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ON NECESSARY CONDITION OF THE EXISTENCE OF THE SINGLE-VALUED SOLUTIONS OF THE EULER-POISSON EQUATIONS

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From the time of Leonhard Euler till now 14 general and partial solutions of the Euler–Poisson equations were found. Now we do not know the number of partial solutions. Therefore we think that it is interesting to find the full collection of single-valued solutions of the Euler–Poisson equations. In this paper we provide some results that were obtained previously and present some new ones.

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Начиная со времен Леонарда Эйлера и до сих пор найдены 14 общих и частных решения уравнений Эйлера-Пуассона. В данный момент неизвестно, сколько существует частных решений. Представляется интересным найти полный список частных решений уравнений Эйлера-Пуассона. В статье приведены как некоторые новые результаты в этом направлении, так и результаты полученные автором ранее.

Introduction. The problem of the moving of the solid body is described by the Euler–Poisson equations:

$$\begin{cases} A\dot{p} = Ap \times p + \gamma \times r \\ \dot{\gamma} = \gamma \times p, \end{cases} \tag{0.1}$$

here $p = (p_1, p_2, p_3) \in \mathbb{C}^3$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3$, $r = (r_1, r_2, r_3) \in \mathbb{R}^3$ $A = \operatorname{diag}(A_1, A_2, A_3)$, $0 < A_i < A_j + A_k$ for any i, j, k. These equations (see, for example, [1]), called below the Euler–Poisson equations, determine the law of changes of the angle speed p of the rotation of the solid and the vector of gravity γ in the coordinates, associated with the body, in which the inertia operator has the diagonal form.

The Euler–Poisson equations have three first integrals in any initial conditions:

$$\mathcal{H} = \frac{1}{2}\langle Ap, p\rangle + \langle \gamma, r\rangle, \quad \mathcal{M} = \langle Ap, \gamma\rangle, \quad \mathcal{T} = \langle \gamma, \gamma\rangle.$$

In general, we cannot solve the solid body problem with the aid of these integrals, but if some special conditions for the parameters of the problem are set, the complementary forth integral may exist.

Nowadays we know 14 general and partial solutions of the Euler–Poisson equations, which one can see in the treatise [1] and review [2]. Here we give the complete collection of these cases, comprising its authors and dates: Euler (1758), Lagrange (1773), Kovalevskaya (1866),

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Hess (1890), Bobylyev and Steklov (1896), Steklov (1899), Goryachev (1899), Goryachev and Chaplygin (1900), Chaplygin (1904), Kovalevskiy (1908), Grioli (1974), Dokshevich (1966), and Konoshevich and Pozdnyakovich (1968), Dokshevich (1970).

In connection with these results we can ask the natural question concerning the complete classification of the integrable cases. To begin our research, we should find at first all single-valued solutions of the problem ([3]) under discussion. We exploit the approach, that was used by Kovalevskaya, when she applied it in order to find her case ([4]). She considered the solutions of the Euler–Poisson equations as functions of the complex variables at the singular points. It should be noticed that Lyapunov ([5]) proved that Euler, Lagrange and Kovalevskaya cases form the fool collection of the general single-valued solutions. But this Kovalevskaya's idea does not give us an opportunity to find all the partial single-valued solutions. That is why we use a new idea of the factorization of the phase space for its compactification ([6]). By means of this we get the classification of the singular points of the Euler–Poisson equations, that is the basis for the classification of all single-valued solutions ([6]-[8]).

There exist two types of the singular points. They are α and β-singular points.

If we want to avoid awkward definitions we can describe the α-points as singular ones in the solutions of the Euler–Poisson equations in Euler case ($r = 0$) if the condition that all moments of inertia A_i are different is satisfied. Correspondingly the β-points are the singular points of the solutions when $A_i = A_j$, for some i, j. Moreover the small perturbation does not change the type of a singular point, i.e. the asymptotic behaviour of α-points is determined by 5 free parameters having the members $t^{m-2}\ Ln^{n-1}(t)$, $m, n \in \mathbb{N}$, up to coefficients. Similarly, the asymptotic behaviour of β-points is determined by 5 free parameters having the members $t^z, z \in \mathbb{C}$, up to coefficients.

