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DIFFERENTIAL SUBORDINATION THEOREMS FOR MEROMORPHIC FUNCTIONS CONTAINING INTEGRAL OPERATOR

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The purpose of this paper is to introduce some subordination results for meromomphic functions containing integral operator in a punctured unit disk.

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Для мероморфных в проколотом единичном круге функций, содержащих интегральный оператор, доказываются теоремы подчинения.

1. Introduction and preliminaries. Let $\mathcal{E}$ be the class of analytic functions, in the punctured unit disk $U\colon=\{z\in\mathbb{C},\ 0<|z|<1\}$ of the form $f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n$. A function $f\in\mathcal{E}$ is *meromorphic starlike* if $f(z)\neq 0$ and

$$(\forall\ z\in U)\colon\ -\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0.$$

Similarly, the function f is *meromorphic convex* if $f'(z)\neq 0$ and

$$(\forall\ z\in U)\colon\ -\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0.$$

In [1] Ravichandran et al. studied sufficient conditions for subordination for class $\mathcal{E}$ of meromorphic functions

$$(\forall\ z\in U)\colon\ -\frac{zf'(z)}{f(z)}\prec q(z).$$

Let F and G be analytic functions in the unit disk U. The function F is *subordinate* to G, written $F\prec G$, if G is univalent, $F(0)=G(0)$ and $F(U)\subset G(U)$. In general, given two functions F and G, which are analytic in U, the function F is said to be *subordinate* to G in U if there exists a function h, analytic in U with

$$h(0)=0\quad\text{and } (\forall\ z\in U)\colon\ |h(z)|<1,$$

such that

$$(\forall\ z\in U)\colon\ F(z)=G(h(z)).$$

Let $\phi\colon\mathbb{C}^2\to\mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)),zp'(z))\prec h(z)$ then p is called *a solution of the differential subordination.* The univalent function q is called a *dominant* of the solutions of the differential subordination, $p\prec q$.

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Given two functions $f, g \in \mathcal{E}$ such that
$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$$
their convolution or Hadamard product $f(z) * g(z)$ is defined by
$$f(z) * g(z) = (f * g)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n, \ \ z \in U.$$
In [2] Noor defined the function $\lambda(a, c, z)$ by
$$\lambda(a, c, z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(c)_{n+1}} z^n, \ \ z \in U,$$
$c \neq 0, -1, -2, ...,\ \ a > 0$, where $(a)_n$ is the Pochammer symbol (or shifted factorial) defined as $(a)_0 = 1,\ \ (a)_n = a(a+1)...(a+n-1),\ \ n > -1$. Then $\lambda(a, c, z) = \frac{1}{z}\, {}_2F_1(a, 1; c, z)$, where ${}_2F_1(a, b; c, z)$ is Gauss hypergeometric function defined as
$${}_2F_1(a, b; c, z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + ...$$
Let $f \in \mathcal{E}$. Denoted by $\widetilde{L}\colon \mathcal{E} \to \mathcal{E}$ the operator defined by
$$\widetilde{L}(a, c)f(z) = \lambda(a, c, z) * f(z), \ \ z \in U.$$
Defined the function $(\lambda(a, c, z))^{-1}$ given by
$$\lambda(a, c, z) * (\lambda(a, c, z))^{-1} = \frac{1}{z(1-z)^{\mu}}, \ \ \mu > 0, \ \ z \in U.$$
Analogous to $\widetilde{L}(a, c)$, a linear operator $I_\mu(a, c)$ on $\mathcal{E}$ is defined as follows
$$I_\mu(a, c)f(z) = (\lambda(a, c, z))^{-1} * f(z), \ \ \mu > 0, \ a > 0, \ \ c \neq 0, -1, -2, ... \qquad (1)$$
It can be verified that
$$z(I_\mu(a+1, c)f(z))' = aI_\mu(a, c)f(z) - (a+1)I_\mu(a+1, c)f(z)$$
and
$$z(I_\mu(a, c)f(z))' = \mu I_{\mu+1}(a, c)f(z) - (\mu+1)I_\mu(a, c)f(z). \qquad (2)$$

In the present investigation, we obtain sufficient conditions for a function containing Noor Integral operator (1) of meromorphic function f, by applying a method based on the differential subordination,
$$-\Big[\frac{I_\mu(a, c)f(z)}{z}\Big]^{\nu} \prec q(z), \ \ \nu > 0, \ \text{ and } \ -\Big[\frac{I_\mu(a, c)f(z)}{z}\Big]^{\nu} \prec -\Big[\frac{I_\mu(a, c)g(z)}{z}\Big]^{\nu}, \ \ \nu > 0.$$
In order to prove our subordination results, we need the following lemma in the sequel.

