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Ya. S. Mahola, M. M. Sheremeta

PROPERTIES OF ENTIRE SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION OF N-TH ORDER WITH POLYNOMIAL COEFFICIENTS OF N-TH DEGREE

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Properties of entire solutions of the linear differential equation

$$z^n w^{(n)} + \sum_{j=0}^{n-1} (a_{n-j}^{(j)} z^{j+1} + a_{n-j+1}^{(j)} z^j) w^{(j)} = 0$$

are investigated.

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Исследованы свойства целых решений линейного дифференциального уравнения

$$z^n w^{(n)} + \sum_{j=0}^{n-1} (a_{n-j}^{(j)} z^{j+1} + a_{n-j+1}^{(j)} z^j) w^{(j)} = 0.$$

1. Introduction. An analytic univalent in $\mathbb{D} = \{z : |z| < 1\}$ function f is said to be *convex* in $\mathbb{D}$ if $f(\mathbb{D})$ is a convex domain. It is well known [1, p. 203] that the condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. According to W. Kaplan [2], a function f is said to be *close-to-convex* in $\mathbb{D}$ (see also [1, p.583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0\,(z \in \mathbb{D})$. A close-to-convex function f has the characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in the complement of G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence, it follows that a function f is close-to-convex in $\mathbb{D}$ if and only if the function $(f(z) - f(0))/f'(0)$ is close-to-convex in $\mathbb{D}$. S. M. Shah ([3]) studied the close-to-convexity of entire solutions $f(z) = \sum_{s=0}^{\infty} f_s z^s$ of the differential equation

$$z^2 w'' + (a_1^{(1)} z^2 + a_2^{(1)} z) w' + (a_1^{(0)} z^2 + a_2^{(0)} z + a_3^{(0)}) w = 0. \quad (1)$$

Substituting f in (1) we obtain $a_3^{(0)} f_0 = 0$, $(a_2^{(1)} + a_3^{(0)}) f_1 + a_2^{(0)} f_0 = 0$ and

$$f_s = -\frac{a_1^{(1)}(s-1) + a_2^{(0)}}{s(s + a_2^{(1)} - 1) + a_3^{(0)}} f_{s-1} - \frac{a_1^{(0)}}{s(s + a_2^{(1)} - 1) + a_3^{(0)}} f_{s-2} \ (s \geq 2). \quad (2)$$

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If either $a_1^{(1)} = a_2^{(0)} = 0$ or $a_1^{(0)} = 0$ then two-term recurrent formula (2) reduces to a one-term recurrent formula, and in this case S.M. Shah has indicated conditions on the rest coefficients $a_m^{(n)}$ under which f and all its derivatives are close-to-convex in $\mathbb{D}$. In particular he has obtained the following result.

Theorem A. *Let* $a_1^{(0)} = 0$, $a_2^{(1)} > 0$ *and* $-1 \le a_1^{(1)} < 0$. *If either* $a_3^{(0)} = 0$ *and* $-a_2^{(1)} \le a_2^{(0)} < 0$ *or* $a_2^{(1)} + a_3^{(0)} = 0$ *and* $-a_2^{(1)} \le 2a_2^{(0)} < 0$ *then differential equation (1) has an entire solution* f *such that all its derivatives* $f^{(j)}\,(j \in \mathbb{Z}_+)$ *are close-to-convex in* $\mathbb{D}$ *and* $\ln M_f(r) = (1+o(1))|a_1^{(1)}|r$ *as* $r \to +\infty$, *where* $M_f(r) = \max\{|f(z)| : |z| = r\}$.

We remark that since $a_1^{(0)} = 0$ in Theorem A, differential equation (1) in this case has the following form

$$z^2 w'' + (a_1^{(1)} z^2 + a_2^{(1)} z) w' + (a_2^{(0)} z + a_3^{(0)}) w = 0. \tag{3}$$

For complex $a_1^{(1)}$, $a_2^{(1)}$, $a_2^{(0)}$, $a_3^{(0)}$ in [4] the following theorem is proved.

Theorem B. *Let* $a_1^{(1)} \ne 0$, $a_2^{(1)} + a_3^{(0)} = 0$, $|a_2^{(1)}| < 2$ *and* $\dfrac{2(|a_1^{(1)}| + |a_2^{(0)}|)}{2 - |a_2^{(1)}|} < \ln 2$. *Then differential equation (3) has an entire solution* f *such that all its derivatives* $f^{(j)}\,(j \in \mathbb{Z}_+)$ *are close-to-convex in* $\mathbb{D}$ *and* $\ln M_f(r) = (1+o(1))|a_1^{(1)}|r$ *as* $r \to +\infty$.

We remark also that in the general case of two-member recurrent formula (2), the close-to-convexity of entire solutions of (1) is investigated in [4-8].

Here we consider a generalization of Shah's differential equation (1)

$$z^n w^{(n)} + \sum_{j=1}^{n} \Big(\sum_{k=1}^{j+1} a_k^{(n-j)} z^{n-k+1} \Big) w^{(n-j)} = 0, \quad (n \ge 2) \tag{4}$$

and study properties of its solutions under some conditions on coefficients $a_k^{(n-j)}$, when $a_k^{(n-j)}$ is not necessarily real.

