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THE GROWTH OF AN ENTIRE FUNCTION WITH A GIVEN SEQUENCE OF ZEROS

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Let $\zeta=(\zeta_n)$ be a sequence of complex numbers tending to ∞, and $A(\zeta)$ be the class of entire functions with zeros at the points ζ_n and only at them. We investigate the problem on minimal growth of functions from the class $A(\zeta)$. In particular, we prove the existence of an entire function $f\in A(\zeta)$ such that

$$\varliminf_{r\to+\infty}\frac{\ln\ln M_f(r)}{\ln N_\zeta(r)}=1,$$

where $M_f(r)$ is the maximum modulus of f, and $N_\zeta(r)$ denotes the integrated counting function of ζ.

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Пусть $\zeta=(\zeta_n)$ – стремящаяся к ∞ последовательность комплексных чисел, а $A(\zeta)$ – класс целых функций с нулями в точках ζ_n и только в них. Исследуется вопрос о минимальном росте функций из класса $A(\zeta)$, в частности, доказано существование целой функции $f\in A(\zeta)$ такой, что

$$\varliminf_{r\to+\infty}\frac{\ln\ln M_f(r)}{\ln N_\zeta(r)}=1,$$

где $M_f(r)$ — максимум модуля f, а $N_\zeta(r)$ — усредненная считающая функция ζ.

1. Introduction. Denote by A the class of transcendental entire functions such that $f(z)\not\equiv 0$. For any $f\in A$ and each $r\ge 0$ put

$$M_f(r)=\max\{|f(z)|:|z|=r\},\quad T_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\ln^+|f(re^{i\theta})|d\theta.$$

Let L be the class of function that are positive continuous and increasing to $+\infty$ on $\mathbb{R}$.

By $\mathcal{Z}$ we denote the class of complex sequences $\zeta=(\zeta_n)$ such that $0<|\zeta_0|\le|\zeta_1|\le\ldots$ and $\zeta_n\to\infty$, $n\to\infty$. Let $n_\zeta(r)=\sum_{|\zeta_n|\le r}1$ be the counting function, and

$$N_\zeta(r)=\int_0^r\frac{n_\zeta(t)-n_\zeta(0)}{t}dt+n_\zeta(0)\ln r$$

be the integrated counting function of the sequence $\zeta\in\mathcal{Z}$. We say that $f\in A(\zeta)$ if and only if $f\in A$ and the sequence of zeros of the function f such that their moduli form a nondecreasing sequence coinciding with ζ; in this case by the Jensen formula (see [1, p.24])

$$N_\zeta(r)=\frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta})|d\theta-\ln\frac{|f^{(\lambda)}(0)|}{\lambda!},$$

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where $\lambda = \min\{n \in \mathbb{Z}_+ : f^{(n)}(0) \neq 0\}$, we have $N_\zeta(r) \leq \ln M_f(r) + O(1)$ $(r \to +\infty)$.

By the classical Weierstrass theorem, $A(\zeta) \neq \varnothing$ for any sequence $\zeta \in \mathcal{Z}$. Moreover, if $g \in A(\zeta)$, $h \in A$, $f(z) = g(z)e^{h(z)}$, then $f \in A(\zeta)$. Using this fact, it is easily seen that the following statement is true: for any sequence $\zeta \in \mathcal{Z}$ and any function $l \in L$ there exists an entire function $f \in A(\zeta)$ such that

$$l(N_\zeta(r)) = o(\ln M_f(r)) \quad (r \to +\infty).$$

In other words, one cannot specify any restriction on the growth from above for entire functions with a given sequence of zeros.

The converse problem considered by A. A. Gol'dberg is fundamentally more important ([2]) : *how slow is the growth of* $\ln M_f(r)$ *in comparison with the growth of* $N_\zeta(r)$ *for entire functions* $f \in A(\zeta)$*?* In particular, in [2] the following theorems are proved.

Theorem A. *For any sequence* $\zeta \in \mathcal{Z}$ *there exists an entire function* $f \in A(\zeta)$ *such that*

$$\ln\ln M_f(r) = o(N_\zeta(r)) \quad (E_f \not\ni r \to +\infty), \tag{1}$$

where $E_f \subset (1, +\infty)$ *is an exceptional set, which has a finite logarithmic measure (i.e.,* $\int_{E_f} \frac{dr}{r} < +\infty$*).*