Lemma 1 ([3]). *Let q be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ for $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that: 1) Q is starlike univalent in U; 2) $(\forall\, z \in U)\colon\ \operatorname{Re}\frac{zh'(z)}{Q(z)} > 0$. If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$ then $p(z) \prec q(z)$ and q is the best dominant.*

Lemma 2 ([4]). *Let q be convex univalent in the unit disk U for ψ and $\gamma \in \mathbb{C}$ with*

$$\operatorname{Re}\Big\{1+\frac{zq''(z)}{q'(z)}+\frac{\psi}{\gamma}\Big\}>0. \tag{3}$$

If p is analytic in U and $\psi p(z)+\gamma zp'(z) \prec \psi q(z)+\gamma zq'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.

2. Subordination results. In this section by using Lemma 1 we prove the following subordination result.

Theorem 1. *Let $q(z) \neq 0$ be univalent in U such that $\dfrac{zq'(z)}{q(z)}$ is starlike univalent in U and*

$$\operatorname{Re}\Big\{1+\frac{\alpha}{\gamma}q(z)+\frac{2\beta}{\gamma}q^2(z)+\frac{3\delta}{\gamma}q^3(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\Big\}>0,\ \ \alpha,\ \gamma\ \in \mathbb{C},\ \ \gamma \neq 0. \tag{4}$$

If $f \in \mathcal{E}$ satisfies the subordination

$$\alpha\Big[-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^\nu\Big]+\beta\Big[\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^\nu\Big]^2+\delta\Big[-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^\nu\Big]^3-$$

$$-\nu\gamma\Big[\frac{z(I_\mu(a,c)f(z))'}{I_\mu(a,c)f(z)}-1\Big] \prec \alpha q(z)+\beta q^2(z)+\delta q^3(z)+\frac{\gamma zq'(z)}{q(z)}$$

then

$$-\Big[\frac{I_\mu(a,c)f(z)}{z}\Big]^\nu \prec q(z) \tag{5}$$

and q is the best dominant.

Proof. Our aim is to apply Lemma 1. Setting $p(z)\colon = -\Big[\dfrac{I_\mu(a,c)f(z)}{z}\Big]^\nu$. Computation shows that $\dfrac{zp'(z)}{p(z)}=-\nu\Big[\dfrac{z(I_\mu(a,c)f(z))'}{I_\mu(a,c)f(z)}-1\Big]$ which yields the following subordination

$$\alpha p(z)+\beta p^2(z)+\delta p^3(z)+\frac{\gamma zp'(z)}{p(z)} \prec \alpha q(z)+\beta q^2(z)+\delta q^3(z)+\frac{\gamma zq'(z)}{q(z)},\ \ \alpha,\ \gamma\ \in \mathbb{C}.$$

By setting $\theta(\omega)\colon = \alpha\omega+\beta\omega^2+\delta\omega^3$ and $\phi(\omega)\colon = \gamma/\omega$, $\gamma \neq 0$, it can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$ and $\phi(\omega)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(\omega) \neq 0$ for $\omega \in \mathbb{C}\backslash\{0\}$. Also, by setting

$$Q(z)=zq'(z)\phi(q(z))=\gamma z\frac{q'(z)}{q(z)},\ \ h(z)=\theta(q(z))+Q(z)=\alpha q(z)+\beta q^2(z)+\delta q^3(z)+\gamma z\frac{q'(z)}{q(z)},$$

we find that Q is starlike univalent in U and that

$$\operatorname{Re}\Big\{\frac{zh'(z)}{Q(z)}\Big\}=\Big\{1+\frac{\alpha}{\gamma}q(z)+\frac{2\beta}{\gamma}q^2(z)+\frac{3\delta}{\gamma}q^3(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\Big\}>0.$$

Then the relation (5) follows by an application of Lemma 1. □

Corollary 1. *Suppose that $f \in \mathcal{E}$ and (3) holds then*

$$-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^\nu+\Big[\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^\nu\Big]^2+\Big[-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^\nu\Big]^3-$$

$$-\nu\Big[\frac{z(I_\mu(a,c)f(z))'}{I_\mu(a,c)f(z)}-1\Big] \prec \frac{1+Az}{1+Bz}+\Big[\frac{1+Az}{1+Bz}\Big]^2+\Big[\frac{1+Az}{1+Bz}\Big]^3+\frac{(A-B)z}{(1+Az)(1+Bz)}$$

implies $-\Big(\dfrac{I_\mu(a,c)f(z)}{z}\Big)^\nu \prec \dfrac{1+Az}{1+Bz}$, $\ -1 \le B < A \le 1$, *and* $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. By setting $\alpha=\beta=\delta=\gamma=1$ and $q(z):=\frac{1+Az}{1+Bz}$ where $-1\le B<A\le 1$. $\square$

