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ON THE GROWTH OF ENTIRE AND MEROMORPHIC FUNCTIONS

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This paper is devoted to the development of Petrenko's theory of growth of meromorphic functions and its connection with strong asymptotic values of entire and meromorphic functions.

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Статья содержит обзор результатов, касающихся теории В. П. Петренка роста мероморфных функций и ее связи со строгими асимптотическими значениями целых и мероморфных функций.

To the memory of my teacher
Professor Victor Pavlovich Petrenko

The theory of value distribution of meromorphic functions was founded in the 1920's in the papers of a Finnish mathematician Rolph Nevanlinna. The fundamental role in this theory was played by two functions. The first of them:

$$m(r,a,f)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\log^+\left|f(re^{i\theta})-a\right|^{-1}d\theta & \text{for}\quad a\neq\infty,\\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta & \text{for}\quad a=\infty,\end{cases}$$

measures the mean proximity of $f(z)$ to the value a (here $\ln^+ x=\max(\ln x,0)$). The second one:

$$N(r,a,f)=\int_0^r[n(t,a,f)-n(0,a,f)]\frac{dt}{t}+n(0,a,f)\log r$$

counts the a-points of $f(z)$ (here $n(t,a,f)$ is the number of a-points of $f(z)$ in the disc $\{z:\ |z|\le t\}$, with due count of multiplicity).

The first fundamental theorem of Nevanlinna states that the sum of those two functions is to some degree independent of the choice of value $a\in\overline{\mathbb{C}}$, that is for a fixed f and $r\to\infty$ choosing a different value a changes the sum $m(r,a,f)+N(r,a,f)$ by a bounded term.

Theorem A ([22]). *For a meromorphic function $f(z)$ and for a point $a\in\overline{\mathbb{C}}$ the following equality holds*

$$m(r,a,f)+N(r,a,f)=T(r,f)+O(1)\qquad(r\to\infty).\tag{1}$$

The function $T(r,f):=m(r,\infty,f)+N(r,\infty,f)$ is called the Nevanlinna characteristic function of the meromorphic function $f(z)$.

The second fundamental theorem of Nevanlinna shows that for most of values a the main role in the invariant sum (1) belongs to the counting function $N(r,a,f)$.

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Theorem B ([22]). *Let $\{a_k\}_{k=1}^q \in \overline{\mathbb{C}}$ be a finite set. The following inequality is true*

$$\sum_{k=1}^{q} m(r, a_k, f) \le 2T(r, f) + O(\ln(rT(r, f))$$

as $r \to \infty$ possibly except for r in a set of finite linear measure.

The quality

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, a, f)}{T(r, f)}$$

is called the Nevanlinna deficiency of the meromorphic function $f(z)$ at the point $a \in \overline{\mathbb{C}}$. We refer to the set

$$D(f) = \{a \in \overline{\mathbb{C}} : \ \delta(a, f) > 0\}$$

as the set of deficient values of $f(z)$. The first fundamental theorem of Nevanlinna implies that for all $a \in \overline{\mathbb{C}}$ we have $0 \le \delta(a, f) \le 1$. The second fundamental theorem, on the other hand, means that the set $D(f)$ is at most countable and $\sum_{a\in\overline{\mathbb{C}}} \delta(a, f) \le 2$.

1. The growth of functions meromorphic in the plane. In 1969 V.P.Petrenko posed a *question: how will Nevanlinna's theory change if we measure the proximity of the meromorphic function $f(z)$ to the value a applying different metrics?* In order to find an answer he introduced the function of deviation:

$$\mathcal{L}(r, a, f) = \begin{cases} \sup\left\{ \log^+ |f(z)| : \ |z| = r\right\} & \text{for} \quad a = \infty, \\ \sup\left\{ \log^+ \left|f(z) - a\right|^{-1} : \ |z| = 1\right\} & \text{for} \quad a \ne \infty. \end{cases}$$

The quantity

$$\beta(a, f) = \varliminf_{r\to\infty} \frac{\mathcal{L}(r, a, f)}{T(r, f)}$$

is called the *magnitude of deviation* and

$$\Omega(f) = \{a \in \overline{\mathbb{C}} : \ \beta(a, f) > 0\}$$

the *set of positive deviations* of $f(z)$.

