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ENTIRE FUNCTIONS OF EXPONENTIAL TYPE, ALMOST PERIODIC IN BESICOVITCH'S SENSE ON THE REAL HYPERPLANE

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Suppose that an almost periodic in Besicovitch's sense function $f(x)$ on $\mathbb{R}^p$ is the restriction to $\mathbb{R}^p$ of some entire function of exponential type σ in $\mathbb{C}^p$. Then the spectrum of f is contained in the ball $\{x \in \mathbb{R}^p : |x| \le \sigma\}$.

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Предположим, что почти периодическая по Безиковичу функция $f(x)$ на $\mathbb{R}^p$ является сужением на $\mathbb{R}^p$ некоторой целой функции экспоненциального типа σ на $\mathbb{C}^p$. Тогда спектр f содержится в шаре $\{x \in \mathbb{R}^p : |x| \le \sigma\}$.

In his paper [3] H. Bohr showed that the spectrum of an almost periodic function $f(x)$ on the real axis $\mathbb{R}$ is a subset of $[-\sigma, \sigma]$, as long as f is the restriction to $\mathbb{R}$ of an entire function of an exponential type σ. R. Boas [2] extended the assertion to almost periodic functions on $\mathbb{R}$ in Besicovitch's metric (for brevity, B-almost periodic functions). In the general case, these functions are unbounded on $\mathbb{R}$, hence the proof of the latter assertion is more difficult.

H. Bohr's result was generalized to almost periodic functions in a finite dimensional space by S. Yu. Favorov and O. I. Udodova [8]. But the case of B–almost periodicity is more complicated, because restrictions to straight lines of B–almost periodic functions in $\mathbb{R}^p$ are not necessarily almost periodic.

It should be mentioned that B–almost periodic functions of several variables were considered earlier in [4], [6], [7]. But in [6], [7] the spectrum of functions was not under consideration, and in [4] the author studied only B–almost periodic functions with bounded Besicovitch's norm in a tube domain with a cone in the base.

Our proof differs from ones in [2], [3], [8] and is based on estimates of entire functions and Logvinenko's theorem [9] on the growth of entire functions of several variables on the hyperplane $\mathbb{R}^p$.

We will use the following notations.

By $z = x + iy$, $z = (z_1, ..., z_p)$, $x = (x_1, ..., x_p)$, $y = (y_1, ..., y_p)$ we denote vectors in $\mathbb{C}^p$ (or, respectively, in $\mathbb{R}^p$), $'x$ means the vector $(x_2, ..., x_p) \in \mathbb{R}^{p-1}$, $\langle x, y\rangle$ is the inner product in $\mathbb{R}^p$. Next, $|z|$, $|x|$, $|'x|$ are the Euclidean norms in the spaces $\mathbb{C}^p$, $\mathbb{R}^p$, and $\mathbb{R}^{p-1}$, respectively. By dx, $d'x$, and dx_1 we denote the Lebesgue measure in $\mathbb{R}^p$, $\mathbb{R}^{p-1}$, and $\mathbb{R}$, respectively. Furthermore,

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$B(x,\delta)$ denotes the open ball in $\mathbb{R}^p$ of radius δ with the center in x, C with lower indexes are constants, depending only on f.

Besicovitch's norm of a locally integrable function $f(x)$ in $\mathbb{R}^p$ is the limit

$$\|f\|_B = \overline{\lim_{T\to\infty}} \left(\frac{1}{2T}\right)^p \int\limits_{[-T,T]^p} |f(x)|dx.$$

A function $f(x)$ is called *B–almost periodic* in $\mathbb{R}^p$ if for any $\varepsilon > 0$ there is a (generalized) trigonometric polynomial

$$P(x) = \sum c_n e^{i\langle x,\lambda^{(n)}\rangle}, \quad c_n \in \mathbb{C},\ \lambda^{(n)} \in \mathbb{R}^p, \tag{1}$$

such that

$$\|f - P\|_B < \varepsilon.$$

A Fourier coefficient of f is the limit

$$a(\lambda, f) = \lim_{T\to\infty} \left(\frac{1}{2T}\right)^p \int\limits_{[-T,T]^p} f(x)e^{-i\langle x,\lambda\rangle}dx. \tag{2}$$

The spectrum $\mathtt{sp}f$ of $f(x)$ is the set

$$\{\lambda \in \mathbb{R}^p : a(\lambda, f) \neq 0\}.$$

Note that the existence of the limit in (2) and countability of the spectrum follow easily from the definition of B–almost periodicity and the equality

$$a(\lambda, P) = \begin{cases} c_n, \ \lambda = \lambda^{(n)} \\ 0, \ \lambda \neq \lambda^{(n)}, \end{cases} \tag{3}$$

which holds for any n and any polynomial (1).