Corollary 2. *Suppose that* $f\in\mathcal{E}$ *and (3) holds then*

$$-\Big(\frac{f(z)}{z}\Big)^{\nu}+\Big[\Big(\frac{f(z)}{z}\Big)^{\nu}\Big]^2+\Big[-\Big(\frac{f(z)}{z}\Big)^{\nu}\Big]^3-\nu\Big[\frac{zf'(z)}{f(z)}-1\Big]\prec$$

$$\prec\frac{1+Az}{1+Bz}+\Big[\frac{1+Az}{1+Bz}\Big]^2+\Big[\frac{1+Az}{1+Bz}\Big]^3+\frac{(A-B)z}{(1+Az)(1+Bz)}$$

implies $-(\frac{f(z)}{z})^{\nu}\prec\frac{1+Az}{1+Bz},\quad -1\le B<A\le 1,$ *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. By setting $\alpha=\beta=\delta=\gamma=1$ and $q(z):=\frac{1+Az}{1+Bz}$ where $-1\le B<A\le 1$ and $\mu=2,\ a=2,\ c=1.$ $\square$

Corollary 3. *Suppose that* $f\in\mathcal{E}$ *and (3) holds then*

$$-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}+\Big[\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\Big]^2+\Big[-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\Big]^3-$$

$$-\nu\Big[\frac{z(I_\mu(a,c)f(z))'}{I_\mu(a,c)f(z)}-1\Big]\prec\frac{1+z}{1-z}+\Big[\frac{1+z}{1-z}\Big]^2+\Big[\frac{1+z}{1-z}\Big]^3+\frac{2z}{1-z^2}$$

implies $-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\prec\frac{1+z}{1-z},$ *and* $\frac{1+z}{1-z}$ *is the best dominant.*

Proof. By setting $\alpha=\beta=\delta=\gamma=1$ and $q(z):=\frac{1+z}{1-z}$. $\square$

Corollary 4. *Suppose that* $f\in\mathcal{E}$ *and (3) holds then*

$$-\Big(\frac{f(z)}{z}\Big)^{\nu}+\Big[\Big(\frac{f(z)}{z}\Big)^{\nu}\Big]^2+\Big[-\Big(\frac{f(z)}{z}\Big)^{\nu}\Big]^3-\nu\Big[\frac{zf'(z)}{f(z)}-1\Big]\prec$$

$$\prec\frac{1+z}{1-z}+\Big[\frac{1+z}{1-z}\Big]^2+\Big[\frac{1+z}{1-z}\Big]^3+\frac{2z}{1-z^2}$$

implies $-\Big(\frac{f(z)}{z}\Big)^{\nu}\prec\frac{1+z}{1-z},$ *and* $\frac{1+z}{1-z}$ *is the best dominant.*

Proof. By setting $\mu=2,\ a=2,\ c=1,\ \alpha=\beta=\delta=\gamma=1$ and $q(z):=\frac{1+z}{1-z}$. $\square$

Corollary 5. *Suppose that* $f\in\mathcal{E}$ *and (3) holds then*

$$-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}+\Big[\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\Big]^2+\Big[-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\Big]^3-$$

$$-\nu\Big[\frac{z(I_\mu(a,c)f(z))'}{I_\mu(a,c)f(z)}-1\Big]\prec e^{\nu Az}+e^{2\nu Az}+e^{3\nu Az}+\nu Az$$

implies $-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\prec e^{\nu Az},$ *and* $e^{\nu Az}$ *is the best dominant.*

Proof. By setting $\alpha=\beta=\delta=\gamma=1$ and $q(z):=e^{\nu Az},\ |\nu A|<\pi.$ $\square$

Corollary 6. *Suppose that $f \in \mathcal{E}$ and (3) holds then*

$$-\Big(\frac{f(z)}{z}\Big)^{\nu}+\Big[\Big(\frac{f(z)}{z}\Big)^{\nu}\Big]^{2}+\Big[-\Big(\frac{f(z)}{z}\Big)^{\nu}\Big]^{3}-\nu\Big[\frac{zf'(z)}{f(z)}-1\Big]\prec$$

$$\prec e^{\nu Az}+e^{2\nu Az}+e^{3\nu Az}+\nu Az$$

implies $-(\frac{f(z)}{z})^{\nu}\prec e^{\nu Az}$, *and* $e^{\nu Az}$ *is the best dominant.*