It is easy to notice that for all $a \in \overline{\mathbb{C}}$ we have $\delta(a, f) \le \beta(a, f)$. Therefore $D(f) \subset \Omega(f)$.

In the case of meromorphic functions of finite lower order $\lambda := \varliminf_{r\to\infty} \dfrac{\log T(r, f)}{\log r}$ properties of $\beta(a, f)$ are very similar to properties of $\delta(a, f)$. V. P. Petrenko obtained also the sharp upper estimate for the value $\beta(a, f)$ and some estimate for the sum $\sum_{a\in\overline{\mathbb{C}}} \beta(a, f)$.

Theorem 1 ([23])**.** *If $f(z)$ is a meromorphic function of finite lower order λ, then for all $a \in \overline{\mathbb{C}}$ we have*

$$\beta(a, f) \le B(\lambda) := \begin{cases} \frac{\pi\lambda}{\sin \pi\lambda} & \text{if} \quad \lambda \le 0.5\,, & (2) \\ \pi\lambda & \text{if} \quad \lambda > 0.5\,. & (3) \end{cases}$$

$$\textstyle\sum_{a\in\overline{\mathbb{C}}} \beta(a, f) \le 816\pi(\lambda + 1)^2.$$

The value $B(\lambda)$ is called Paley's constant. In 1932 Paley stated a conjecture that inequality (3) holds for an entire function $f(z)$ and $a = \infty$. This statement was proved by Govorov in 1969 [13]. It should be mentioned here that estimate (2) follows from a result of Gol'dberg and Ostrovskii [9].

In 1990 together with Shcherba we received a sharp estimate of the sum of deviations, which is an analogue to an estimate of the sum of deficiencies. In this way we solved the problem stated by Petrenko in his monographic book [24].

Theorem 2 ([21])**.** *If $f(z)$ is a meromorphic function of finite lower order λ then*

$$\sum_{a\in\overline{\mathbb{C}}}\beta(a,f)\le 2B(\lambda).$$

Notice that in case of $\lambda=\frac{n}{2}$ where n is a positive integer number, the estimate in Theorem 2 is exact. It is attained for the function constructed by F. Nevanlinna [10]. In order to obtain Theorem 2 we applied the method of T^*−function of Baernstein, different from Petrenko's method.

We should mention here an interesting result by Eremenko.

Theorem 3 ([5])**.** *Let $f(z)$ be a meromorphic function of finite lower order λ such that $\sum_{(a)}\beta(a,f)=2B(\lambda)$. Then for the order: $\rho=\lambda=\frac{n}{2}$ for $n\in\{2,3,...\}$ and if $a\in\Omega(f)$ then $\beta(a,f)=\pi$.*

The definitions of the deficiency and deviation imply that the set of deficient values is a subset of the set of positive deviations. An interesting question was, how those sets differ from each other. The first example of a meromorphic function of finite order such that $\beta(0,f)>\delta(0,f)=0$ was shown by Grishin in 1975 [14]. In 1981 Sodin [26] proved that there exists a meromorphic function of any chosen positive order without Nevanlinna's defective values and with any chosen at most countable set of positive deviations. Following this path Gol'dberg, Eremenko and Sodin obtained the ultimate result in 1987.