2. Recurrent formula for coefficients. Putting at first $a_1^{(n)} = 1$ we prove the following theorem.

Theorem 1. *A function* $f(z) = \sum_{s=0}^{\infty} f_s z^s$, *analytic at the origin, is a solution of differential equation (4) if and only if for each* $s \in \mathbb{Z}_+$

$$\sum_{m=0}^{\min\{s,n\}} \sum_{k=0}^{\min\{s,n\}-m} a_{n+1-k-m}^{(k)} \frac{(s-m)!}{(s-k-m)!} f_{s-m} = 0. \tag{5}$$

Proof. Rewriting differential equation (4) in the form

$$a_1^{(n)} z^n w^{(n)} + \big(a_1^{(n-1)} z^n + a_2^{(n-1)} z^{n-1}\big) w^{(n-1)} + \big(a_1^{(n-2)} z^n + a_2^{(n-2)} z^{n-1} + a_3^{(n-2)} z^{n-2}\big) w^{(n-2)} + \\ + \dots + \Big(\sum_{k=1}^{n+1-j} a_k^{(j)} z^{n-k+1} \Big) w^{(j)} + \dots + \Big(\sum_{k=1}^{n+1} a_k^{(0)} z^{n-k+1} \Big) w = 0$$

and regrouping members we obtain

$$\Big(a_1^{(n)} z^n w^{(n)} + a_2^{(n-1)} z^{n-1} w^{(n-1)} + a_3^{(n-2)} z^{n-2} w^{(n-2)} + \dots + a_{n+1-j}^{(j)} z^j w^{(j)} + \dots + \\ + a_n^{(1)} z w' + a_{n+1}^{(0)} w \Big) +$$

$$+\left(a_1^{(n-1)}z^n w^{(n-1)} + a_2^{(n-2)}z^{n-1}w^{(n-2)} + \cdots + a_{n-j}^{(j)}z^{j+1}w^{(j)} + \cdots + a_n^{(0)}zw\right) +$$
$$+\left(a_1^{(n-2)}z^n w^{(n-2)} + \cdots + a_{n-1-j}^{(j)}z^{j+2}w^{(j)} + \cdots + a_{n-1}^{(0)}z^2 w\right) + \cdots +$$
$$+\left(a_1^{(1)}z^n w' + a_2^{(0)}z^{n-1}w\right) + a_1^{(0)}z^n w = 0,$$

that is

$$\sum_{k=0}^{n} a_{n+1-k}^{(k)} z^k w^{(k)} + \sum_{k=0}^{n-1} a_{n-k}^{(k)} z^{k+1} w^{(k)} + \sum_{k=0}^{n-2} a_{n-1-k}^{(k)} z^{k+2} w^{(k)} + \cdots +$$
$$+\sum_{k=0}^{n-m} a_{n+1-k-m}^{(k)} z^{k+m} w^{(k)} + \cdots + \sum_{k=0}^{1} a_{2-k}^{(k)} z^{k+n-1} w^{(k)} + a_1^{(0)} z^n w = 0$$

and, thus

$$\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{n+1-k-m}^{(k)} z^{k+m} w^{(k)} = 0. \tag{6}$$

We suppose that a function $f(z) = \sum_{s=0}^{\infty} f_s z^s$, analytic at the origin, is a solution of differential equation (4). Since $f^{(k)}(z) = \sum_{s=k}^{\infty} \frac{s!}{(s-k)!} f_s z^{s-k}$, from (6) we obtain the identity

$$\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{n+1-k-m}^{(k)} \sum_{s=k}^{\infty} \frac{s!}{(s-k)!} f_s z^{s+m} \equiv 0,$$

that is

$$\Phi(z) := \sum_{m=0}^{n}\sum_{k=0}^{n-m}\sum_{s=k+m}^{\infty} a_{n+1-k-m}^{(k)} \frac{(s-m)!}{(s-k-m)!} f_{s-m} z^s \equiv 0. \tag{7}$$