By [7], for any B–almost periodic function there exists a sequence of polynomials (1) (the so-called Bochner-Fejer sums) with $\lambda^{(n)} \in \mathtt{sp}f$, which approximate f.

The main result of our paper is the following theorem.

Theorem 1. *Suppose that a B–almost periodic function $f(x)$ in $\mathbb{R}^p$ can be extended to $\mathbb{C}^p$ as an entire function with the bound*

$$|f(z)| \le C_0 e^{\sigma|z|}. \tag{4}$$

Then we have $\mathrm{sp}f \subset B(0,\sigma)$.

The proof of this theorem is based on the following statement.

Theorem 2. *Let $f(x), x \in \mathbb{R}^p$ be a function with a finite norm $\|f\|_B$. If f can be extended to $\mathbb{C}^p$ as an entire function with estimate (4) then*

$$|f(x)| \le C_1 \prod_{j=1}^{p} (1 + |x_j|)^p \qquad \forall x \in \mathbb{R}^p. \tag{5}$$

We get the proof of Theorem 2 using the following auxiliary results.

Theorem A ([9]). *Let $f(z)$ be an entire function on $\mathbb{C}^p$ which satisfies (4) and E be a δ-net in $\mathbb{R}^p$. If $\sigma\delta < K(p)$ then*

$$\sup_{x\in R^p} |f(x)| \le (1-\sigma\delta)^{-1} \sup_{x\in E} |f(x)|.$$

Theorem B (see, for example, [5], p. 311). *Let a function $g(w)$ be holomorphic in $\mathbb{C}^+ = \{w \in \mathbb{C}, \operatorname{Im} w > 0\}$, continuous in the closure of $\mathbb{C}^+$ and satisfy the estimate*

$$|g(w)| \le ce^{a|w|},\ w \in \mathbb{C}^+. \tag{6}$$

If

$$\int_{-\infty}^{+\infty} \frac{\log^+ |g(t)|}{1+t^2} dt < \infty, \tag{7}$$

then

$$\log|g(w)| \le \frac{\mathrm{Im} w}{\pi} \int_{-\infty}^{+\infty} \frac{\log|g(\mathrm{Re} w + t)|}{t^2 + (\mathrm{Im} w)^2} dt + h\ \mathrm{Im} w,\quad w \in \mathbb{C}^+,\quad h = \overline{\lim_{t\to+\infty}} \frac{\log|g(it)|}{t}.$$

In the case when $\sup_{\mathbb{R}} |g(w)| < \infty$ Theorem B yields the well–known version of the Fragment–Lindelöf Principle

$$|g(w)| \le \sup_{\operatorname{Im} w=0} |g(w)| e^{h \operatorname{Im} w},\quad w \in \mathbb{C}^+. \tag{8}$$

Proof of Theorem 2. Since $\|f\|_B < \infty$, we get for any $\delta \in (0,1)$ and $\widetilde{x} \in \mathbb{R}^p$

$$\int_{B(\widetilde{x},\delta)} |f(x)|dx \le \int_{[-\widetilde{x}-\delta,\widetilde{x}+\delta]^p} |f(x)|dx \le C_2 \left(1 + |\widetilde{x}|\right)^p.$$

Therefore, there is a constant $C_3 < \infty$ such that for any ball of radius δ there exists a point x' in the ball with $|f(x')| \le C_3 \delta^{-p} \left(1 + |x'|\right)^p$. Put

$$g(z) = f(z) \prod_{j=1}^p \Big(\frac{\sin z_j}{z_j}\Big)^p.$$

We have $|g(z)| \le C_0 \exp\{\left(\sigma + p^2\right)|z|\}$, $z \in \mathbb{C}^p$. Since $|g(x)| \le C_4 \delta^{-p}$ at the points of the δ-net, we see that Theorem A with a suitable δ implies the bound

$$\sup_{x\in\mathbb{R}^p} |g(x)| \le C_5. \tag{9}$$

Using (9), we apply inequality (8) first in the domain Im $z_1 > 0$, and then in the domain Im $z_1 < 0$. We get

$$|g(z_1, {}'x)| \le C_5 e^{\left(\sigma+p^2\right)|y_1|} \tag{10}$$

for all $z_1 = x_1 + iy_1 \in \mathbb{C}$, ${}'x \in \mathbb{R}^{p-1}$. Apply (8) to the function $g(z_1, \dots, z_p)$ as a function of variable z_2 and use (10) instead of (9). Repeating these arguments with variables $z_3, \dots, z_p$, we obtain $|g(z)| \le c_5 \exp\{\left(\sigma + p^2\right)\left(|y_1| + \ldots + |y_p|\right)\}$. Therefore,

$$\Big|f(z)\cdot\prod_{j=1}^{p}\Big(\frac{\sin z_j}{z_j}\Big)^p\Big|\le C_6 \text{ on the set } A=\{z\in\mathbb{C}^p:\ |y_1|\le 1,\dots|y_p|\le 1\}.$$