Proof. By setting $\mu=2$, $a=2$, $c=1$, $\alpha=\beta=\delta=\gamma=1$ and $q(z):=e^{\nu Az}$, $|\nu A|<\pi$. □

Corollary 7. *Let the assumptions of Theorem 1 hold. Then* $-\Big(\frac{f(z)}{z}\Big)^{\nu}\prec q(z)$ *and q is the best dominant.*

Proof. By setting $\mu=2$, $a=2$, $c=1$. □

Corollary 8. *Let the assumptions of Theorem 1 hold. Then* $-\Big(f'(z)+\frac{2f(z)}{z}\Big)^{\nu}\prec q(z)$ *and q is the best dominant.*

Proof. By setting $\mu=2$, $a=1$, $c=1$. □

Note that by using the identity (2), Theorem 1 becomes

Theorem 2. *Let $q(z)\neq 0$ be univalent in U such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U and satisfies (3). If $f\in\mathcal{E}$ satisfies the subordination*

$$\alpha\Big[-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\Big]+\beta\Big[\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\Big]^{2}+\delta\Big[-\Big(\frac{I_\mu(a,c)f(z)}{z}\Big)^{\nu}\Big]^{3}-$$

$$-\nu\gamma\Big[\frac{\mu I_{\mu+1}(a,c)f(z)}{I_\mu(a,c)f(z)}-(\mu+1)-1\Big]\prec\alpha q(z)+\beta q^{2}(z)+\delta q^{3}(z)+\frac{\gamma zq'(z)}{q(z)}$$

then $-\Big[\frac{I_\mu(a,c)f(z)}{z}\Big]^{\nu}\prec q(z)$ *and q is the best dominant.*

Theorem 3. *Let* $-\Big[\frac{I_\mu(a,c)g(z)}{z}\Big]^{\nu}$ *be convex univalent in the unit disk U and ψ and $\gamma\in\mathbb{C}$ with*

$$\operatorname{Re}\Big\{1+\frac{\psi}{\gamma}+\frac{zG'(z)}{G(z)}+\nu zG(z)\Big\}>0 \tag{6}$$

where $G(z):=\Big[\frac{(I_\mu(a,c)g(z))'}{I_\mu(a,c)g(z)}-\frac{1}{z}\Big]$. *If* $-\Big[\frac{I_\mu(a,c)f(z)}{z}\Big]^{\nu}$ *is analytic in U and*

$$-\Big[\frac{I_\mu(a,c)f(z)}{z}\Big]^{\nu}\Big\{\psi-\gamma\nu\Big[\frac{z(I_\mu(a,c)f(z))'}{I_\mu(a,c)f(z)}-1\Big]\Big\}\prec$$

$$\prec-\Big[\frac{I_\mu(a,c)g(z)}{z}\Big]^{\nu}\Big\{\psi-\gamma\nu\Big[\frac{z(I_\mu(a,c)g(z))'}{I_\mu(a,c)g(z)}-1\Big]\Big\},$$

then

$$-\Big[\frac{I_\mu(a,c)f(z)}{z}\Big]^{\nu}\prec-\Big[\frac{I_\mu(a,c)g(z)}{z}\Big]^{\nu} \tag{7}$$

and $-\Big[\frac{I_\mu(a,c)g(z)}{z}\Big]^{\nu}$ *is the best dominant.*

Proof. Our aim is to apply Lemma 2. Setting

$$p(z):\ = -\Big[\frac{I_\mu(a,c)f(z)}{z}\Big]^\nu \text{ and } q(z):\ = -\Big[\frac{I_\mu(a,c)g(z)}{z}\Big]^\nu.$$

Computation shows that $zp'(z) = -\nu\Big[\frac{I_\mu(a,c)f(z)}{z}\Big]^\nu\Big[\frac{z(I_\mu(a,c)f(z))'}{I_\mu(a,c)f(z)} - 1\Big]$ which yields the following subordination $\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z),\ \psi,\ \gamma \in \mathbb{C}$. Moreover, we have

$$\operatorname{Re}\Big\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\Big\} = \operatorname{Re}\Big\{1 + \frac{\psi}{\gamma} + \frac{zG'(z)}{G(z)} + \nu z G(z)\Big\} > 0.$$

Then the relation (7) comes by an application of Lemma 2. □

Corollary 9. *Let the assumptions of Theorem 3 hold. Then* $-\Big(\frac{f(z)}{z}\Big)^\nu \prec -\Big(\frac{g(z)}{z}\Big)^\nu$ *and* $-\Big(\frac{g(z)}{z}\Big)^\nu$ *is the best dominant.*

Proof. By setting $\mu = 2,\ \ a = 2,\ c = 1.$ □

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