УДК 517.5

S. I. Fedynyak

ON MAXIMUM MODULUS POINTS AND ZERO SET FOR AN ENTIRE FUNCTION

S. I. Fedynyak. *On maximum modulus points and zero set of an entire function*, Matematychni Studii, **30** (2008) 169–172.

Let f be an entire function. Let w be a point such that $|f(w)| = \max\{|f(z)| : |z| = |w|\}$ and $R(w, f)$ be the distance between the point w and the zero set of f. We obtain asymptotic estimates for the function $R(w, f), |w| \to \infty$, when f is an entire function of arbitrary growth.

С. И. Федыняк. *О точках максимума модуля и множестве нулей целой функции.* // Математичні Студії. – 2008. – Т.30, №2. – C.169–172.

Пусть f — целая функция. Точку w, для которой $|f(w)| = \max\{|f(z)| : |z| = |w|\}$, называем точкой максимума модуля функции f. Исследуется оценка снизу расстояния между точками максимума модуля и множеством нулей функции f.

1. Introduction. Let f be an entire function. A point w is called a *maximum modulus point* if $|f(w)| = M(|w|, f)$, where $M(r, f) = \max\{|f(z)| : \ |z| = r\}$. We denote by $R(w, f)$ the distance between a maximum modulus point w and the zero set of f, i.e.

$$R(w, f) = \inf\{|w - z| : \ f(z) = 0\}.$$

Let $\rho(r)$ be a proximate order such that $\lim\limits_{r\to\infty} \rho(r) = \rho \in (0, \infty)$. Denote by $[\rho(r), \sigma]$ the set of all entire functions of proximate order $\rho(r)$ and positive type σ with respect to $\rho(r)$.

I. V. Ostrovskii and A. E. Üreyen [1] showed that if $f \in [\rho(r), \sigma]$, then

$$\varliminf_{|w|\to\infty} |w|^{\rho(|w|)-1} R(w, f) \geq \frac{1}{e^2\rho\sigma}.$$

A. E. Üreyen [2] got similar results for the entire functions of zero or infinite order.

In particulary, if entire function f satisfies the condition

$$0 < \sigma = \varlimsup_{r\to+\infty} \frac{\ln M(r, f)}{(\ln r)^p} < +\infty, \quad (p > 1), \tag{1}$$

then

$$\varliminf_{|w|\to\infty} \frac{p \cdot (\ln |w|)^{p-1}}{|w|} \cdot R(w, f) \geq \frac{1}{e^2\sigma}. \tag{2}$$

This paper is devoted to the study of the lower asymptotic of $R(w, f)$ for entire functions of arbitrary growth.

2000 *Mathematics Subject Classification:* 30D20.

2. Auxiliary lemmas. Let Ω be the class of functions Φ positive on $(-\infty,+\infty)$, with positive continuous increasing to $+\infty$ derivatives Φ'. For $\Phi\in\Omega$ we denote $\Psi(x)=x-\Phi(x)/\Phi'(x)$. In [3] it is shown that Ψ is an increasing to $+\infty$ function on $(-\infty,+\infty)$. By Ψ^{-1} we denote the inverse function to Ψ.

Let

$$K(r,f):=r\frac{d}{dr}\ln M(r,f)=\frac{d}{d\ln r}\ln M(r,f).$$

Lemma 1. *If $\Phi\in\Omega$ and an entire function f satisfies the condition*

$$\varlimsup_{r\to+\infty}\frac{\ln M(r,f)}{\Phi(\ln r)}=1, \tag{3}$$

then

$$\varlimsup_{r\to+\infty}\frac{K(r,f)}{\Phi'(\Psi^{-1}(\ln r))}\le 1. \tag{4}$$

Proof. Without loss of generality, we assume that $f(0)=1$. For $r<R$ we have

$$\ln M(R,f)=\int\limits_0^R\frac{d}{dt}\ln M(t,f)\;dt=\int_0^R\frac{K(t,f)}{t}dt\ge\int_r^R\frac{K(t,f)}{t}dt\ge$$
$$\ge K(r,f)\int_r^R\frac{dt}{t}=K(r,f)\ln\frac{R}{r}.$$

Thus

$$\frac{K(r,f)}{\Phi'(\Psi^{-1}(\ln r))}\le\frac{\ln M(R,f)}{\Phi(\ln R)}\cdot\frac{\Phi(\ln R)}{\Phi'(\Psi^{-1}(\ln r))\cdot\ln\frac{R}{r}}.$$