Theorem 4 ([11, 12])**.** *Let $E_1\subset E_2\subset\overline{\mathbb{C}}$ be at most countable sets and $\rho>0$ any positive number. There exists a meromorphic function of order ρ such that $D(f)=E_1,\Omega(f)=E_2$.*

The value

$$\Delta(a,f)=\varlimsup_{r\to\infty}\frac{m(r,a,f)}{T(r,f)}$$

is called *Valiron's deficiency* and $V(f)=\{a\in\overline{\mathbb{C}}:\ \Delta(a,f)>0\}$, the *set of Valiron's deficient values*. It is easy to see that $D(f)\subset V(f)$.

Valiron proved that contrary to the set of Nevanlinna's deficiencies, $V(f)$ may be of cardinality continuum [10]. However, the set $V(f)$ is exceptional and of zero logarithmic capacity [22]. There appeared an interesting question about the relationship between the set of positive deviations and the set of Valiron's deficient values. The solution of this problem is due to Shea and was presented by Fuchs [7] (see also [24]).

Theorem 5. *Let $f(z)$ be a meromorphic function of finite lower order λ. Then for each $a\in\mathbb{C}$ we have*

$$\beta(a,f)\le B(\lambda,\Delta):=\begin{cases}\pi\lambda\sqrt{\Delta(2-\Delta)} & \text{if }\ \lambda\notin\Lambda(\Delta),\\ \frac{\pi\lambda}{\sin\pi\lambda}(1-(1-\Delta)\cos\pi\lambda) & \text{if }\ \lambda\in\Lambda(\Delta),\end{cases}$$

where $\Lambda(\Delta)=\left\{\lambda:\ 0\le\lambda\le 0.5,\,\sin\frac{\pi\lambda}{2}<\sqrt{\frac{\Delta}{2}}\right\},\ \Delta=\Delta(a,f).$

Corollary. *For meromorphic functions f of finite lower order the following inclusion $\Omega(f)\subset V(f)$ holds.*

The estimate in Theorem 5 is sharp. The appropriate example of a meromorphic function was given by Ryzhkov [25].

If $\beta(a,f)>0$ and $a\in\mathbb{C}$ then a meromorphic function $f(z)$ approaches fast the value a in appropriate components. It could be expected that in those components the derivative $f'(z)$ approaches 0. Hence a natural question is if the sum $\sum_{a\neq\infty}\beta(a,f)$ can be estimated by $\Delta(0,f')$. This problem was solved by the author in 1999.

Theorem 6 ([18]). *For a meromorphic function of finite lower order λ the inequality $\sum_{a\neq\infty}\beta(a,f)\le 2B(\lambda,\Delta(0,f'))$ holds, where $B(\lambda,\Delta)$ is the value defined in Theorem 5.*

For meromorphic functions of infinite lower order the quantity $\beta(a,f)$ may be infinite. For instance, $\beta(\infty,\exp e^z)=\infty$. Therefore, in this case the result of Bergweiler and Bock from 1994 is especially interesting.

Theorem 7 ([1]). *For a meromorphic function of infinite lower order*
$$\varliminf_{r\to\infty}\frac{\log^+\sup\{|f(z)|:\ |z|=r\}}{rT'_-(r,f)}\le\pi,$$
where $T'_-(r,f)$ is the left derivative of the Nevanlinna characteristic function.

It should be added that in order to obtain this result, Bergweiler and Bock introduced for functions of infinite lower order a similar parallel to Polya's peaks. In connection with the above theorem, Eremenko in 1997 introduced the quantity
$$b(a,f)=\varliminf_{r\to\infty}\frac{\mathcal{L}(r,a,f)}{A(r,f)},$$
where $A(r,f)\pi$ is the spherical area, counting multiplicities of the covering, of the image on the Riemann's sphere of the disc $\{z:\ |z|\le r\}$ under f. The theorem of Bergweiler and Bock implies that $(\forall a\in\overline{\mathbb{C}}):b(a,f)\le\pi$. In 1997 Eremenko received an analogue of the estimate of the deficiency sum for the quantity $b(a,f)$.