Besides

$$\Phi(z) = \sum_{k=0}^{n}\sum_{s=k}^{\infty} a_{n+1-k}^{(k)} \frac{s!}{(s-k)!} f_s z^s + \cdots + \sum_{k=0}^{n-m}\sum_{s=k+m}^{\infty} a_{n+1-k-m}^{(k)} \frac{(s-m)!}{(s-k-m)!} f_{s-m} z^s + \ldots$$
$$\cdots + \sum_{k=0}^{1}\sum_{s=k+n-1}^{\infty} a_{2-k}^{(k)} \frac{(s-n+1)!}{(s-k-n+1)!} f_{s-n+1} z^s + \sum_{s=n}^{\infty} a_1^{(0)} f_{s-n} z^s =$$
$$= \sum_{s=n}^{\infty} a_1^{(n)} \frac{s!}{(s-n)!} f_s z^s + \sum_{k=0}^{n-1}\sum_{s=k}^{\infty} a_{n+1-k}^{(k)} \frac{s!}{(s-k)!} f_s z^s + \ldots$$
$$\cdots + \sum_{s=n}^{\infty} a_1^{(n-m)} \frac{(s-m)!}{(s-n)!} f_{s-m} z^s + \sum_{k=0}^{n-m-1}\sum_{s=k+m}^{\infty} a_{n+1-k-m}^{(k)} \frac{(s-m)!}{(s-k-m)!} f_{s-m} z^s + \ldots$$
$$\cdots + \sum_{s=n}^{\infty} a_1^{(1)} \frac{(s-n+1)!}{(s-n)!} f_{s-n+1} z^s + \sum_{s=n-1}^{\infty} a_2^{(0)} \frac{(s-n+1)!}{(s-n+1)!} f_{s-n+1} z^s + \sum_{s=n}^{\infty} a_1^{(0)} f_{s-n} z^s =$$
$$= \sum_{s=n}^{\infty}\left(\sum_{m=0}^{n} a_1^{(n-m)} \frac{(s-m)!}{(s-n)!} f_{s-m}\right) z^s + \sum_{k=0}^{n-1}\left(\sum_{s=n}^{\infty} + \sum_{s=k}^{n-1}\right) a_{n+1-k}^{(k)} \frac{s!}{(s-k)!} f_s z^s + \ldots$$
$$\cdots + \sum_{k=0}^{n-m-1}\left(\sum_{s=n}^{\infty} + \sum_{s=k+m}^{n-1}\right) a_{n+1-k-m}^{(k)} \frac{(s-m)!}{(s-k-m)!} f_{s-m} z^s + \ldots$$
$$\cdots + \left(\sum_{s=n}^{\infty} a_2^{(0)} f_{s-n+1} z^s + a_2^{(0)} f_0 z^{n-1}\right) =$$

$$= \sum_{s=n}^{\infty}\Big(\sum_{m=0}^{n} a_1^{(n-m)}\frac{(s-m)!}{(s-n)!}f_{s-m}\Big)z^s + \sum_{s=n}^{\infty}\Big(\sum_{m=0}^{n-1}\sum_{k=0}^{n-m-1} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}\Big)z^s +$$

$$+\sum_{k=0}^{n-1}\sum_{s=k}^{n-1} a_{n+1-k}^{(k)}\frac{s!}{(s-k)!}f_s z^s + \dots + \sum_{k=0}^{n-m-1}\sum_{s=k+m}^{n-1} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}z^s + \dots$$

$$\dots + a_2^{(0)}f_0 z^{n-1} = \sum_{s=n}^{\infty}\Big(\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}\Big)z^s + \Phi_1(z),$$

where

$$\Phi_1(z) = \sum_{k=0}^{n-1}\sum_{s=k}^{n-1} a_{n+1-k}^{(k)}\frac{s!}{(s-k)!}f_s z^s + \dots + \sum_{k=0}^{n-m-1}\sum_{s=k+m}^{n-1} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}z^s + \dots$$

$$+a_2^{(0)}f_0 z^{n-1} = \sum_{m=0}^{n-1}\sum_{k=0}^{n-m-1}\sum_{s=k+m}^{n-1} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}z^s.$$

Since Φ_1 is a polynomial of degree at most $n-1$, identity (6) is true if and only if $\Phi_1(z)\equiv 0$ and for all $s\ge n$

$$\sum_{m=0}^{n}\sum_{k=0}^{n-m} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m} = 0. \tag{8}$$

The equation (8) is a recurrent formula for f_s by $f_{s-1}, f_{s-2}, \dots, f_{s-n}$ for all $s\ge n$. It remains to find formulas for $f_0, f_1, \dots, f_{n-1}$, using the identity $\Phi_1(z)\equiv 0$. Since

$$\sum_{k=0}^{n-m-1}\sum_{s=k+m}^{n-1} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}z^s = \sum_{s=m}^{n-1} a_{n+1-m}^{(0)}\frac{(s-m)!}{(s-m)!}f_{s-m}z^s + \dots$$

$$\dots + \sum_{s=k+m}^{n-1} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}z^s + \dots + \sum_{s=n-2}^{n-1} a_3^{(n-m-2)}\frac{(s-m)!}{(s-n+2)!}f_{s-m}z^s +$$

$$+a_2^{(n-m-1)}\frac{(n-1-m)!}{0!}f_{n-1-m}z^{n-1} = a_{n+1-m}^{(0)}f_{n-1-m}z^{n-1} + \dots$$

$$\dots + a_{n+1-k-m}^{(k)}\frac{(n-1-m)!}{(n-1-k-m)!}f_{n-1-m}z^{n-1} + \dots + a_3^{(n-m-2)}\frac{(n-1-m)!}{1!}f_{n-1-m}z^{n-1} +$$

$$+a_2^{(n-m-1)}\frac{(n-1-m)!}{0!}f_{n-1-m}z^{n-1} + \sum_{s=m}^{n-2} a_{n+1-m}^{(0)}\frac{(s-m)!}{(s-m)!}f_{s-m}z^s + \dots$$

$$\dots + \sum_{s=k+m}^{n-2} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}z^s \dots + a_3^{(n-m-2)}\frac{(n-2-m)!}{0!}f_{n-2-m}z^{n-2} =$$

$$= \Big(\sum_{k=0}^{n-m-1} a_{n+1-k-m}^{(k)}\frac{(n-1-m)!}{(n-1-k-m)!}\Big)f_{n-1-m}z^{n-1} + \sum_{k=0}^{n-m-2}\sum_{s=k+m}^{n-2} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}z^s,$$