We set $R=r\cdot\exp\left\{\dfrac{\Phi(\Psi^{-1}(\ln r))}{\Phi'(\Psi^{-1}(\ln r))}\right\}$. Then

$$\frac{\Phi(\ln R)}{\Phi'(\Psi^{-1}(\ln r))\cdot\ln\frac{R}{r}}=\frac{\Phi\left(\ln r+\dfrac{\Phi(\Psi^{-1}(\ln r))}{\Phi'(\Psi^{-1}(\ln r))}\right)}{\Phi(\Psi^{-1}(\ln r))}=$$
$$=\frac{\Phi\left(\ln r+\Psi^{-1}(\ln r)-\left(\Psi^{-1}(\ln r)-\dfrac{\Phi(\Psi^{-1}(\ln r))}{\Phi'(\Psi^{-1}(\ln r))}\right)\right)}{\Phi(\Psi^{-1}(\ln r))}=$$
$$=\frac{\Phi\left(\ln r+\Psi^{-1}(\ln r)-\Psi\left(\Psi^{-1}(\ln r)\right)\right)}{\Phi(\Psi^{-1}(\ln r))}=\frac{\Phi\left(\Psi^{-1}(\ln r)\right)}{\Phi(\Psi^{-1}(\ln r))}=1.$$

Hence we obtain inequality (4). □

We will use the following fact from [1] (substantially due to Macintyre [4]).

Lemma 2. *Let f be an arbitrary entire function and suppose that $h>0$. Then*

$$R(w,f)\ge h\exp\left\{-K(|w|+h,f)\cdot\frac{h}{|w|}\right\}.$$

3. Main result.

Theorem 1. *If $\Phi \in \Omega$ and an entire function f satisfies condition (3), then*

$$\varliminf_{|w|\to\infty} \frac{R(w,f)}{|w|} \cdot \Phi'\left(\Psi^{-1}\left(\ln|w| + \frac{1}{\Phi'(\Psi^{-1}(\ln|w|))}\right)\right) \geq \frac{1}{e}. \tag{5}$$

Proof. By Lemma 1 and 2, for arbitrary positive ε, we have

$$R(w,f) \geq h\exp\left\{-(1+\varepsilon)\Phi'(\Psi^{-1}(\ln(|w|+h)))\cdot\frac{h}{|w|}\right\},$$

when $|w|$ is sufficiently large. Setting

$$h = \frac{|w|}{\Phi'\left(\Psi^{-1}\left(\ln|w| + \frac{1}{\Phi'(\Psi^{-1}(\ln|w|))}\right)\right)},$$

we obtain

$$\frac{R(w,f)}{|w|}\Phi'\left(\Psi^{-1}\left(\ln|w| + \frac{1}{\Phi'(\Psi^{-1}(\ln|w|))}\right)\right) \geq$$
$$\geq \exp\left\{-(1+\varepsilon)\frac{\Phi'\left(\Psi^{-1}\left(\ln|w|+\frac{h}{|w|}\right)\right)}{\Phi'\left(\Psi^{-1}\left(\ln|w| + \frac{1}{\Phi'(\Psi^{-1}(\ln|w|))}\right)\right)}\right\} \geq \exp\{-(1+\varepsilon)\},$$

which implies (5). □

The following corollary improves inequality (2).

Corollary 1. *If an entire function f satisfies the condition (1), then*

$$\varliminf_{|w|\to\infty} \frac{p\cdot(\ln|w|)^{p-1}}{|w|}\cdot R(w,f) \geq \frac{1}{e\sigma}\cdot\left(\frac{p-1}{p}\right)^{p-1}.$$

Proof. For $\Phi(x) = \sigma x^p$ we have $\Psi^{-1}(x) = \frac{p}{p-1}\cdot x$. Hence

$$\Phi'\left(\Psi^{-1}\left(\ln|w|\right)\right) = \sigma p\left(\ln|w|\right)^{p-1}\left(\frac{p}{p-1}\right)^{p-1}.$$

Thus we obtain the assertion of the corollary. □

REFERENCES

1. Ostrovskii I.V., Üreyen A.E. *Distance between maximum modulus point and zero set for entire function*// Complex Variables. – 2003. – V.48. – P. 583–598.
2. Üreyen A.E. *On maximum modulus point and the zero set for entire function*// Computational Methods and Function Theory. – 2004. – V.4, №2. – P.341–354.
3. Шеремета М.М. *Про похідну цілої функції*// Укр. мат. журн. – 1988. – Т.40, №2. – P.188–192.
4. Macintyre A.J. *Wiman's method and "flat regions"of integral functions*// Quart. J. Math. – 1938. – V.9. – P.81–88.

Faculty of Mechanics and Mathematics
Ivan Franko National University of Lviv
fedynyak@yahoo.com

Received 1.07.2008