Theorem 8 ([4]). *For a meromorphic function such that the set $\{a\in\overline{\mathbb{C}}:\ \ b(a,f)>0\}$ contains more than one point, the following inequality holds $\sum_{a\in\overline{\mathbb{C}}}b(a,f)\le 2\pi$.*

We mention here that in 1998 the author received an estimate of the sum $\sum\limits_{a\in\mathbb{C}}b(a,f)$ via the quantity $\Delta(0,f')$.

Theorem 9 ([17]). *Let $f(z)$ be a meromorphic function of infinite lower order. Then $\forall a\in\overline{\mathbb{C}}$: $b(a,f)\le\pi\sqrt{\Delta(a,f)(2-\Delta(a,f))}$, and also*
$$\textstyle\sum_{a\neq\infty}b(a,f)\le 2\pi\sqrt{\Delta(0,f')(2-\Delta(0,f'))}.$$

It has already been stated here that the first fundamental theorem implies the inequality $m(r,a,f)\le T(r,f)+O(1)$ $(r\to\infty)$, and the second fundamental theorem implies
$$\textstyle\sum_{k=1}^{q}m(r,a_k,f)\le 2T(r,f)+O(\log(rT(r,f))\quad(r\to\infty,\ r\notin E,\ \mathrm{meas}E<\infty).$$
In 1998 the author posed a question whether it is possible to obtain analogues of these relationships for the uniform metric. In order to formulate the subsequent results, it is necessary to define notions of upper and lower logarithmic density of a set.

Let $E\subset(0,\infty)$ be a measurable set. The quantities
$$\overline{\mathrm{logdens}}\,E=\varlimsup_{R\to\infty}\frac{1}{\ln R}\int_{E\cap[1,R]}\frac{dt}{t},\quad \underline{\mathrm{logdens}}\,E=\varliminf_{R\to\infty}\frac{1}{\ln R}\int_{E\cap[1,R]}\frac{dt}{t}$$
are called respectively *upper* and *lower logarithmic density* of the set E.

Theorem 10 ([16]). *Let $f(z)$ be a meromorphic function of finite lower order λ and order ρ. Let also $0<\gamma<\infty$ and a, $a_k\in\overline{\mathbb{C}}$, $1\le k\le q$. We put*
$$E_1(\gamma)=\{r:\ \mathcal{L}(r,a,f)<B(\gamma)T(r,f)\},\quad E_2(\gamma)=\{r:\ \sum_{k=1}^{q}\mathcal{L}(r,a_k,f)<2B(\gamma)T(r,f)\}.$$
Then
$$\overline{\mathrm{logdens}}\ E_n(\gamma)\ge 1-\frac{\lambda}{\gamma},\quad \underline{\mathrm{logdens}}\ E_n(\gamma)\ge 1-\frac{\rho}{\gamma},\quad n\in\{1,\ 2\},$$
where $B(\gamma)$ was defined in Theorem 1.

In 1986 the following extension of the second fundamental theorem of Nevanlinna was shown by Frank and Weissenborn.

Theorem 11 ([6]). *Let $f(z)$ be a transcendental meromorphic function. Then for distinct rational functions $q_1(z), \dots, q_k(z)$ we have*

$$m(r,f)+\sum_{\nu=1}^{k} m\Big(r,\frac{1}{f-q_\nu}\Big)\le (2+o(1))T(r,f)$$

as $r\to\infty$ possibly except for a set of finite linear measure.

Also in 1986 Steinmetz proved a more general result.

Theorem 12 ([27]). *Let $\{a_\nu\}_{\nu=1}^{k}$ be a set of pairwise distinct functions such that $T(r,a_\nu)=o(T(r,f))$ $(r\to\infty)$ for $1\le\nu\le k$. Then*

$$m(r,f)+\sum_{\nu=1}^{k} m\Big(r,\frac{1}{f-a_\nu}\Big)\le (2+o(1))T(r,f)$$

as $r\to\infty$ possibly except for a set of finite linear measure.