we have

$$\Phi_1(z) = \Big(\sum_{k=0}^{n-1} a_{n+1-k}^{(k)}\frac{(n-1)!}{(n-1-k)!}\Big)f_{n-1}z^{n-1} + \sum_{k=0}^{n-2}\sum_{s=k}^{n-2} a_{n+1-k}^{(k)}\frac{s!}{(s-k)!}f_s z^s + \dots +$$

$$+\Big(\sum_{k=0}^{n-m-1} a_{n+1-k-m}^{(k)}\frac{(n-1-m)!}{(n-1-k-m)!}\Big)f_{n-1-m}z^{n-1} +$$

$$+\sum_{k=0}^{n-m-2}\sum_{s=k+m}^{n-2} a_{n+1-k-m}^{(k)}\frac{(s-m)!}{(s-k-m)!}f_{s-m}z^s + \dots + (a_3^{(0)} + a_2^{(1)})f_1 z^{n-1} + a_3^{(0)}f_0 z^{n-2} +$$

$$+a_2^{(0)} f_0 z^{n-1} = \Big(\sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} a_{n+1-k-m}^{(k)} \frac{(n-1-m)!}{(n-1-k-m)!} f_{n-1-m} \Big) z^{n-1} + \Phi_2(z),$$

where $\Phi_2(z) = \sum_{m=0}^{n-2} \sum_{k=0}^{n-m-2} \sum_{s=k+m}^{n-2} a_{n+1-k-m}^{(k)} \frac{(s-m)!}{(s-k-m)!} f_{s-m} z^s$ is a polynomial of the degree $\leq n-2$. Therefore, $\Phi_1(z) \equiv 0$ if and only if $\Phi_2(z) \equiv 0$ and

$$\sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} a_{n+1-k-m}^{(k)} \frac{(n-1-m)!}{(n-1-k-m)!} f_{n-1-m} = 0. \tag{9}$$

Repeating the reasoning we come to the conclusion that $\Phi_2(z) \equiv 0$ if and only if

$$\sum_{m=0}^{n-2} \sum_{k=0}^{n-m-2} a_{n+1-k-m}^{(k)} \frac{(n-2-m)!}{(n-2-k-m)!} f_{n-2-m} = 0,$$

....

$$\sum_{m=0}^{n-j} \sum_{k=0}^{n-m-j} a_{n+1-k-m}^{(k)} \frac{(n-j-m)!}{(n-j-k-m)!} f_{n-j-m} = 0,$$

....

$$a_{n+1}^{(0)} f_0 = 0.$$

Hence and from (8) and (9) we obtain that f is a solution of differential equation (4) if and only if for each $s \in \{0, 1, \ldots, n-1\}$

$$\sum_{m=0}^{s} \sum_{k=0}^{s-m} a_{n+1-k-m}^{(k)} \frac{(s-m)!}{(s-k-m)!} f_{s-m} = 0, \tag{10}$$

and for all $s \geq n$ equality (8) holds. Theorem 1 is proved. □

If $a_k^{(j)} = 0$ for $k \in \{1, 2, \ldots, n-j-1\}$ and $j \in \{0, 1, \ldots, n-2\}$, i. e. $a_1^{(0)} = a_2^{(0)} = \cdots = a_{n-1}^{(0)} = 0, \ldots, a_1^{(n-3)} = a_2^{(n-3)} = 0, a_1^{(n-2)} = 0$ then, since (4) is equivalent to

$$z^n w^{(n)} + \sum_{j=0}^{n-1} \Big(\sum_{k=1}^{n-j+1} a_k^{(j)} z^{n-k+1} \Big) w^{(j)} = 0,$$

from (4) we have

$$z^n w^{(n)} + \sum_{j=0}^{n-1} \left(a_{n-j}^{(j)} z^{j+1} + a_{n-j+1}^{(j)} z^j \right) w^{(j)} = 0. \tag{11}$$

Differential equation (11) is a generalization of differential equation (3) and from Theorem 1 we obtain the following proposition.

Corollary 1. *A function $f(z) = \sum_{s=0}^{\infty} f_s z^s$, analytic at the origin, is a solution of the differential equation (11) if and only if*

$$a_{n+1}^{(0)} f_0 = 0, \tag{12}$$

$$(a_{n+1}^{(0)}+a_n^{(1)})f_1+a_n^{(0)}f_0=0 \tag{13}$$

and for $s\ge 2$

$$\sum_{k=0}^{\min\{s,n\}} a_{n+1-k}^{(k)}\frac{s!}{(s-k)!}f_s+\sum_{k=0}^{\min\{s,n\}-1} a_{n-k}^{(k)}\frac{(s-1)!}{(s-k-1)!}f_{s-1}=0. \tag{14}$$

Indeed, for $s=0$ and $s=1$ from (5) we obtain (12) and (13). If $s\ge 2$, then

$$0=\sum_{k=0}^{\min\{s,n\}} a_{n+1-k}^{(k)}\frac{s!}{(s-k)!}f_s+\sum_{k=0}^{\min\{s,n\}-1} a_{n-k}^{(k)}\frac{(s-1)!}{(s-k-1)!}f_{s-1}+$$
$$+\sum_{k=0}^{\min\{s,n\}-2} a_{n-k-1}^{(k)}\frac{(s-2)!}{(s-k-2)!}f_{s-2}+\dots+a_{n+1-\min\{s,n\}}^{(0)}f_{s-\min\{s,n\}}=$$
$$=\sum_{k=0}^{\min\{s,n\}} a_{n+1-k}^{(k)}\frac{s!}{(s-k)!}f_s+\sum_{k=0}^{\min\{s,n\}-1} a_{n-k}^{(k)}\frac{(s-1)!}{(s-k-1)!}f_{s-1}.$$

Further we study properties of entire solutions of differential equation (11) only.