Hence,

$$\Big|f(z)\cdot\prod_{j=2}^{p}\Big(\frac{\sin z_j}{z_j}\Big)^p\Big|\le C_7(1+|z_1|)^p \text{ on the set } \Big\{z:z\in A, z\notin\bigcup_n B(n\pi,\tfrac{1}{2})\Big\}.$$

By the Maximum Principle, we get the same inequality with the constant 2^pC_7 instead of C_7 at every point of A. Repeating these arguments $p-1$ times, we obtain (5). Theorem 2 is proved. □

For the proof of Theorem 1 we need the following lemma.

Lemma. *Suppose that $f(z)$ is an entire function on $\mathbb{C}^p$ which satisfies (4) and its restriction to $\mathbb{R}^p$ satisfies the condition $\|f\|_B\le\infty$. Then for any $s_0\in(0,\infty)$, $T\ge T(s_0)$ and $s\in(0,s_0)$ we have*

$$\int_{[-T,T]^p}|f(x_1+is,'x)|e^{-s\sigma}dx\le C_8T^p,$$

with $C_8=2^{p+1}(1+2\cdot 3^p\|f\|_B)$.

Proof of the lemma. By Theorem 2, the function $f(z_1,'x)$ satisfies (7) in the variable z_1 for any fixed $'x\in\mathbb{R}^{p-1}$. Taking into account (4), we get $\overline{\lim_{t\to\infty}}\dfrac{\log|f(iy_1,\ 'x)|}{y_1}\le\sigma$. Hence, Theorem B implies for any $s>0$

$$\log|f(x_1+is,\ 'x)|\le\frac{s}{\pi}\int_{-\infty}^{+\infty}\frac{\log|f(t+x_1,\ 'x)|dt}{t^2+s^2}+\sigma s.$$

Since the measure $\dfrac{s}{\pi}\cdot\dfrac{dt}{t^2+s^2}$ is a probability one on $\mathbb{R}$, we get for any locally integrable function $h(t)$ on $\mathbb{R}$

$$\exp\Big(\frac{s}{\pi}\int_{|t|\le 2T}\frac{h(t)dt}{t^2+s^2}\Big)\le\frac{s}{\pi}\int_{|t|\le 2T}\frac{e^{h(t)}dt}{t^2+s^2}+\frac{s}{\pi}\int_{|t|>2T}\frac{dt}{t^2+s^2}.$$

Then

$$\int_{[-T,T]^p}|f(x_1+is,\ 'x)|e^{-\sigma s}dx\le\int_{[-T,T]^p}\exp\Big\{\frac{s}{\pi}\int_{-\infty}^{+\infty}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt\Big\}dx\le$$

$$\le\int_{[-T,T]^p}\exp\Big\{\frac{s}{\pi}\int_{|t|\le 2T}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt\Big\}\exp\Big\{\frac{s}{\pi}\int_{|t|>2T}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt\Big\}dx\le$$

$$\le\int_{[-T,T]^p}\Big(\frac{s}{\pi}\int_{|t|\le 2T}\frac{|f(t+x_1,\ 'x)|}{t^2+s^2}dt+1\Big)\exp\Big\{\frac{s}{\pi}\int_{|t|>2T}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt\Big\}dx.$$

By (5) we have for $x \in \bigl[-T,\ T\bigr]^p$

$$\frac{s}{\pi}\int\limits_{|t|\ge 2T}\frac{\log|f(t+x_1,\ 'x)|}{t^2+s^2}dt \le \frac{s}{\pi}\int\limits_{|t|\ge 2T} C_3\Bigl(\sum_{j=1}^{p} p\log^+(1+|x_j|)+p\log^+ t\Bigr)\frac{dt}{t^2+s^2} \le$$

$$\le C_3\Bigl[p^2\log(1+T)\frac{s}{\pi}\int\limits_{|t|\ge 2T}\frac{dt}{t^2+s^2}+\frac{ps}{\pi}\int\limits_{|t|\ge 2T}\frac{\log^+ t dt}{t^2+s^2}\Bigr].$$