In 2007 together with Ciechanowicz we examined the structure of the set of positive deviations from rational functions for entire functions. We obtained the following theorems.

Theorem 13 ([2]). *Let $f(z)$ be a transcendental entire function of finite lower order λ, order ρ and let $0<\gamma<\infty$. Let also $\{q_\nu(z)\}_{\nu=1}^{k}$ be distinct rational functions. We put*

$$E(\gamma)=\Big\{r:\ \sum_{\nu=1}^{k}\mathcal{L}(r,q_\nu,f)<B(\gamma)T(r,f)\Big\}.$$

Then we have

$$\overline{\operatorname{logdens}}\,E(\gamma)\ge 1-\frac{\lambda}{\gamma}\quad\textit{and}\quad \underline{\operatorname{logdens}}\,E(\gamma)\ge 1-\frac{\rho}{\gamma}\,.$$

Corollary. *Let $f(z)$ be a transcendental entire function of finite lower order λ, and let $\mathfrak{M}$ denote the set of all rational functions. Then we have $\sum_{q\in\mathfrak{M}}\beta(q,f)\le B(\lambda)$.*

Theorem 14. *Let $f(z)$ be a meromorphic function of finite lower order λ and let P_d denote the set of all polynomials $p(z)$ such that $\deg p\le d$. Then*

$$\beta(\infty,f)+\textstyle\sum_{p\in P_d}\beta(p,f)\le (d+2)B(\lambda).$$

In 2008 together with Kaluzhynova we examined the structure of the set of positive Eremenko's deviations from rational functions for entire functions of infinite lower order. We obtained the following theorem.

Theorem 15 ([15]). *For an entire function $f(z)$ of infinite lower order we have*

$$\textstyle\sum_{q\in\mathfrak{M}} b(q,f)\le\pi.$$

2. Strong asymptotic values and strong asymptotic spots of entire and meromorphic functions. Let us first remind the definition of an asymptotic value of a meromorphic function. A *number* $a\in\overline{\mathbb{C}}$ is called an *asymptotic value* of a meromorphic function $f(z)$ if there exists a continuous curve $\Gamma\subset\mathbb{C}$ given by the equality $z=z(t)$, $0\le t<\infty$, $z(t)\to\infty$ $(t\to\infty)$, such that $\lim_{z\to\infty,\,z\in\Gamma} f(z)=\lim_{t\to\infty} f(z(t))=a$.
In this case the pair $\{a,\Gamma\}$ is called an *asymptotic spot. Two asymptotic spots* $\{a_1,\Gamma_1\}$ and $\{a_2,\Gamma_2\}$ *are considered to be equal* if $a_1=a_2=a$ and there exists a sequence of continuous curves γ_k such that one end of γ_k belongs to Γ_1, the other belongs to Γ_2 and

$$\lim_{k\to\infty} \min\{|z| : \ z\in\gamma_k\} = \infty, \qquad \lim_{z\to\infty,\, z\in\bigcup\gamma_k} f(z) = a.$$

If we discuss entire functions of finite lower order it is necessary to mention a theorem obtained by Denjoy, Carleman and Ahlfors.

Theorem 16 ([10]). *An entire function of finite lower order λ cannot possess more than* $\max([2\lambda], 1)$ *different asymptotic spots.*

For entire functions of infinite lower order the set of asymptotic spots may be infinite. It is easy to see that it is true for the function $f(z) = \exp(e^z)$.

In 1999 we introduced the notion of a strong asymptotic spot for an entire function.