Теорема B. *For any function* $\psi \in L$ *such that* $\psi(x) = o(x)$, $x \to +\infty$, *there exist a sequence* $\zeta \in \mathcal{Z}$ *and a set* $E \subset (0, +\infty)$ *of upper linear density* 1 *(i.e.,* $\varlimsup\limits_{r\to+\infty} \frac{1}{r}\int_{E\cap(0,r)} dt = 1$*) such that*

$$\psi(N_\zeta(r)) = o(\ln\ln M_f(r)) \quad (E \ni r \to +\infty) \tag{2}$$

for each entire function $f \in A(\zeta)$*.*

Another version of statement similar to Theorem A is obtained in [3]. Actually, in [3] some weaker estimate of $\ln M_f(r)$ is established, but this estimate holds outside a smaller exceptional set, namely: for any sequence $\zeta \in \mathcal{Z}$ there exists an entire function $f \in A(\zeta)$ such that (1) holds with "O" instead of "o" and with the set E_f of finite measure.

In connection with the stated results, the following *Problems* arise, which are the objects of consideration for our paper.

Problem 1. *To what extent the estimate of the exceptional set* E_f *in Theorem A can be improved with preservation of the statement of the theorem?*

Problem 2. *Find a necessary and sufficient condition on a function* $\varphi \in L$ *under which for any sequence* $\zeta \in \mathcal{Z}$ *there exists an entire function* $f \in A(\zeta)$ *such that*

$$\ln\ln M_f(r) \leq \varphi(N_\zeta(r)), \tag{3}$$

for all $r \geq r_f$ *outside the exceptional set of a finite logarithmic measure (or a finite measure).*

Problem 3. *Does there exist a function* $\varphi(x)$ *positive on* $\mathbb{R}$ *and increasing much more slowly than* x *as* $x \to +\infty$ *such that for any sequence* $\zeta \in \mathcal{Z}$ *there exists an entire function* $f \in A(\zeta)$ *and we have*

$$\varliminf_{r\to+\infty} \frac{\ln\ln M_f(r)}{\varphi(N_\zeta(r))} = 0?$$

Problem 4. *Does there exist a function* $\varphi \in L$ *such that for any sequence* $\zeta \in \mathcal{Z}$ *there exists an entire function* $f \in A(\zeta)$ *satisfying the relation (3) for all* $r \geq r_f$ *(without the exceptional set)?*

Concerning Problem 1, the following theorem is true.

Theorem 1. *For any sequence $\zeta \in \mathcal{Z}$ there exist an entire function $f \in A(\zeta)$ and a function $\alpha \in L$ such that (1) holds with the exceptional set $E_f \subset (0, +\infty)$ that satisfy $\int_{E_f} r^{\alpha(r)} dr < +\infty$.*

The following theorem is a slight generalization of Theorem B.

Theorem 2. *For any function $\psi \in L$ such that*

$$\lim_{x \to +\infty} \frac{\psi(x)}{x} = 0, \tag{4}$$

there exist a sequence $\zeta \in \mathcal{Z}$ and a set $E = \bigcup_{n=0}^{\infty} (x_n, y_n)$ such that

$$1 < x_0 < y_0 < x_1 < y_1 < \dots, \quad \lim_{n \to \infty} \frac{\ln y_n}{\ln x_n} = +\infty \tag{5}$$

and (2) holds for any entire function $f \in A(\zeta)$.

It is easy to show that the set E from Theorem 2 has upper density 1 (even upper logarithmic density 1). Besides, for this set one has

$$\int_E \frac{dr}{r \ln r} = \sum \ln \frac{\ln y_n}{\ln x_n} = +\infty.$$

As a consequence from Theorems 1 and 2 we obtain the following statement which solves Problem 2.

Theorem 3. *Let $\varphi \in L$, and h be a function positive on $\mathbb{R}$ such that*

$$\frac{c_1}{r \ln r} \le h(r) \le r^{c_2} \quad (r \ge r_0),$$

where c_1 and c_2 are positive constants. For any sequence $\zeta \in \mathcal{Z}$ there exists an entire function $f \in A(\zeta)$ such that relation (3) holds for all $r > 0$ outside the set $E_f \subset (0, +\infty)$ satisfying $\int_{E_f} h(r) dr < +\infty$ if and only if the condition (4) holds.

A positive answer to Problem 3 follows from the following statement.

Theorem 4. *For any sequence $\zeta \in \mathcal{Z}$ there exists an entire function $f \in A(\zeta)$ such that*

$$\underline{\lim}_{r \to +\infty} \frac{\ln \ln M_f(r)}{\ln N_\zeta(r)} = 1. \tag{6}$$

The answer to Problem 4 is negative.

