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ASYMPTOTIC PROPERTIES OF HOLOMORPHIC FUNCTIONS IN THE HALF-PLANE OF IMPROVED REGULAR GROWTH OF ORDER LESS THAN ONE

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If f be a holomorphic function in the half-plane of order $\rho\in(0;1)$ with the indicator h, and for some $\rho_1\in(0;\rho)$ there exists a sequence (r_k) such that $0<r_k\uparrow+\infty$, $r_{k+1}^\rho-r_k^\rho=O(r_k^{\rho_1})$ and $\ln|f(r_ke^{i\varphi})|=r_k^\rho h(\varphi)+O\left(\frac{r_k^{\rho_1}}{\sin\varphi}\right)$ as $k\to+\infty$ uniformly with respect to $\varphi\in(0;\pi)$, then $\int_0^\pi\ln|f(re^{i\varphi})|\sin\varphi\,d\varphi=r^\rho\int_0^\pi h(\varphi)\sin\varphi\,d\varphi+O(r^{\rho_1})$ $(r\to+\infty)$ and there exists $\rho_2\in(0;\rho)$ such that $\sum\limits_{1<|\lambda_n|<r}|\lambda_n|\sin\varphi_n-\frac{1}{2\pi}\int_1^r\ln|f(t)f(-t)|\,dt-\frac{1}{2\pi}\int_1^r d[s(t)-s(-t)]=\frac{\rho-1}{2\pi}r^{\rho+1}\int_0^\pi h(\varphi)\sin\varphi\,d\varphi+O(r^{\rho_2+1})$, $\sum\limits_{1<|\lambda_n|<r}\sin\varphi_n-\frac{1}{2\pi}\int_1^r\frac{\ln|f(t)f(-t)|}{t}dt-\frac{1}{2\pi}\int_1^r\frac{d[s(t)-s(-t)]}{t}=\frac{\rho^2-1}{2\pi\rho}r^\rho\int_0^\pi h(\varphi)\sin\varphi d\varphi+O(r^{\rho_2})$ as $r\to+\infty$.

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Если для голоморфной в верхней полуплоскости функции f формального порядка $\rho\in(0;1)$ с индикатором h для некоторого $\rho_1\in(0;\rho)$ существует последовательность (r_k) такая, что $0<r_k\uparrow+\infty$, $r_{k+1}^\rho-r_k^\rho=O(r_k^{\rho_1})$ и $\ln|f(z)|=|z|^\rho h(\varphi)+O(|z|^{\rho_1})$ $(k\to+\infty)$ равномерно по $\varphi\in(0;\pi)$, то $\int_0^\pi\ln|f(re^{i\varphi})|\sin\varphi\,d\varphi=r^\rho\int_0^\pi h(\varphi)\sin\varphi\,d\varphi+O(r^{\rho_1})$ $(r\to+\infty)$ и существует $\rho_2\in(0;\rho)$ такое, что $\sum\limits_{1<|\lambda_n|<r}|\lambda_n|\sin\varphi_n-\frac{1}{2\pi}\int_1^r\ln|f(t)f(-t)|\,dt-\frac{1}{2\pi}\int_1^r d[s(t)-s(-t)]=\frac{\rho-1}{2\pi}r^{\rho+1}\int_0^\pi h(\varphi)\sin\varphi\,d\varphi+O(r^{\rho_2+1})$, $\sum\limits_{1<|\lambda_n|<r}\sin\varphi_n-\frac{1}{2\pi}\int_1^r\frac{\ln|f(t)f(-t)|}{t}dt-\frac{1}{2\pi}\int_1^r\frac{d[s(t)-s(-t)]}{t}=\frac{\rho^2-1}{2\pi\rho}r^\rho\int_0^\pi h(\varphi)\sin\varphi d\varphi+O(r^{\rho_2})$, $r\to+\infty$.