Assume that the coefficients at f_s and f_{s-1} are not equal to zero for all $s\ge 2$ in equality (14), and we will find solutions of equation (11) in the form $f(z)=z+\sum\limits_{s=2}^{\infty}f_sz^s$ under the condition $a_{n+1}^{(0)}+a_n^{(1)}=0$ and in the form $f(z)=-\frac{a_n^{(1)}}{a_n^{(0)}}+z+\sum\limits_{s=2}^{\infty}f_sz^s$ when $a_{n+1}^{(0)}=0$ and $a_n^{(0)}\ne 0$.

3. Growth. The following theorem describes the growth of entire solution f of differential equation (11).

Theorem 2. *For a transcendental entire solution f of differential equation (11) one has* $\ln M_f(r)=(1+o(1))q\sqrt[q]{|a_q^{(n-q)}|r}$ $(r\to+\infty)$, *where* $q=\min\{j\in\{1,2,\dots,n\}:\ a_j^{(n-j)}\ne 0\}$.

Proof. For $n\ge 2$ and $s\ge n$ from (14) we obtain

$$\sum_{k=0}^{n}\frac{a_{n+1-k}^{(k)}}{(s-k)!}sf_s+\sum_{k=0}^{n-1}\frac{a_{n-k}^{(k)}}{(s-k-1)!}f_{s-1}=0. \tag{15}$$

Since $a_1^{(n)}=1$, we have $\sum\limits_{k=0}^{n}\frac{a_{n+1-k}^{(k)}}{(s-k)!}=\frac{a_1^{(n)}}{(s-n)!}+\frac{a_2^{(n-1)}}{(s-n+1)!}+\dots+\frac{a_{n+1}^{(0)}}{s!}=\frac{1+o(1)}{(s-n)!}$ $(s\to\infty)$. On the other hand, if $a_1^{(n-1)}\ne 0$ then

$$\sum_{k=0}^{n-1}\frac{a_{n-k}^{(k)}}{(s-k-1)!}=\frac{a_1^{(n-1)}}{(s-n)!}+\frac{a_2^{(n-2)}}{(s-n+1)!}+\dots+\frac{a_n^{(0)}}{(s-1)!}=\frac{(1+o(1))a_1^{(n-1)}}{(s-n)!}\quad(s\to\infty).$$

If $a_1^{(n-1)}=\dots=a_{q-1}^{(n-q+1)}=0$ and $a_q^{(n-q)}\ne 0$ $(2\le q\le n)$ then analogously

$$\sum_{k=0}^{n-1}\frac{a_{n-k}^{(k)}}{(s-k-1)!}=\frac{a_q^{(n-q)}}{(s-n+q-1)!}+\frac{a_{q+1}^{(n-q-1)}}{(s-n+q)!}+\dots+\frac{a_n^{(0)}}{(s-1)!}=\frac{(1+o(1))a_q^{(n-q)}}{(s-n+q-1)!}$$

as $s\to\infty$. Therefore, from (15) we have $f_s=-\frac{1+o(1)}{s}a_1^{(n-1)}f_{s-1}$ $(s\to\infty)$ provided $q=1$ and $f_s=-\dfrac{1+o(1)}{s}\dfrac{(s-n)!}{(s-n+q-1)!}a_q^{(n-q)}f_{s-1}=-\dfrac{1+o(1)}{s^q}a_q^{(n-q)}f_{s-1}$ $(s\to\infty)$, provided $2\le q\le n$. Hence it follows that for every $\varepsilon\in(0,1)$ and all $s\ge s_0(\varepsilon)$

$$\frac{1-\varepsilon}{s^q}|a_q^{(n-q)}||f_{s-1}|\le|f_s|\le\frac{1+\varepsilon}{s^q}|a_q^{(n-q)}||f_{s-1}|$$

and, therefore,

$$\frac{K_1}{(s!)^q}\left((1-\varepsilon)|a_q^{(n-q)}|\right)^s\le|f_s|\le\frac{K_2}{(s!)^q}\left((1-\varepsilon)|a_q^{(n-q)}|\right)^s, \tag{16}$$

where $K_1=K_1(\varepsilon)$ and $K_2=K_2(\varepsilon)$ are positive constants.

