}^n$ is a solution of the equation*

$$g_0(z)\frac{\partial^p F(z)}{\partial \mathbf{b}^n} + g_1(z)\frac{\partial^{p-1} F(z)}{\partial \mathbf{b}^{p-1}} + \ldots + g_p(z)F(z) = h(z),$$

then $F(z)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ and for all $z^0 \in \mathbb{C}^n$

$$\varlimsup_{r\to+\infty} \frac{\ln M_{\mathbf{b}}(r, F, z^0)}{L^*(z^0 + r\mathbf{b})} \leq \max\{1, C\},$$

where $L^*(z^0 + r\mathbf{b}) = \int_0^r L(z^0 + t\mathbf{b})dt$, $C = \sum_{j=1}^p M_j + (M+1)\sum_{j=1}^p m_j + M$.

Proof. The claim of the theorem follows similarly to [4] from Lemma 7, Lemma 8 and Theorem 1. □

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