Let $f\colon\mathbb{C}^+\to\mathbb{C}$ be a holomorphic function in the half-plane $\mathbb{C}^+=\{z\colon \operatorname{Im} z>0\}$ of finite formal order $\rho\in(0;1)$, $\lambda_n=|\lambda_n|e^{i\varphi_n}$ zeroes of the function f in $\mathbb{C}^+$, $f(t)$, $t\in\mathbb{R}$, the angular boundary values of f on real axis, $s(t)$ the singular boundary function of f,

$$\sigma(r)=\sum_{1<|\lambda_n|<r}|\lambda_n|\sin\varphi_n-\frac{1}{2\pi}\int_1^r\ln|f(t)f(-t)|\,dt-\frac{1}{2\pi}\int_1^r d[s(t)-s(-t)],\tag{1}$$

$$a(r)=\sum_{1<|\lambda_n|<r}\sin\varphi_n-\frac{1}{2\pi}\int_1^r\ln|f(t)f(-t)|\frac{1}{t}\,dt-\frac{1}{2\pi}\int_1^r\frac{1}{t}\,d[s(t)-s(-t)].\tag{2}$$

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In the present paper we obtain an analogue to M. Govorov's results ([1]) on holomorphic functions of completely regular growth in the half-plane and results of B. Vynnyts'kyi, R. Khats on entire functions of improved regular growth ([2]).

Theorem 1. *Let for a holomorphic in $\mathbb{C}^+$ function $f\colon \mathbb{C}^+ \to \mathbb{C}$ of finite formal order $\rho \in (0;1)$ with some trigonometric ρ-convex on $(0;\pi)$ function $h\colon (0;\pi) \to \mathbb{R}$ there exist $\rho_1 \in (0;\rho)$ and sequence (r_k) such that*

$$0 < r_k \uparrow +\infty, \quad r_{k+1}^{\rho} - r_k^{\rho} = O(r_k^{\rho_1}), \quad k \to +\infty, \tag{3}$$

and

$$\ln|f(r_k e^{i\varphi})| = r_k^{\rho} h(\varphi) + O\left(\frac{r_k^{\rho_1}}{\sin\varphi}\right), \quad k \to +\infty. \tag{4}$$

uniformly with respect to $\varphi \in (0;\pi)$. Then

$$\int_0^{\pi} \ln|f(re^{i\varphi})| \sin\varphi \, d\varphi = r^{\rho} \int_0^{\pi} h(\varphi) \sin\varphi \, d\varphi + O(r^{\rho_1}), \quad r \to +\infty, \tag{5}$$

and for some $\rho_2 \in (0;\rho)$

$$\sigma(r) = \frac{\rho - 1}{2\pi} r^{\rho+1} \int_0^{\pi} h(\varphi) \sin\varphi \, d\varphi + O(r^{\rho_2+1}), \quad r \to +\infty, \tag{6}$$

$$a(r) = \frac{\rho^2 - 1}{2\pi\rho} r^{\rho} \int_0^{\pi} h(\varphi) \sin\varphi \, d\varphi + O(r^{\rho_2}), \quad r \to +\infty. \tag{7}$$

Proof. Suppose that $|f(t)| \le 1$, $t \in \mathbb{R}$. Then a and σ are non-decreasing on $[1;+\infty)$ functions. According to Carleman's general formula ([1, p.26]), we have

$$\sum_{1<|\lambda_n|<r} \left(\frac{1}{|\lambda_n|} - \frac{|\lambda_n|}{r^2}\right) \sin\varphi_n - \frac{1}{2\pi} \int_1^r \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln|f(t)f(-t)| \, dt -$$
$$-\frac{1}{2\pi} \int_1^r \left(\frac{1}{t^2} - \frac{1}{r^2}\right) d[s(t) - s(-t)] = \frac{1}{\pi r} \int_0^{\pi} \ln|f(re^{i\varphi})| \sin\varphi \, d\varphi + c_1 + \frac{c_2}{r^2},$$

where c_1, c_2 some positive constants. Therefore,

$$\int_1^r \left(\frac{1}{t^2} - \frac{1}{r^2}\right) d\sigma(t) = \frac{1}{\pi r} \int_0^{\pi} \ln|f(re^{i\varphi})| \sin\varphi \, d\varphi + c_1 + \frac{c_2}{r^2},$$

after integrations by parts, we obtain $\int_1^r t^{-3}\sigma(t) \, dt = \frac{1}{2\pi r} \int_0^{\pi} \ln|f(re^{i\varphi})| \sin\varphi \, d\varphi + \frac{c_1}{2} + \frac{c_2}{2r^2}$.
Using (4) we get

$$\int_1^{r_k} \frac{\sigma(t)}{t^3} dt = \frac{1}{2\pi r_k} \int_0^{\pi} \ln\left|f\left(r_k e^{i\varphi}\right)\right| \sin\varphi d\varphi + \frac{c_1}{2} + \frac{c_2}{2r_k^2} =$$
$$= \frac{r_k^{\rho-1}}{2\pi} \int_0^{\pi} h(\varphi) \sin\varphi d\varphi + O\left(r_k^{\rho_1 - 1}\right) + \frac{c_1}{2} \quad (k \to +\infty).$$