In order to obtain an asymptotic behavior of f from (16) we consider the function $f^*(r)=\sum\limits_{s=0}^{\infty}\frac{r^s}{(s!)^q}$ $r\ge0$. Let $\mu_{f^*}(r)=\max\{r^s/(s!)^q:\ s\ge0\}$ be maximal term of the last series and $\nu_{f^*}(r)=\max\{s:\ r^s/(s!)^q=\mu_{f^*}(r)\}$ be its central index. Then $\nu_{f^*}(r)=s$ for $s^q\le r<(s+1)^q$, whence $\nu_{f^*}(r)=(1+o(1))r^{1/q}$ $(r\to+\infty)$. Therefore,

$$\ln\mu_{f^*}(r)=\ln\mu_{f^*}(1)+\int\limits_1^r\frac{\nu_{f^*}(t)}{t}dt=(1+o(1))qr^{1/q},\quad r\to+\infty,$$

and by the Borel theorem, $\ln M_{f^*}(r)=(1+o(1))\ln\mu_{f^*}(r)=(1+o(1))qr^{1/q}$ $(r\to+\infty)$. Hence and from (16), in view of the arbitrariness of ε, we obtain

$$\ln M_f(r)=(1+o(1))q\left(|a_q^{(n-q)}|r\right)^{1/q}\ (r\to+\infty).$$

□

4. Close-to-convexity. For investigation of the close-to-convexity of a solution of differential equation (11) we use the following lemma ([4, 9]).

Lemma 1. *If* $\sum\limits_{s=2}^{\infty}s|f_s|<1$ *then the function*

$$f(z)=z+\sum_{s=2}^{\infty}f_sz^s \tag{17}$$

is close-to-convex in $\mathbb{D}$.

In view of Lemma 1, we search a solution of differential equation (11) in the form (17). Then $f_0=0$, $f_1=1$ and (13) implies $a_{n+1}^{(0)}+a_n^{(1)}=0$. Further we suppose that $\sum\limits_{k=0}^{\min\{s,n\}}a_{n+1-k}^{(k)}\dfrac{s!}{(s-k)!}\ne0$ for all $s\ge2$. Then for $2\le s\le n$ from (14) we obtain

$$f_s=-\frac{\sum\limits_{k=0}^{s-1}\frac{a_{n-k}^{(k)}}{(s-k-1)!}}{s\sum\limits_{k=0}^{s}\frac{a_{n+1-k}^{(k)}}{(s-k)!}}f_{s-1}=(-1)^2\frac{\sum\limits_{k=0}^{s-1}\frac{a_{n-k}^{(k)}}{(s-k-1)!}}{s\sum\limits_{k=0}^{s}\frac{a_{n+1-k}^{(k)}}{(s-k)!}}\frac{\sum\limits_{k=0}^{s-2}\frac{a_{n-k}^{(k)}}{(s-k-2)!}}{(s-1)\sum\limits_{k=0}^{s-1}\frac{a_{n+1-k}^{(k)}}{(s-k-1)!}}f_{s-2}=$$

$$=(-1)^{s-1}\frac{\sum\limits_{k=0}^{s-1}\frac{a_{n-k}^{(k)}}{(s-k-1)!}}{s\sum\limits_{k=0}^{s}\frac{a_{n+1-k}^{(k)}}{(s-k)!}}\frac{\sum\limits_{k=0}^{s-2}\frac{a_{n-k}^{(k)}}{(s-k-2)!}}{(s-1)\sum\limits_{k=0}^{s-1}\frac{a_{n+1-k}^{(k)}}{(s-k-1)!}}\cdots\frac{\sum\limits_{k=0}^{1}\frac{a_{n-k}^{(k)}}{(1-k)!}}{2\sum\limits_{k=0}^{2}\frac{a_{n+1-k}^{(k)}}{(2-k)!}}f_1=\frac{(-1)^{s-1}}{s!}\prod_{j=2}^{s}\frac{\sum\limits_{k=0}^{j-1}\frac{a_{n-k}^{(k)}}{(j-k-1)!}}{\sum\limits_{k=0}^{j}\frac{a_{n+1-k}^{(k)}}{(j-k)!}}. \tag{18}$$

If $s \geq n+1$ then

$$f_s = -\frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(s-k-1)!}}{s\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(s-k)!}} f_{s-1} = (-1)^2 \frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(s-k-1)!}}{s\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(s-k)!}} \frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(s-k-2)!}}{(s-1)\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(s-k-1)!}} f_{s-2} =$$

$$= (-1)^{s-n} \frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(s-k-1)!}}{s\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(s-k)!}} \frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(s-k-2)!}}{(s-1)\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(s-k-1)!}} \cdots \frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(n-k)!}}{(n+1)\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(n+1-k)!}} f_n =$$

$$= \frac{(-1)^{s-n}}{s(s-1)\dots(n+1)} \prod_{j=n+1}^{s} \frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(j-k-1)!}}{\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(j-k)!}} f_n,$$

that is, in view of (18) with $s = n$, we have

$$f_s = \frac{(-1)^{s-1}}{s!} \prod_{j=n+1}^{s} \left(\frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(j-k-1)!}}{\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(j-k)!}} \right) \prod_{j=2}^{n} \left(\frac{\sum\limits_{k=0}^{j-1} \frac{a_{n-k}^{(k)}}{(j-k-1)!}}{\sum\limits_{k=0}^{j} \frac{a_{n+1-k}^{(k)}}{(j-k)!}} \right). \tag{19}$$