Theorem 5. *For any function $\varphi \in L$ there exists a sequence $\zeta \in \mathcal{Z}$ such that for all entire functions $f \in A(\zeta)$ the following relation holds*

$$\overline{\lim}_{r \to +\infty} \frac{\ln \ln M_f(r)}{\varphi(\ln N_\zeta(r))} = \infty. \tag{7}$$

We remark that Theorems 1–4 remain true if they are reformulated for the Nevanlinna characteristic $T_f(r)$ instead of $\ln M_f(r)$ (this is obvious concerning Theorems 1 and 4; concerning Theorems 2 and 3 see their proofs). The following *Question* remains open: *is it allowed to replace* $\ln M_f(r)$ *with* $T_f(r)$ *in Theorem 5?*

By $\mathcal{Z}^*$ we denote the class of finite or countable complex sequences $\zeta = (\zeta_n)$ such that $0 \le |\zeta_0| \le |\zeta_1| \le \dots$ and, in the case of the countable sequences, $\zeta_n \to \infty$ as $n \to \infty$. For

$\zeta \in \mathcal{Z}^*$ we say that $f \in A(\zeta)$ if and only if $f \in A$ and the sequence of zeros of the function f such that moduli form a nondecreasing sequence coinciding with ζ. It is easily seen that all theorems stated above remain true if $\mathcal{Z}$ is replaced with $\mathcal{Z}^*$ in them.

Remark also that problems, similar to ones considered here, but with $n_\zeta(r)$ instead of $N_\zeta(r)$, are studied in the papers [2], [4–8].

2. Auxiliary results. Suppose that $p \in \mathbb{Z}_+$, and $E(z,p)$ is the Weierstrass primary factor, that is

$$E(z,0) = 1 - z; \qquad E(z,p) = (1-z)e^{z+\frac{z^2}{2}+\dots+\frac{z^p}{p}}, \quad p \in \mathbb{N}.$$

The following statement of O. Blumenthal ([9]) is true.

Lemma A. *For all $z \in \mathbb{C}$ and $p \in \mathbb{Z}_+$ the inequality $\ln|E(z,p)| \le |z|^{p+1}$ holds.*

Lemma 1. *For any sequence $\zeta \in \mathcal{Z}$ there exists a nonnegative sequence $\lambda = (\lambda_n)$ such that $\lambda_n \sim \frac{\ln n}{\ln|\zeta_n|}$, $n \to \infty$, and for any sequence of nonnegative integers (p_n) such that $p_n \ge [\lambda_n]$, $n \ge n_0$, the product*

$$f(z) = \prod_{n=0}^{\infty} E\left(\frac{z}{\zeta_n}, p_n\right) \tag{8}$$

specifies an entire function $f \in A(\zeta)$. Besides, $\ln M_f(r) \le G_f(r) := \sum_{n=0}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{p_n+1}$.

Proof. Suppose that (α_n) is any positive sequence increasing to $+\infty$ such that $\ln \alpha_n = o(\ln|\zeta_n|)$, $n \to \infty$. Put

$$\lambda_n = \begin{cases} 0, & \text{if } \alpha_n \ge |\zeta_n| \text{ or } n < 3; \\ \dfrac{\ln n + 2\ln\ln n}{\ln|\zeta_n| - \ln \alpha_n}, & \text{if } \alpha_n < |\zeta_n| \text{ and } n \ge 3. \end{cases}$$

Then the sequence $\lambda = (\lambda_n)$ is nonnegative, $\lambda_n \sim \dfrac{\ln n}{\ln|\zeta_n|}$, $n \to \infty$, and $\left(\dfrac{\alpha_n}{|\zeta_n|}\right)^{\lambda_n} = \dfrac{1}{n \ln^2 n}$ $(n \ge n_0)$. Since $\alpha_n \to +\infty$ $(n \to \infty)$, the series $\sum_{n=0}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{\lambda_n}$ converges for each fixed $r \ge 0$. According to the inequalities $\lambda_n \le p_n + 1$, $n \ge n_0$, the series $G_f(r) = \sum_{n=0}^{\infty} \left(\dfrac{r}{|\zeta_n|}\right)^{p_n+1}$ also converges for each fixed $r \ge 0$. If $|z| \le r$ then by Lemma A $\ln|f(z)| \le G_f(r)$, that is the product in (8) is convergent uniformly and absolutely on each compact set from $\mathbb{C}$. Therefore, it specifies a entire function $f \in A(\zeta)$, besides, $\ln M_f(r) \le G_f(r)$. □

The following statement is well known ([10, 11]).