Under the above-stated assumption we sort all single-valued solutions of the Euler–Poisson equations into four groups which can intersect:

1) the solutions which have no singular points, i.e. are entire;

2) the solutions which have single-valued $\alpha-$points;

3) the solutions which have single-valued $\beta-$points, due to the special choice of the parameters of solid body;

4) the solutions which have single-valued $\beta-$points, due to the special choice of the parameters of asymptotics.

All the entire solutions and the proof of their completeness can be found in the works [6], [9]. They are the particular cases of Euler ([10]), of Lagrange ([11]) and of Grioli ([12]). All the solutions having single-valued $\alpha-$points are the partial solutions describing the motion of the Hess solid. However, we know all the solutions only in the Hess case ([13]), i.e. when the Hess integral equals zero ([14]). The proof of completeness of the list see in [15].

The main result of this paper comprises finding the exact list of single-valued solutions of the forth group, which consists of the cases of Bobylyov–Steklov ([16], [17]) and Steklov ([18]). We refer to the papers [19], [20], which contain some part of the discussed problem. The exact formulation of the main result of the paper we present in the preliminary.

1. Preliminary. Now we give the definitions and the formulations of the theorems, which we want to use. We use the notations: $z(t) = (p(t), \gamma(t))$, $B_{ij} = A_i - A_j$, $C_{ij} = 2A_i - A_j$. We define $\mathbb{C}$-scalar product in $\mathbb{C}^3$: $\langle x, y\rangle = \sum_{i=1}^{3} x_i y_i$. We use the circle permutation of the indices $\sigma = (1, 2, 3)$ for writing the product or sums (for example, $\sum_\sigma A_1A_2 = A_1A_2 + A_2A_3 + A_3A_1$, $\prod_\sigma A_1 = A_1A_2A_3$) and expressions which differ one from another only in the circle permutation of the indices ($\dot{\gamma}=\gamma\times p$ can be written as $\gamma_1=p_3\gamma_2-p_2\gamma_3,\sigma$).

Definition 1. The following algebraic system

$$\begin{cases} A\overset{\sim 0}{p}\times\overset{\sim 0}{p}+\overset{\sim 0}{\gamma}\times r+A\overset{\sim 0}{p}=0,\\ \overset{\sim 0}{\gamma}\times\overset{\sim 0}{p}+2\overset{\sim 0}{\gamma}=0.\end{cases}\tag{1.1}$$

will be called a *characteristic system for the Euler-Poisson equations.*

Theorem 1 ([7]). *Let the condition $\prod_\sigma(B_{12}r_1)\neq 0$ satisfied. Then the characteristic system (1.1) has 8 solutions (taking into account the multiplicities of the roots) which can be obtained as follows: if ϱ is a root of the polynomial $\mathcal{P}(\varrho)=\sum_\sigma[r_1^4B_{23}^2(A_1-\varrho)^4(2A_2-\varrho)^2(2A_3-\varrho)^2-2r_2^2r_3^2B_{12}B_{31}(A_2-\varrho)^2(A_3-\varrho)^2(2A_1-\varrho)\prod_\sigma(2A_1-\varrho)]$,*

$$\overset{\sim 0}{p}_1=\sqrt{\frac{(2A_2-\varrho)(2A_3-\varrho)}{B_{12}B_{31}}},\ \sigma,\quad \overset{\sim 0}{\gamma}=-(A\overset{\sim 0}{p}\times\overset{\sim 0}{p})\left\langle\overset{\sim 0}{p},r\right\rangle^{-1}$$