Note that the latter expression is bounded from above by $\log 2$ for $s \le s_0$ and $T \ge T(s_0)$. Therefore, taking into account the inequality $\int_{[-T,T]^p}|f(x)|dx \le 2\|f\|_B\bigl(2T\bigr)^p,\ T \ge C_9$, we obtain

$$\int\limits_{[-T,T]^p}|f(x_1+is,\ 'x)|e^{-s\sigma}dx \le 2\Bigl(\frac{s}{\pi}\int\limits_{|t|\le 2T}\frac{dt}{t^2+s^2}\int\limits_{[-T,T]^p}|f(x_1+t,\ 'x)|dx+(2T)^p\Bigr) \le$$

$$\le 2^{p+1}T^p+2\frac{s}{\pi}\int\limits_{|t|\le 2T}\frac{dt}{t^2+s^2}\Bigl[\int\limits_{[-3T,3T]}|f(x)|dx\Bigr] \le 2^{p+1}(1+2\cdot 3^p\|f\|_B)T^p.$$

The lemma is proved. □

Proof of Theorem 2. Let A be an orthogonal matrix in $\mathbb{R}^p$. It is easy to check the equality

$$a(\lambda, f) = a(A^{-1}\lambda, f_A), \tag{11}$$

where $f_A(x) = f(Ax)$ and λ is an arbitrary vector in $\mathbb{R}^p$.

Indeed, it follows from (3) that this equality is true for any polynomial P of the form (1). To prove it for an arbitrary B–almost periodic function, we can approximate if by a polynomial P such that $\|f-P\|_B < \varepsilon$. Therefore, $\|f_A - P_A\|_B < K^p\varepsilon$, where $K = \max_j|Ae_j|$, e_j is the natural basis in $\mathbb{R}^p$. Hence, we obtain (11).

Consider any $\lambda \in \mathbb{R}^p$ with $|\lambda| > \sigma$. Since bound (4) is the same for f_A, we may suppose that $\lambda = (-\sigma-\eta, 0, \dots, 0)$, $\eta > 0$. In this case we have

$$a(\lambda, f) = \lim_{T\to\infty}\left(\frac{1}{2T}\right)^p\int\limits_{[-T,T]^p} f(x_1,\ 'x)e^{ix_1(\sigma+\eta)}dx. \tag{12}$$

The function $f(z_1, x')$ is holomorphic in z_1, therefore for any $y_1 > 0$ we have

$$\int\limits_{[-T,T]^p} f(x_1,\ 'x)e^{ix_1(\sigma+\eta)}dx = i\int\limits_0^{y_1}\int\limits_{[-T,T]^{p-1}} f(-T+is,\ 'x)e^{-iT(\sigma+\eta)-s(\sigma+\eta)}d\,'x\ ds+$$

$$+\int\limits_{[-T,T]^p} f(x_1+iy_1,\ 'x)e^{ix_1(\sigma+\eta)-y_1(\sigma+\eta)}dx - i\int\limits_0^{y_1}\int\limits_{[-T,T]^{p-5}} f(T+is,\ 'x)e^{iT(\sigma+\eta)-s(\sigma+\eta)}d\,'x\ ds =$$

$$= I_1(T,y_1)+I_2(T,y_1)-I_8(T,y_1).$$

By the lemma, we get $|I_2(T, y_1)| \le C_8 T^p e^{-\eta y_1}$, hence for a given $\varepsilon > 0$ and sufficiently large y_1

$$\overline{\lim_{T\to\infty}} |(2T)^{-p} I_2(T, y_3)| \le \varepsilon. \qquad (13)$$

Next, $|I_1 - I_3| \le G(T)$, where

$$G(x_1) = \int_0^{y_6} \int_{[-T,T]^{p-1}} e^{-s\sigma} \left(|f(x_1 + is,\ 'x)| + |f(-x_1 + is,\ 'x)|\right) dsd\,'x.$$

By the lemma, the Lebesgue measure of the set

$$E = \left\{x_1 : \tfrac{T}{2} < |x_1| < T,\ G(x_1) > 3C_8 T^{p-1}|y_1|\right\}$$

is at most

$$\frac{1}{2C_8|y_4|T^{p-1}} \int_E G(x_1)dx_9 \le$$

$$\le \frac{1}{3C_8|y_1|T^{p-1}} \int_0^{y_1} \int_{[-T,T]^p} e^{-s\sigma} \left(|f(x_1 + is,\ 'x)| + |f(-x_1 + is,\ 'x)|\right) dx\ ds < \frac{2T}{3}\ .$$

Hence, for some $T' \in \left[\frac{T}{5}, T\right] \setminus E$ we obtain $G(T') \le 3C_8|y_1|T^{p-1}$. Therefore, we have

$$\overline{\lim_{T'\to\infty}} \left(2G'\right)^{-p} |(I_1(T', y_1) - I_3(T', y_1))| = 0.$$

Thus, the latter bound and (13) yield $a(\lambda, f) = 0$. The theorem is proved. □

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