Lemma B. *Let a function $l \in L$ have the right-hand derivative $l'_+(x)$ at each point $x \in \mathbb{R}$, and functions h and ψ be positive integrated on each finite segment. Then for the set $E(a) = \{x \in \mathbb{R} : \ l(x) > a, \ l'_+(x) > h(x)\psi(l(x))\}$, where $a \in \mathbb{R}$, the following estimate holds*

$$\int_{E(a)} h(x)dx \le \int_a^{\infty} \frac{dy}{\psi(y)}.$$

Lemma 2. *Let $l \in L$ and $\varliminf\limits_{r\to+\infty} \dfrac{l(r)}{\ln r} = 0$. Then there exists a set $E = \bigcup\limits_{n=0}^{\infty} (x_n, y_n)$ such that relation (5) and $l(r) = o(\ln r)$ $(E \ni r \to +\infty)$ hold.*

Proof. The statement of lemma is trivial if $l(r) = o(\ln r)$ $(r \to +\infty)$. In the other case the conditions of our lemma imply the existence of a function $\gamma \in L$ such that

$$\varliminf_{r\to+\infty} \frac{\gamma(r)l(r)}{\ln r} = 0, \qquad \varlimsup_{r\to+\infty} \frac{\gamma(r)l(r)}{\ln r} = +\infty.$$

Consider the set $E_0 = \{r > 1 : \gamma(r)l(r) < \ln r\}$. Then E_0 and $E_0 \backslash (1, +\infty)$ are unbounded from above sets. Besides, E_0 is open. Thus, the set E_0 is a countable union of intervals. Let us select from this union a sequence of intervals $(t_n; y_n)$ so that for all n the inequality $2y_n < t_{n+1}$ holds and there exists a point $x_n \in (t_n; y_n)$ at which $\sqrt{\gamma(x_n)}l(x_n) = \ln x_n$.

Suppose $E = \bigcup(x_n, y_n)$. Since $E \subset E_0$ and $l(r) = o(\ln r)$, $E_0 \ni r \to +\infty$, $l(r) = o(\ln r)$, $E \ni r \to +\infty$. Further, $\dfrac{\ln y_n}{\ln x_n} = \dfrac{\gamma(y_n)l(y_n)}{\sqrt{\gamma(x_n)}l(x_n)} \geq \sqrt{\gamma(y_n)} \to +\infty, \quad n \to \infty$. The lemma is proved. □

The following lemma ([1, c. 338–341]) is used in the proof of Theorem 2.

Lemma C. *If a sequence $\zeta \in \mathcal{Z}$ is positive and $\sum_{n=0}^{\infty} \frac{1}{\zeta_n} = \infty$, then $r = o(T_f(r))$, $r \to +\infty$, for any entire function $f \in A(\zeta)$.*

Suppose $f(z) \in A$, $p \in \mathbb{Z}$, $c_p(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ip\theta} \ln|f(re^{i\theta})| d\theta$ is the p-th Fourier coefficient of the function $\ln|f(re^{i\theta})|$, $d_p(r) = \operatorname{Re} c_p(r)$.

If $\zeta = (\zeta_n)$ is the sequence of zeros of the function f, $f(0) \neq 0$ and $\ln f(z) = \sum_{p=0}^{\infty} a_p z^p$ in a neighborhood of the point 0, then for each $p \in \mathbb{N}$ by the Poisson–Jensen formula ([1, p. 16–17]) one has

$$c_p(r) = \frac{1}{2}a_p r^p + \frac{1}{2p}\sum_{|\zeta_n|<r}\left(\left(\frac{r}{\zeta_n}\right)^p - \left(\frac{\overline{\zeta_n}}{r}\right)^p\right).$$

Further, for $R > r$ we obtain

$$c_p(R) - \left(\frac{R}{r}\right)^p c_p(r) = \frac{1}{2p}\sum_{|\zeta_n|<R}\left(\left(\frac{R}{\zeta_n}\right)^p - \left(\frac{\overline{\zeta_n}}{R}\right)^p\right) - \frac{1}{2p}\sum_{|\zeta_n|<r}\left(\left(\frac{R}{\zeta_n}\right)^p - \left(\frac{\overline{\zeta_n}R}{r^2}\right)^p\right) =$$

$$= \frac{1}{2p}\sum_{r\leq|\zeta_n|<R}\left(\left(\frac{R}{\zeta_n}\right)^p - \left(\frac{\overline{\zeta_n}}{R}\right)^p\right) + \frac{1}{2p}\sum_{|\zeta_n|<r}\left(\left(\frac{\overline{\zeta_n}R}{r^2}\right)^p - \left(\frac{\overline{\zeta_n}}{R}\right)^p\right).$$

From this the sequent statement immediately follows.