or if ξ is a root of the polynomial $\mathcal{Q}(\xi)=(\xi^2+1)(r_1B_{23}(C_{21}B_{31}+C_{31}B_{12}+C_{31}B_{12}\xi^2)-$ $-r_3B_{12}\xi(C_{13}B_{23}+C_{13}B_{23}\xi^2+C_{23}B_{31}\xi^2))^2+(r_2B_{31}\xi(C_{32}B_{12}\xi^2-C_{12}B_{23}))^2$, then

$$\xi_0=\sqrt{-1-\xi^2},\ \overset{\sim 0}{p}_1=\frac{B_{23}\,\xi_0}{B_{12}\xi^2-B_{23}},\quad \overset{\sim 0}{p}_2=\frac{B_{31}\,\xi}{B_{12}\xi^2-B_{23}},\quad \overset{\sim 0}{p}_3=\frac{B_{12}\,\xi\,\xi_0}{B_{12}\xi^2-B_{23}},$$

$$2\overset{\sim 0}{\gamma}_3=\left\langle A\overset{\sim 0}{p},\overset{\sim 0}{p}\right\rangle(\xi r_1+\xi_0r_2+r_3)^{-1},\quad \overset{\sim 0}{\gamma}_1=\overset{\sim 0}{\gamma}_3\,\xi,\quad \overset{\sim 0}{\gamma}_2=\overset{\sim 0}{\gamma}_3\,\xi_0.$$

In this case every root ϱ_k or ξ_k corresponds to exactly one solution of the characteristic system $(\overset{\sim 0}{p},\overset{\sim 0}{\gamma})$ and the following relations take place:

$$\langle A\overset{\sim 0}{p},\overset{\sim 0}{p}\rangle=-2\varrho,\quad \langle\overset{\sim 0}{p},\overset{\sim 0}{p}\rangle=-4,\quad \langle A\overset{\sim 0}{p},r\rangle=\varrho\langle\overset{\sim 0}{p},r\rangle,$$

$$\sum_\sigma r_1B_{23}\overset{\sim 0}{\gamma}_2\overset{\sim 0}{\gamma}_3(C_{21}B_{31}(\overset{\sim 0}{\gamma}_3)^2-C_{31}B_{12}(\overset{\sim 0}{\gamma}_2)^2)=0.$$

The proof of Theorem 1 see in [7].

Let $(p(t),\gamma(t))$ be a solution of the Euler - Poisson's flow. Then $(\alpha p(\alpha t),\ \alpha^2\gamma(\alpha t))$ is also a solution. This circumstance gives the possibility for a factorization of the set of the trajectories (0.1).

Theorem 2 ([6]). *Let $\mathbb{C}$ act in $\mathbb{C}^n$ by the following form:*

$$\alpha\colon(z_1,...,z_n)\to(\alpha^{k_1}z_1,...,\alpha^{k_n}z_n),\quad k=(k_1,...,k_n)\in\mathbb{N}^n.$$

Then the quotient-space $P_k^{n-1}=\left\{(z_1^{(k_1)}\colon...\colon z_n^{(k_n)})\right\}$ is a compact holomorphic manifold under this action ([21]). The canonical projection $\pi\colon\mathbb{C}^6\to P_*^5$, where $*=(1,1,1,2,2,2)$, $\mathbb{C}^6=\{(p_1,p_2,p_3,\gamma_1,\gamma_2,\gamma_3)\}$ maps the foliation ([22]), induced by flow (1.1) onto the foliation F of the compact holomorphic manifold P_*^5.

Remark 1. The foliation F is integrable becouse, there exists the fibre invariant mapping $J\colon P^5_* \setminus X^2 \to P^2$,

$$J\colon (p_1^{(1)}: p_2^{(1)}: p_3^{(1)}: \gamma_1^{(2)}: \gamma_2^{(2)}: \gamma_3^{(2)}) \to \left(H^6(p,\gamma)\colon M^4(p,\gamma)\colon T^3(\gamma)\right),$$

where $X^2 = \{\pi(z)\colon H(z) = M(z) = T(z) = 0\}$. Moreover, the surface X^2 is also fibre invariant for the foliation F.