Let $p \in \mathbb{N}$ be a fixed number. Since $f^{(p)}(z) = \sum\limits_{s=0}^{\infty} f_s^{(p)} z^s$ and $f_s^{(p)} = \dfrac{(s+p)!}{s!} f_{s+p}$ and $f_1^{(p)} \neq 0$ provided $f^{(p)}(z)$ is close-to-convex in $\mathbb{D}$, the derivative $f^{(p)}$ is close-to-convex in $\mathbb{D}$ if and only if the function $F_p(z) = (f^{(p)}(z) - f_0^{(p)})/f_1^{(p)} = z + \sum\limits_{s=2}^{\infty} f_{s,p} z^s$ is close-to-convex in $\mathbb{D}$, where $f_{0,p} = 0$, $f_{1,p} = 1$ and $f_{s,p} = \dfrac{f_s^{(p)}}{f_1^{(p)}} = \dfrac{(s+p)!}{s!(1+p)!} \dfrac{f_{s+p}}{f_{1+p}}$ for $s \geq 2$. Therefore, if $s + p \leq n$ then from (18) we have

$$f_{s,p} = \frac{(-1)^{s-1}}{s!} \prod_{j=2+p}^{s+p} \left(\sum_{k=0}^{j-1} \frac{a_{n-k}^{(k)}}{(j-k-1)!} \Big(\sum_{k=0}^{j} \frac{a_{n+1-k}^{(k)}}{(j-k)!} \Big)^{-1} \right). \tag{20}$$

If $1 + p < n$ and $s + p \geq n + 1$ then from (18) and (19) we obtain

$$f_{s,p} = \frac{(-1)^{s-1}}{s!} \prod_{j=n+1}^{s+p} \left(\frac{\sum\limits_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(j-k-1)!}}{\sum\limits_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(j-k)!}} \right) \prod_{j=p+2}^{n} \left(\frac{\sum\limits_{k=0}^{j-1} \frac{a_{n-k}^{(k)}}{(j-k-1)!}}{\sum\limits_{k=0}^{j} \frac{a_{n+1-k}^{(k)}}{(j-k)!}} \right) \tag{21}$$

and if $1+p = n$, $s+p \geq n+1$, then $f_{s,p} = \dfrac{(-1)^{s-1}}{s!} \prod\limits_{j=n+1}^{s+p} \left(\sum\limits_{k=0}^{n-1} \dfrac{a_{n-k}^{(k)}}{(j-k-1)!} \Big(\sum\limits_{k=0}^{n} \dfrac{a_{n+1-k}^{(k)}}{(j-k)!} \Big)^{-1} \right)$. Finally, if $1 + p \geq n + 1$ then, in view of (19),

$$f_{s,p} = \frac{(-1)^{s-1}}{s!} \prod_{j=2+p}^{s+p} \left(\sum_{k=0}^{n-1} \frac{a_{n-k}^{(k)}}{(j-k-1)!} \Big(\sum_{k=0}^{n} \frac{a_{n+1-k}^{(k)}}{(j-k)!} \Big)^{-1} \right). \tag{22}$$

Now we put $\varkappa = \max\left\{\left|\sum_{k=0}^{\min\{j,n\}-1}\frac{a_{n-k}^{(k)}}{(j-k-1)!}\right|\left|\sum_{k=0}^{\min\{j,n\}}\frac{a_{n+1-k}^{(k)}}{(j-k)!}\right|^{-1} : j \geq 2\right\}$. Then the following theorem is true.

Theorem 3. *If $a_{n+1}^{(0)} + a_n^{(1)} = 0$ and $\varkappa < \ln 2$ then differential equation (11) has an entire solution (17) such that all derivatives $f^{(p)}$ $(p \geq 0)$ are close-to-convex in $\mathbb{D}$.*

Proof. From (18)–(22) we obtain $|f_{s,p}| \leq \varkappa^{s-1}/s!$ for all $s \geq 2$ and $p \geq 0$, where $f_{s,0} = f_s$. Therefore, if $\varkappa < \ln 2$ and $p \geq 0$ then $\sum_{s=2}^{\infty} s|f_{s,p}| \leq \sum_{s=2}^{\infty}\dfrac{\varkappa^{s-1}}{(s-1)!} = e^{\varkappa} - 1 < 1$. In view of Lemma 1, Theorem 3 is proved. □

Now we suppose that $a_{n+1}^{(0)} = 0$ and $a_n^{(0)} \neq 0$. Then (12) holds and (13) implies that $f_0 = -a_n^{(1)}/a_n^{(0)}$ provided $f_1 = 1$. Therefore, we search a solution of differential equation (11) in the form

$$f(z) = -\frac{a_n^{(1)}}{a_n^{(0)}} + z + \sum_{s=2}^{\infty} f_s z^s. \tag{23}$$

For $s \geq 2$ recurrent formula (14) is true and thus, equalities (18)–(22) hold. Therefore, as above, we obtain the following proposition.