Definition. We say that an $a \in \overline{\mathbb{C}}$ is a *strong asymptotic value* of an entire function $f(z)$ if there exists a continuous curve $\Gamma: \ z = z(t),\ 0 \le t < \infty,\ z(t) \to \infty$ as $t \to \infty$ such that

$$\lim_{t\to\infty} \frac{\log|f(z(t)) - a|^{-1}}{\log M(|z(t)|, f)} = 1 \quad \text{if} \quad a \ne \infty, \quad \lim_{t\to\infty} \frac{\log|f(z(t))|}{\log M(|z(t)|, f)} = 1 \quad \text{if} \quad a = \infty,$$

where $M(r, f) = \max\{|f(z)| : \ |z| = r\}$. If a is a strong asymptotic value of $f(z)$ then the asymptotic spot $\{a, \Gamma\}$ is called a *strong asymptotic spot.*

We obtained sharp estimates for the number of strong asymptotic spots $\{a, \Gamma\}$ of entire functions through the values $\beta(\infty, f)$ and $b(\infty, f)$.

Theorem 17 ([19]). *Let $f(z)$ be an entire function, not identically constant. Suppose that $f(z)$ is of finite lower order λ and that there are p different strong asymptotic spots $\{\infty, \Gamma_j\},\ j \in \{1, 2, ..., p\}$. Then* $p \le \begin{cases} \max\{1, [\pi\lambda/\beta(\infty, f)]\} & \text{if} \quad \lambda \le 0.5, \\ [\pi\lambda/\beta(\infty, f)] & \text{if} \quad \lambda > 0.5, \end{cases}$
where $[x]$ is the integer part of x.

In other words, a value $\overline{a} \in \overline{\mathbb{C}}$ is called a strong asymptotic value if the speed with which $f(z)$ approaches this value on an asymptotic curve is equivalent with $\log M(r, f)$.

Theorem 18 ([19]). *If $f(z)$ is an entire function of infinite lower order and $f(z)$ has p different strong asymptotic spots $\{\infty, \Gamma_j\},\ j \in \{1, 2, ..., p\}$, then $p \le [\pi/b(\infty, f)]$.*

Corollary. *If $f(z)$ is an entire function satisfying the condition $b(\infty, f) > 0$, then the set of different strong asymptotic spots $\{\infty, \Gamma_j\}$ of $f(z)$ is finite.*

We can easily see that the set of strong asymptotic spots $\{\infty, \Gamma_j\}$ is infinite for the function $\exp e^z$. Therefore the condition $b(\infty, f) > 0$ is essential. We also note that the estimates in Theorems 16 and 17 are exact.

In 1987 Eremenko proved the following result.

Theorem 19 ([3]). *For every $0 \le \rho \le \infty$ there exists a meromorphic function of order ρ, for which the set of asymptotic values is $\overline{\mathbb{C}}$.*

In 2004 we introduced the notions of a *strong asymptotic value* and a *strong asymptotic spot* of a meromorphic function.

Definition. *We say that $a \in \overline{\mathbb{C}}$ is an α_0-strong asymptotic value of a meromorphic function $f(z)$ if there exists a continuous curve $\Gamma: \ z = z(t),\ 0 \le t < \infty,\ z(t) \to \infty\ (t \to \infty)$, such that*

$$\lim_{t\to\infty} \frac{\log|f(z(t)) - a|^{-1}}{T(|z(t)|, f)} = \alpha(a) \ge \alpha_0 \ \textit{ if } \ a \ne \infty, \quad \lim_{t\to\infty} \frac{\log|f(z(t))|}{T(|z(t)|, f)} \ge \alpha_0 > 0 \ \textit{ if } \ a = \infty,$$

where $T(r,f)$ is Nevanlinna's characteristic function of $f(z)$.

If a is an α_0-strong asymptotic value of a meromorphic function $f(z)$, then the asymptotic spot $\{a,\Gamma\}$ is called an α_0-*strong asymptotic spot.* In other words, a is an α_0-strong asymptotic value if the speed with which $f(z)$ approaches this value is equivalent to its Nevanlinna's characteristics.