Lemma 3. *If an entire function f has only positive zeros $\zeta_0, \zeta_1, \ldots$, then*

$$d_p(R) - \left(\frac{R}{r}\right)^p d_p(r) \geq \frac{1}{2p}\sum_{r\leq\zeta_n<R}\left(\left(\frac{R}{\zeta_n}\right)^p - \left(\frac{\zeta_n}{R}\right)^p\right), \quad R > r.$$

Lemma D ([6, 4]). *For any entire function $f(z) \not\equiv 0$ and each $s \in \mathbb{N}$ the following inequalities hold*

$$d_0(r) + 2\sum_{p=1}^{s}\left(1 - \frac{p}{s}\right) d_p(r) \leq \ln M_f(r), \tag{9}$$

$$|c_s(r)| \leq \ln M_f(r), \quad r \geq r_0. \tag{10}$$

3. Proof of the theorems.

Proof of Theorem 1. Suppose that $\zeta \in \mathcal{Z}$. Then, as it is known, $\ln r = o(N_\zeta(r))$, $r \to +\infty$. Thus, there exists a function $\alpha \in L$ such that $\alpha(r)\ln r = o(N_\zeta(r))$, $r \to +\infty$. Let $h(r) = r^{\alpha(r)}$. For each $n \in \mathbb{Z}_+$ put $p_n = [\sqrt{n}]$ and consider the product in (8). By Lemma 1, this product defines an entire function $f \in A(\zeta)$ such that $\ln M_f(r) \le G(r)$, where

$$G(r) = \sum_{n=0}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{p_n+1}.$$

Put $k(r) = (\ln G(r))'$, $E_f = \{r > 0: \ \ln G(r) > 1, \ k(r) > h(r)\ln^2 G(r)\}$.

Let us show that for the functions $\alpha \in L$, $f \in A(\zeta)$ defined above and the set E_f, Theorem 1 is valid.

First, applying Lemma B with $l(r) = \ln G(r)$, $a = 1$ and $\psi(y) = y^2$, we obtain

$$\int_{E_f} r^{\alpha(r)}dr = \int_{E_f} h(r)dr \le \int_1^{\infty} \frac{dy}{y^2} < +\infty.$$

Further, introduce the notation

$$\mu(r) = \max\left\{\left(\frac{r}{|\zeta_n|}\right)^{p_n+1} : n \in \mathbb{Z}_+\right\}, \quad \nu(r) = \max\left\{n \in \mathbb{Z}_+ : \left(\frac{r}{|\zeta_n|}\right)^{p_n+1} = \mu(r)\right\}.$$

It is easy to see that $\mu(r) \to +\infty$, $\nu(r) \to +\infty$, if $r \to +\infty$. Therefore as $r \to +\infty$ we have

$$\ln\mu(r) = (p_{\nu(r)}+1)\ln\frac{r}{r_{\nu(r)}} = \frac{p_{\nu(r)}+1}{\nu(r)+1} n_\zeta(r_{\nu(r)})\ln\frac{r}{r_{\nu(r)}} \le \frac{p_{\nu(r)}+1}{\nu(r)+1}\int_{r_{\nu(r)}}^{r} \frac{n_\zeta(t)}{t}dt =$$
$$= \frac{p_{\nu(r)}+1}{\nu(r)+1}(N_\zeta(r) - N_\zeta(r_{\nu(r)})) = o(N_\zeta(r)).$$

In addition, for all $r > 0$ and $c > 0$ we have

$$G(r) - \sum_{\sqrt{n}\le c}\left(\frac{r}{|\zeta_n|}\right)^{p_n+1} = \sum_{\sqrt{n}>c}\left(\frac{r}{|\zeta_n|}\right)^{p_n+1} \le \frac{1}{c}\sum_{\sqrt{n}>c}(p_n+1)\left(\frac{r}{|\zeta_n|}\right)^{p_n+1} \le \frac{r}{c}G'(r).$$

Putting here $c = 2rk(r) = 2r\frac{G'(r)}{G(r)}$, we get

$$G(r) \le 2\sum_{\sqrt{n}\le 2rk(r)}\left(\frac{r}{|\zeta_n|}\right)^{p_n+1} \le 2(2rk(r)+1)^2\mu(r),$$

hence if $E_f \not\ni r \to +\infty$, then

$$\ln G(r) \le \ln 2 + 2\ln(rh(r)\ln^2 G(r)+1) + \ln\mu(r) \le$$
$$\le 4(\ln r + \ln h(r) + 2\ln\ln G(r)) + \ln\mu(r) = 8\ln\ln G(r) + o(N_\zeta(r)).$$

Thus,

$$\ln G(r) = o(N_\zeta(r)), \quad E_f \not\ni r \to +\infty,$$

whence, in view of the inequality $\ln M_f(r) \le G(r)$, we obtain (1). Theorem 1 is proved. □