Theorem 4. *Let $a_{n+1}^{(0)} = 0$ and $a_n^{(0)} \neq 0$. If $\varkappa < \ln 2$ then differential equation (11) has an entire solution (24) such that all derivatives $f^{(p)}$ $(p \geq 0)$ are close-to-convex in $\mathbb{D}$.*

5. Convexity. For investigation of the convexity of the solution of differential equation (11), as in [10], we use the following lemma ([11]).

Lemma 2. *If $\sum\limits_{s=2}^{\infty} s^2|f_s| < 1$ then the function (17) is convex in $\mathbb{D}$.*

As in the proofs of Theorems 3 and 4, we have

$$\sum_{s=2}^{\infty} s^2|f_{s,p}| \leq \sum_{s=2}^{\infty}\frac{s\varkappa^{s-1}}{(s-1)!} = \sum_{s=2}^{\infty}\frac{\varkappa^{s-1}}{(s-2)!} + \sum_{s=2}^{\infty}\tfrac{\varkappa^{s-1}}{(s-1)!} = (\varkappa+1)e^{\varkappa} - 1 < 1$$

provided $\varkappa < (\ln 2)/2$. Thus, in view of Lemma 2, the following theorem is true.

Theorem 5. *Let $\varkappa < (\ln 2)/2$. If either $a_{n+1}^{(0)} + a_n^{(1)} = 0$ or $a_{n+1}^{(0)} = 0$ and $a_n^{(0)} \neq 0$ then differential equation (11) has an entire solution (17) or an entire solution (24) respectively, such that all derivatives $f^{(p)}$ $(p \geq 0)$ are convex in $\mathbb{D}$.*

6. Supplement. First we remark that it is possible to find sufficient conditions under which $\varkappa < \ln 2$ or $\varkappa < (\ln 2)/2$, that is, $\varkappa \leq \eta$ for $\eta > 0$. For example, let $\sum_{k=0}^{n-1}|a_{n-k}^{(k)}|^2 > 0$ and $a_{n-k}^{(k)} = |a_{n-k}^{(k)}|e^{i\theta}$ provided $|a_{n-k}^{(k)}| > 0$, and let $a_{n-k+1}^{(k)} \geq 0$ and $|a_{n-k}^{(k)}| \leq \eta a_{n-k}^{(k+1)}$ for all $0 \leq k \leq n-1$. Then

$$\left|\sum_{k=0}^{\min\{j,n\}-1}\frac{a_{n-k}^{(k)}}{(j-k-1)!}\right| \leq \eta\sum_{k=0}^{\min\{j,n\}-1}\frac{a_{n-k}^{(k+1)}}{(j-k-1)!} \leq \eta\sum_{k=0}^{\min\{j,n\}}\frac{a_{n+1-k}^{(k)}}{(j-k)!}.$$

In each concrete case it is possible to find more precise sufficient conditions under which $\varkappa \leq \eta$. For example, let $n = 3$. Then

$$\varkappa=\max\left\{\frac{|a_3^{(0)}+a_2^{(1)}|}{\left|a_2^{(2)}+a_3^{(1)}+\frac{a_4^{(0)}}{2}\right|},\ \left\{\frac{\left|a_1^{(2)}+\frac{a_2^{(1)}}{j-2}+\frac{a_3^{(0)}}{(j-2)(j-1)}\right|}{\left|1+\frac{a_2^{(2)}}{j-2}+\frac{a_3^{(1)}}{(j-2)(j-1)}+\frac{a_4^{(0)}}{(j-2)(j-1)j}\right|}:\ j\ge 3\right\}\right\}\le$$

$$\le\max\left\{\frac{|a_3^{(0)}+a_2^{(1)}|}{|a_2^{(2)}+a_3^{(1)}+a_4^{(0)}/2|},\ \frac{|a_1^{(2)}|+|a_2^{(1)}|+|a_3^{(0)}|/2}{1-(|a_2^{(2)}|+|a_3^{(1)}|/2+|a_4^{(0)}|/6)}\right\}\le\eta$$

provided $a_4^{(0)}=0$, $|a_3^{(0)}+a_2^{(1)}|\le\eta|a_2^{(2)}+a_3^{(1)}|$, $2|a_2^{(2)}|+|a_3^{(1)}|<2$ and $2(|a_1^{(2)}|+|a_2^{(1)}|)+|a_3^{(0)}|\le\le\eta(2-2|a_2^{(2)}|-|a_3^{(1)}|)$. Finally, we remark that the condition $\varkappa<\ln 2$ appeared in Theorems 3 and 4 is a consequence of the applied method. It is impossible to replace $\ln 2$ with a constant $c>\pi$. Indeed, for example, let $f(z)=\frac{1}{c}e^{cz}$, $c>0$. Then the function f is univalent in the disk $\{z:\ |z|<\pi/c\}$ and is not univalent in the closed disk $\{z:\ |z|\le\pi/c\}$. Therefore, if $c>\pi$, then f is not close-to-convex in $\mathbb{D}$. On the other hand, f is a solution of differential equation (11) with $a_{n+1}^{(0)}=0$, $a_{n-j}^{(j+1)}\ge 0$ and $a_{n-j}^{(j)}=-ca_{n-j}^{(j+1)}$, whence $\varkappa=c$.

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Faculty of Mechanics and Mathematics,
Lviv Ivan Franko National University,
Universytets'ka 1, 79000, Lviv

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