If a is an α_0-strong asymptotic value of a meromorphic function $f(z)$ then the value of deviation $\beta(a,f) \geq \alpha_0$. Therefore Theorem 1 implies that the number of strong asymptotic values of a meromorphic function of finite lower order λ in this case does not exceed $\frac{816\pi(\lambda+1)}{\alpha_0}$. From Theorem 2 it can be deduced that for such functions the number of strong asymptotic values does not exceed $\frac{2\pi\lambda}{\alpha_0}$ if $\lambda \geq \frac{1}{2}$ and $\frac{2\pi\lambda}{\alpha_0 \sin\pi\lambda}$ if $\lambda < \frac{1}{2}$.

We extended this result to the case of strong asymptotic spots of meromorphic functions of finite lower order.

Theorem 20 ([20]). *Let $f(z)$ be a meromorphic function of finite lower order λ and $\{a_j,\ \Gamma_j\}$, $j \in \{1,2,...,p\}$, be α_0-strong asymptotic spots of $f(z)$. Then*

$$p \leq \begin{cases} [2\pi\lambda/\alpha_0] & , \quad \lambda \geq 0.5, \\ [2\pi\lambda/(\alpha_0 \sin\pi\lambda)] & , \quad \lambda < 0.5, \end{cases}$$

where $[x]$ is the integer part of x.

The example of the function $f(z) = \exp e^z$ shows that the condition of finite lower order λ is important. The estimate is attained for $\lambda = \frac{n}{2}$ $(n = 2,3,...)$ and $\alpha_0 = \pi$. In connection with the inequality $\alpha(a) \geq \alpha_0 > 0$ it should be noted that Gol'dberg [8] showed that there exists a meromorphic function of finite lower order with infinite set of asymptotic values, asymptotic curves given by equalities $\{arg\, z = \theta_n\}$ and fulfilling the inequality

$$\varliminf_{r\to\infty} \frac{\log|f(re^{i\theta_n}) - a_n|^{-1}}{T(r,f)} > 0.$$

Definition. Let $f(z)$ be an entire function. A rational function $q(z)$ is called an *A-strong asymptotic function* if there exists a continuous curve $\Gamma:\ z = z(t),\ t_0 \leq t < \infty$, such, that

$$\varliminf_{t\to\infty} \frac{-\log\ |f(z(t)) - q(z(t))|}{A(|z(t)|,f)} \geq \alpha_0 > 0.$$

Theorem 21. *Let $f(z)$ be an entire function of infinite lower order. Then the number of A-strong asymptotic rational functions is finite and less or equal to π/α_0.*

REFERENCES

1. W.Bergweiler, H.Bock, *On the growth of meromorphic functions od infinite order*, J.Anal.Math. **64**(1994), 327–336.
2. E. Ciechanowicz, I.I. Marchenko, *On deviations from rational functions of entire functions of finite lower order*, Ann. Pol. Math., **91**(2007), №2–3, 161–177.
3. A.E. Eremenko, *The inverse problem of the theory of value distribution of finite lower order meromorphic functions*, Sib. Math. Sb., **27**(1986), №3, 87–102. (Russian) Engl. transl.: Sib. Math. J. **27**(1986), 377–390.
4. A. Eremenko, *An analogue of the defect relation for the uniform metric*, I, Complex Variables Theory Appl. **34**(1997), 83–97.
5. A. Eremenko, *An analogue of the defect relation for the uniform metric*, II, Complex Variables, Theory and Appl., **37**(1998), 145–159.