Proof of Theorem 2. Let for a function $\psi \in L$ (4) holds. Then it is clear that for any constant $c > 0$ we have

$$\lim_{x\to+\infty} \frac{\psi(c\ln[x])}{\ln[x]} = 0,$$

whence it follows that there exist a sequence (n_k) of integers such that $n_0 \geq 1$, and

$$\psi((n_0 + \dots + n_k)\ln n_{k+1}) = o(\ln n_{k+1}), \quad k \to \infty. \tag{11}$$

Construct the sequence $\zeta = (\zeta_n)$ as follows:

$$\underbrace{n_0, \dots, n_0}_{n_0 \text{ times}}, \underbrace{n_1, \dots, n_1}_{n_1 \text{ times}}, \dots, \underbrace{n_k, \dots, n_k}_{n_k \text{ times}}, \dots .$$

For this sequence

$$\sum_{n=0}^{\infty} \frac{1}{\zeta_n} = \sum_{k=0}^{\infty} \Bigg(\underbrace{\frac{1}{n_k} + \ldots + \frac{1}{n_k}}_{n_k \text{ times}} \Bigg) = \sum_{k=0}^{\infty} 1 = \infty.$$

Then for each entire function $f \in A(\zeta)$, by Lemma C we obtain

$$\ln r \leq \ln T_f(r), \quad r \geq r_f. \tag{12}$$

On the other hand, since $n_\zeta(r) = n_0 + \dots + n_k$ for all $r \in [n_k, n_{k+1})$ and $k \in \mathbb{Z}_+$, we have

$$N_\zeta(n_{k+1}) = \int_{n_0}^{n_{k+1}} \frac{n_\zeta(t)}{t} dt \leq n_\zeta(n_{k+1} - 0) \int_1^{n_{k+1}} \frac{dt}{t} = (n_0 + \dots + n_k)\ln n_{k+1}.$$

Therefore, in view of (11),

$$\varliminf_{r\to+\infty} \frac{\psi(N_\zeta(r))}{\ln r} = 0.$$

By Lemma 2, there exists a set $E = \bigcup_{n=0}^{\infty}(x_n, y_n)$ such that relations (5) and $\psi(N_\zeta(r)) = o(\ln r)$, $E \ni r \to +\infty$ are valid. Then, by (12) for each entire function $f \in A(\zeta)$, we have $\psi(N_\zeta(r)) = o(\ln T_f(r))$, $E \ni r \to +\infty$, whence we obtain (2). Theorem 2 is proved. □

Proof of Theorem 4. Consider any sequence $\zeta \in \mathcal{Z}$ and let us prove that there exists an entire function $f \in A(\zeta)$ such that relation (6) holds. Let

$$\rho_\zeta := \varlimsup_{r\to+\infty} \frac{\ln n_\zeta(r)}{\ln r} = \varlimsup_{n\to\infty} \frac{\ln n}{\ln|\zeta_n|} \tag{13}$$

be the order of the counting function for the sequence ζ. As it is well known [1, p. 63], $n_\zeta(r)$ in (13) can be replaced with $N_\zeta(r)$.

First suppose that $\rho_\zeta < +\infty$. Let p be the genus of the sequence ζ. Consider the entire function (8) with $p_n = p$, $n \in \mathbb{Z}_+$, that is the Weierstrass canonical product of genus p. For this product in the case of $p = 0$ we have (see [1, p. 273])

$$\varliminf_{r\to+\infty} \frac{\ln M_f(r)}{N_\zeta(r)} = 1,$$

whence we derive (6). And if $p > 0$ (thus $\rho_\zeta > 0$), and (x_n) is a sequence increasing to $+\infty$ such that $\ln N_\zeta(x_n) \sim \rho_\zeta \ln x_n$, $n \to \infty$, then by Borel's theorem [1, p. 79], according to which the order of f equals ρ_ζ, we have $\ln\ln M_f(x_n) \le (\rho + o(1)) \ln x_n$, $n \to \infty$, whence we again obtain (6).

Let $\rho_\zeta = +\infty$, and (λ_n) be a nonnegative sequence such that the conclusions of Lemma 1 are true. According to (13) we have $\overline{\lim\limits_{n\to\infty}} \lambda_n = +\infty$, therefore the set

$$N = \{n \in \mathbb{Z}_+ :\ \lambda_k \le \lambda_n \text{ for all } k = 0, \dots n\}$$

is infinite.