It follows that

$$\int_r^{r_k} \frac{\sigma(t)\,dt}{t^3} = \frac{r_k^{\rho-1}}{2\pi}\int_0^{\pi} h(\varphi)\sin\varphi d\varphi - \frac{1}{2\pi r}\int_0^{\pi} \ln\left|f\left(re^{i\varphi}\right)\right|\sin\varphi d\varphi + O\left(r_k^{\rho_1-1}\right) - \frac{c_2}{2r^2}$$

and

$$\Omega(r) := \int_r^{+\infty} \frac{\sigma(t)\,dt}{t^3} = -\frac{1}{2\pi r}\int_0^{\pi} \ln\left|f\left(re^{i\varphi}\right)\right|\sin\varphi d\varphi - \frac{c_2}{2r^2}, \quad r \in (1;+\infty). \tag{8}$$

Then

$$\Omega(r_k) = \int_{r_k}^{+\infty} \frac{\sigma(t)\,dt}{t^3} = -\frac{r_k^{\rho-1}}{2\pi}\int_0^{\pi} h(\varphi)\sin\varphi d\varphi + O\left(r_k^{\rho_1-1}\right) \quad (k \to +\infty),$$

where (r_k) is a sequence with properties (3)–(4). For every $r > r_1$ there exists an m such that $r_m \le r < r_{m+1}$, in view of (3) $r_m/r_{m+1} \to 1$, $r/r_{m+1} \to 1$, $r_m/r \to 1$, $r_m^{\rho-1} - r^{\rho-1} = O(r^{\rho_1-1})$ and $r_{m+1}^{\rho-1} - r^{\rho-1} = O(r^{\rho_1-1})$, if $r \to +\infty$. Moreover, since Ω is a non-increasing function, one has that

$$\Omega(r) \le \Omega(r_m) = -\frac{r^{\rho-1}}{2\pi}\int_0^{\pi} h(\varphi)\sin\varphi d\varphi - \frac{r_m^{\rho-1} - r^{\rho-1}}{2\pi}\int_0^{\pi} h(\varphi)\sin\varphi d\varphi + O\left(r_m^{\rho_1-1}\right) =$$

$$= -\frac{r^{\rho-1}}{2\pi}\int_0^{\pi} h(\varphi)\sin\varphi d\varphi + O\left(r^{\rho_1-1}\right) \quad (r \to +\infty).$$

Similarly, $\Omega(r) \ge \Omega(r_{m+1}) \ge -\frac{r^{\rho-1}}{2\pi}\int_0^{\pi} h(\varphi)\sin\varphi d\varphi + O\left(r^{\rho_1-1}\right)$ $(r \to +\infty)$. Therefore $\Omega(r) = -\frac{r^{\rho-1}}{2\pi}\int_0^{\pi} h(\varphi)\sin\varphi d\varphi + O\left(r^{\rho_1-1}\right)$ $(r \to +\infty)$. Combining this equality with (8), we obtain (5).

Further, according to monotonicity of σ, one has that

$$\sigma(r)\frac{R^2 - r^2}{2R^2r^2} \le \Omega(r) - \Omega(R) = \int_r^{R} \frac{\sigma(t)\,dt}{t^3} \le \sigma(R)\frac{R^2 - r^2}{2R^2r^2}, \quad 0 < r < R < +\infty \tag{9}$$