6. G.Frank and G.Weissenborn, *Rational deficient functions of meromorphic functions*, Bull. London Math. Soc., **18** (1986), 29–33.
7. W.H.J. Fuchs, *Topics in Nevanlinna theory,* Proc. In: NRL Conf. on Classical Function Theory (1970), Naval Res. Lab., Washington, DC, 1–32.
8. A.A. Goldberg, *On defects of meromorphic functions,* Dokl. Akad. Nauk SSSR, **98** 1954, 893–895.
9. A.A. Gol'dberg, I.V. Ostrovskii, *Some theorems on the growth of meromorphic functions*, Zap. Mat. Otd. Kharkov University and Kharkov Math. Soc., ser. **4** (1961), №27, 3–37. (Russian)
10. A.A.Gol'dberg, I.V.Ostrowskii, it Distribution of values of meromorphic functions, Nauka, Moscow, 1970. (Russian)
11. A.A. Gol'dberg, A.E. Eremenko, M.L. Sodin, *Exceptional values in the sense of R. Nevanlinna and in the sense of V. Petrenko,* I, Teor. Funkts., Funkts. Anal. Prilozh., **47** (1987), 41–51. (Russian) Engl. transl. J. Sov. Math. **48** (1990), №6, 648–655.
12. A.A. Gol'dberg, A.E. Eremenko, M.L. Sodin, *Exceptional values in the sense of R. Nevanlinna and in the sense of V. Petrenko,* II, Teor. Funkts., Funkts. Anal. Prilozh., **48** (1987), 58-70. (Russian) Engl. transl. J. Sov. Math. **49** (1990), №2, 891–899.
13. N.V. Govorov, *On Paley's problem*, Funct. Anal. and Its Applications, **3** 1969, №2, 38–43. (Russian)
14. A.F. Grishin, *On comparison of defects* $\delta_p(a)$, Teor. Funkts., Funkts. Anal., Prilozh., **25** (1976), 56-66. (Russian)
15. L.V. Kaluzhynova, I.I. Marchenko, *On deviations of entire functions of infinite lower order from rational functions*, Mat.Zametki, 2008 (Russian) (to appear).
16. I.I.Marchenko, *An analogue of the second main theorem for the uniform metric*, Mat. fizika, analiz, geometriya, **5** (1998), №3/4, 212–227. (Russian)
17. I.I.Marchenko, *On the growth of entire and meromorphic functions*, Mat.Sb. **189** (1998), №6, 59-84. (Russian)
18. I.I. Marchenko, *On the deviations and defects of meromorphic functions of finite lower order*, Ukr. Mat. Journ., **51** (1999), №6, 796–803. (Russian)
19. I.I. Marchenko, *On asymptotic values of entire functions*, Izv. RAN, Ser.Mat. **63**(1999), №3, 133-146. (Russian) Engl. Transl. Izvestiya: Mathematics **63** (1991), №3, 549–560.
20. I.I. Marchenko, *On the strong asymptotic spots of meromorphic functions of finite lower order*, Mat. fizika, analiz, geometriya, **11** (2004), 484–491. (Russian)
21. I.I.Marchenko and A.I.Shcherba, *On the magnitudes of deviations of meromorphic functions*, Mat. Sb. **181** (1990), 3–24. (Russian) Engl. transl. in Math. USSR Sb. **69** (1991), 1–24.
22. R.Nevanlinna, *Single-valfed analytic functions*, OGIZ, Moscow (1941), 388 p. (Russian)
23. V.P.Petrenko, *The growth of meromorphic functions of finite lower order*, Izv.Akad.Nauk SSSR, **33** (1969), №2, 414–454. (Russian)
24. V.P. Petrenko, *Growth of meromorphic functions*, Kharkov, Vyshcha Shkola, 1978, 136 p. (Russian)
25. M.A. Ryzhkov, *On the exactness of the estimate of the value of deviation for a meromorphic function*, Teor. Funkts., Funkts. Anal., Prilozh., **37** (1982), 114–115. (Russian)
26. M.L. Sodin, *On the relationship between the set of defective values and the set of positive deviations for qeromorphic functions of finite lower order*, Sib. Mat. Sb., **22** (1981), №2, 198–205. (Russian)
27. N. Steinmetz, *Eine Verallgemainerung des zweiten Nevanlinnaschen Hauptsatzes*, J. Reine Angew. Math. **368** (1986), 134–141.

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