Consider also the set $E = \{r > 0 :\ N_\zeta(r) > e^2,\ n_\zeta(r) > N_\zeta(r) \ln^2 N_\zeta(r)\}$. Taking into account that $n_\zeta(r) = r(N_\zeta(r))'_+$, by Lemma B we have

$$\int_E \frac{dr}{r} \le \int_{e^2}^{\infty} \frac{dy}{y \ln^2 y} = \frac{1}{2}.$$

Remark that the logarithmic measure of the interval (r, er) is equal to 1. Thus, for any $r > 0$ such that $N_\zeta(r) > e^2$, the interval (r, er) contains a point x that does not belong to the sequence $(|\zeta_n|)$ such that $n_\zeta(x) \le N_\zeta(x) \ln^2 N_\zeta(x)$.

Put

$$m(r) = \min\left\{ m \ge n_\zeta(r) + 1 :\ \sum_{n=m+1}^{\infty} \left(\frac{r}{|\zeta_n|}\right)^{\lambda_n} \le \frac{1}{2} \right\}.$$

Choose $n_0 \in N$ so that $r_{n_0} > 1$, $N(r_{n_0}) > e^2$, and on the interval (r_{n_0}, er_{n_0}) choose a point x_0 that does not belong to the sequence $(|\zeta_n|)$ and such that $n_\zeta(x_0) \le N_\zeta(x_0) \ln^2 N_\zeta(x_0)$.

Suppose that we have already defined integers $n_0 < \dots < n_k$ and real numbers $x_0 < \dots < x_k$ different from the members of the sequence $(|\zeta_n|)$. Notice that $r_{n(x_k)} < x_k < r_{n(x_k)+1}$, and put

$$n_{k+1} = \min\left\{ m \in N :\ m \ge n_k + 1,\ \sum_{n=n_\zeta(x_k)+1}^{m(x_k)} \left(\frac{x_k}{|\zeta_n|}\right)^{\lambda_m} \le \frac{1}{2} \right\}.$$

Then on the interval $(r_{n_{k+1}}, er_{n_{k+1}})$ we select a point x_{k+1} that does not belong to the sequence $(|\zeta_n|)$, and $n_\zeta(x_{k+1}) \le N_\zeta(x_{k+1}) \ln^2 N_\zeta(x_{k+1})$.

For each $k \in \mathbb{Z}_+$ by l_k we denote the largest index of the numbers $\lambda_0, \dots, \lambda_{n_\zeta(x_k)}$:

$$l_k = \max\{n \in \{0, \dots, n_\zeta(x_k)\} : \lambda_n = \max\{\lambda_0, \dots, \lambda_{n_\zeta(x_k)}\}\}.$$

It is clear that $n_k \le l_k \le n_\zeta(x_k)$, $r_{l_k} \le x_k \le er_{l_k}$.

For each $n \in \{0, \dots, n_\zeta(x_0)\}$ put $p_n = [\lambda_{l_0}]$ and let $p_n = [\lambda_{l_{k+1}}]$ for each $n \in \{n_\zeta(x_k) + 1, \dots, n_\zeta(x_{k+1})\}$ and $k \in \mathbb{Z}_+$. Consider the product in (8), which by Lemma 1 specifies the entire function $f \in A(\zeta)$. Let us prove that for this function relation (6) is satisfied.

For any $k \in \mathbb{Z}_+$ we have

$$\ln M_f(x_k) \le \sum_{n=0}^{\infty} \left(\frac{x_k}{|\zeta_n|}\right)^{p_n+1} \le$$

$$\le \sum_{n=0}^{n_\zeta(x_k)} \left(\frac{x_k}{|\zeta_n|}\right)^{p_n+1} + \sum_{n=n_\zeta(x_k)+1}^{m(x_k)} \left(\frac{x_k}{|\zeta_n|}\right)^{\lambda_{n_{k+1}}} + \sum_{n=m(x_k)+1}^{\infty} \left(\frac{x_k}{|\zeta_n|}\right)^{\lambda_n} \le$$

$$\le x_k^{\lambda_{l_k}+1} \sum_{n=0}^{n_\zeta(x_k)} \left(\frac{1}{|\zeta_n|}\right)^{p_n+1} + \frac{1}{2} + \frac{1}{2} < x_k^{\lambda_{l_k}+1} G_f(1) + 1.$$

Then we obtain, as $k \to \infty$

$$\ln\ln M_f(x_k) \le (1+o(1))\lambda_{l_k} \ln x_k = (1+o(1))\frac{\ln l_k}{\ln r_{l_k}} \ln r_{l_k} = (1+o(1))\ln l_k \le$$
$$\le (1+o(1))\ln n_\zeta(x_k) \le (1+o(1))\ln(N_\zeta(x_k)\ln^2 N_\zeta(x_k)) =$$
$$= (1+o(1))\ln N_\zeta(x_k),$$

whence we obviously derive (6). Theorem 4 is proved. □

Proof of Theorem 5. Let $\varphi \in L$. Let us prove that there exists a sequence $\zeta \in \mathcal{Z}$ such that for any entire function $f \in A(\zeta)$ relation (7) holds.