Let $A = -\frac{1}{2\pi}\int_0^{\pi} h(\varphi)\sin\varphi d\varphi$, $1 + \rho_1 - \rho < \alpha < 1$, $\max\{\rho + \alpha - 1, \rho_1 - \alpha + 1\} < \rho_2 < \rho$ and $R = r + r^{\alpha}$. Then $\Omega(r) - \Omega(R) = A\left(r^{\rho-1} - (r + r^{\alpha})^{\rho-1}\right) + O\left(r^{\rho_1-1}\right) = A(1-\rho)\,r^{\rho+\alpha-2} + O\left(r^{\rho+2\alpha-3}\right) + O\left(r^{\rho_1-1}\right)$, $\quad r \to +\infty$. Moreover,

$$\frac{2R^2r^2}{R^2 - r^2} = \frac{2(r + r^{\alpha})^2r^2}{(r + r^{\alpha})^2 - r^2} = r^{3-\alpha} + O(r^2) \quad (r \to +\infty).$$

Hence, from the left-hand-side of (9) we have that

$$\sigma(r) \le \frac{2R^2r^2}{R^2 - r^2}(\Omega(r) - \Omega(R)) = (r^{3-\alpha} + O(r^2))(A(1-\rho)r^{\rho+\alpha-2} + O(r^{\rho+2\alpha-3}) + O(r^{\rho_1-1})) =$$

$$= A(1-\rho)\,r^{\rho+1} + O\left(r^{\rho_2+1}\right) \quad (r \to +\infty). \tag{10}$$

Let $r = R - R^{\alpha}$. Then $\Omega(r) - \Omega(R) = A((R - R^{\alpha})^{\rho-1} - R^{\rho-1}) + O(R^{\rho_1-1}) = A(1-\rho)R^{\rho+\alpha-2} + O(R^{\rho+2\alpha-3}) + O(R^{\rho_1-1})$ $(r \to +\infty)$, and

$$\frac{2R^2r^2}{R^2 - r^2} = \frac{2(R - R^{\alpha})^2R^2}{R^2 - (R - R^{\alpha})^2} = \frac{2(1 - R^{\alpha-1})^2R^2}{1 - (1 - R^{\alpha-1})^2} = \frac{2(1 - 2R^{\alpha-1} + R^{2\alpha-2})R^2}{2R^{\alpha-1} - R^{2\alpha-2}} =$$

$$= \left(R^{3-\alpha} - 2R^2 + R^{1+\alpha}\right)\left(1 + \frac{R^{\alpha-1}}{2} + O\left(R^{2\alpha-2}\right)\right) = R^{3-\alpha} + O\left(R^2\right) \quad (r \to +\infty).$$

Therefore, from right-hand-side of (9) we obtain

$$\sigma(R) \geq \frac{2R^2r^2}{R^2 - r^2}(\Omega(r) - \Omega(R)) = (R^{3-\alpha} + O(R^2))(A(1-\rho)R^{\rho+\alpha-2} + O(R^{\rho+2\alpha-3}) +$$

$$+ O(R^{\rho_1 - 1})) = A\,(1-\rho)\,R^{\rho+1} + O\left(R^{\rho_2+1}\right), \quad r \to +\infty. \tag{11}$$

Then (10) and (11) imply that (6) holds. Besides,

$$a(r) = \int_1^r \frac{d\sigma(t)}{t} = \frac{\sigma(r)}{r} + \int_1^r \frac{\sigma(t)}{t^2} dt.$$

Thus, (7) follows from (6). Theorem 1 is proved in the case when the modules of the angular boundary values does not exceed one. The proof of the theorem in the general case reduces to previous by consideration of the function $F(z) = f(z)\,K_1 \exp\left(-Kz^\rho e^{-i\rho\pi/2}\right)$, where K, K_1 are some non-negative constants. □

REFERENCES

1. Говоров Н. В. Краевая задача Римана с бесконечным индексом. – М.: Наука, 1986. – 240 с.
2. Винницький Б. В., Хаць Р. В. *Про асимптотичну поведінку цілих функцій нецілого порядку* // Матем. студії. – 2004. – Т. 21, № 2. – С. 140–150.
3. Levin B. Ya. Lectures on entire functions. – Transl. Math. Monographs, V. 150. – Amer. Math. Soc., Providence RI, 1996. – 248 p.
4. Левин Б. Я. Распределение корней целых функций. – М.: Гостехиздат, 1956. – 632 с.
5. Гольдберг А. А., Островский И. В. Распределение значений мероморфных функций. – М.: Наука, 1970. – 591 с.
6. Гришин А. Ф. *О регулярности роста субгармонических функций* // Теор. функций, функц. анализ и их прилож.(Харьков). – 1968. – Вып.6. – С. 3–29.

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