Let $h(x) = e^{\varphi^2(x)}$ and $\gamma \in L$ be any function such that

$$\gamma(x) \le \min\left\{\ln x; \frac{1}{3}h^{-1}(\ln x)\right\}, \quad x \ge x_0. \tag{14}$$

Put $n_0 = 3$ and define inductively

$$n_k = \min\left\{n \in \mathbb{N} : m_{k-1} := n_0 + \ldots + n_{k-1} < \frac{n}{k}; km_{k-1} < \min\{h^{-1}(\gamma(n)); \gamma(n)\}\right\} \tag{15}$$

for each $k \in \mathbb{N}$.

Let us generate the sequence ζ in the following way:

$$\underbrace{1,\ldots,1}_{n_0\ \text{times}}, \underbrace{e,\ldots,e}_{n_1\ \text{times}}, \ldots, \underbrace{e^k,\ldots,e^k}_{n_k\ \text{times}}, \ldots.$$

Let $R_k = e^{k+\frac{\gamma(n_k)}{n_k}}$. Then $e^k < R_k$ for each $k \in \mathbb{Z}_+$ and, according to (14), $R_k < e^{k+1}$ for each $k \ge k_0$.

Further, using (15), we obtain

$$N_\zeta(e^k) = \int_1^{e^k} \frac{n(t)}{t}dt \le \int_1^{e^k} \frac{m_{k-1}}{t}dt = km_{k-1}, \quad k \in \mathbb{N}; \tag{16}$$

$$N_\zeta(R_k) = N_\zeta(e^k) + \int_{e^k}^{R_k} \frac{n(t)}{t}dt = N_\zeta(e^k) + \int_{e^k}^{R_k} \frac{m_k}{t}dt \le$$
$$\le km_{k-1} + m_k\frac{\gamma(n_k)}{n_k} \le 3\gamma(n_k), \quad k \ge k_0. \tag{17}$$

Consider any function $f \in A(\zeta)$. The following two cases are possible.

Case 1: there exist infinitely many $k \in \mathbb{Z}_+$ such that $|c_p(e^k)| \ge \frac{1}{4}\gamma(n_k)$ for some integer p. In this case, for all such sufficiently large k, according to (10), (15) and (16), we have

$$\ln M_f(e^k) \ge \frac{1}{4}\gamma(n_k) \ge \frac{1}{4}h(km_{k-1}) \ge \frac{1}{4}h(N_\zeta(e^k)),$$

whence we easily obtain relation (7).

Case 2: $|c_p(e^k)| < \frac{1}{4}\gamma(n_k)$ for all $k \ge k_0$ and each integer p. In this case we put $s_k = \left[\frac{n_k}{\ln^2 n_k}\right]$ and suppose $1 \le p \le s_k$; then, according to (14), we obtain uniformly with respect to such p

$$0 \le \frac{p\gamma(n_k)}{n_k} \le \frac{s_k \ln n_k}{n_k} \le \frac{1}{\ln n_k} \to 0, \quad n \to \infty.$$

Thus, for $k \geq k_0$, accordingly to Lemma 3, we have

$$d_p(R_k) \geq \frac{n_k}{2p}\left(\left(\frac{R_k}{e^k}\right)^p - \left(\frac{e^k}{R_k}\right)^p\right) - \left(\frac{R_k}{e^k}\right)^p |c_p(e^k)| =$$

$$= \frac{n_k}{2p}\left(e^{\frac{p\gamma(n_k)}{n_k}} - e^{-\frac{p\gamma(n_k)}{n_k}}\right) - e^{\frac{p\gamma(n_k)}{n_k}} \cdot \frac{1}{4}\gamma(n_k) \geq \frac{n_k}{2p}\frac{2}{3}\frac{2p\gamma(n_k)}{n_k} - \frac{1}{3}\gamma(n_k) = \frac{1}{3}\gamma(n_k).$$

Therefore, using (9), (14), and (17), we get

$$\ln M_f(R_k) \geq \frac{1}{3}\gamma(n_k) \cdot 2\sum_{p=1}^{s_k}\frac{s_k - p}{p} = \frac{1}{3}\gamma(n_k)(s_k - 1) >$$

$$> \ln n_k \geq h(3\gamma(n_k)) \geq h(N_\zeta(R_k)), \quad k \geq k_0,$$

whence we again obtain (7). Theorem 5 is proved. □

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