Theorem 3 ([8]). *There exist singular β - points of the solutions of the Euler - Poisson problem with the asymptotic behaviour*

$$\begin{cases} p(t) = \dfrac{\overset{\sim 0}{p}}{t} + \beta_2 u_2 t + \beta_3 u_3 t^2 + \beta_4 u_4 t^3 + \ldots + \beta_0 u_0 t^{\lambda_0 - 1} + \\ \quad + \beta^0 u^0 t^{\lambda^0 - 1} + \sum \beta_0^i (\beta^0)^j \psi_{kl} t^{k\lambda^0 + l\lambda_0 - 1} + \ldots \\ \gamma(t) = \dfrac{\overset{\sim 0}{\gamma}}{t^2} + \beta_2 u_2 + \beta_3 u_3 t + \beta_4 u_4 t^2 + \ldots + \beta_0 v_0 t^{\lambda_0 - 2} + \\ \quad + \beta^0 v^0 t^{\lambda^0 - 2} + \sum \beta_0^i (\beta^0)^j \chi_{kl} t^{k\lambda^0 + l\lambda_0 - 2} + \ldots \end{cases} \tag{1.2}$$

here $\beta_2, \beta_3, \beta_4, \beta_0, \beta^0$ are free parameters and $k, l \in \mathbb{N}$, $t = e^{i(\lambda_0 - \lambda^0)\theta}$, $\theta \to +\infty$, $\operatorname{Im}\lambda_0 < 0$, (u_k, v_k), $k = 2, 3, 4$, (u_0, v_0), (u^0, v^0) are the eigenvectors of the operator

$$\mathbf{H}\colon (Ap, \gamma) \to \left(A\overset{\sim 0}{p} \times p + Ap \times \overset{\sim 0}{p} + \gamma \times r + Ap, \overset{\sim 0}{\gamma} \times p + \gamma \times \overset{\sim 0}{p} + 2\gamma\right)$$

with the eigenvalue k, $\lambda_0^{(0)} = 1/2 \overset{(+)}{-} \sqrt{1/4 - S}$, where $S = S_1/S_2$,

$$S_1 = (2\langle A\overset{\sim 0}{\gamma}, \delta\rangle + \langle A\overset{\sim 0}{p}, \overset{\sim 0}{p}\rangle)(\langle A\overset{\sim 0}{\gamma}, \delta\rangle + \langle A\overset{\sim 0}{p}, \overset{\sim 0}{p}\rangle)\langle A\overset{\sim 0}{\gamma}, \overset{\sim 0}{\gamma}\rangle^{-1}$$

$$-3\langle \overset{\sim 0}{p}, r\rangle^2 \langle \overset{\sim 0}{\gamma}, r\rangle^{-1} - 2\langle A\delta, \delta\rangle + 2\langle \delta, r\rangle,$$

$$S_2 = 1/2\langle \overset{\sim 0}{p}, r\rangle^2 \langle \overset{\sim 0}{\gamma}, r\rangle^{-1} - \langle A\overset{\sim 0}{\gamma}, \delta\rangle^2 \langle A\overset{\sim 0}{\gamma}, \overset{\sim 0}{\gamma}\rangle^{-1} + \langle A\delta, \delta\rangle,$$

the vector δ is defined by the conditions $\overset{\sim 0}{p} \times \delta = -2\delta$, $\langle \overset{\sim 0}{\gamma}, \delta\rangle = 2$.

The eigenvector (u_2, v_2) has the following form: $u_2 = \lambda_1 p + \lambda_2 \gamma$, $v_2 = \mu_1 p - \lambda_1 \gamma$, when $\lambda_1, \lambda_2, \mu_1$ are connected by the following relations

$$\lambda_2 \langle A\gamma, \gamma\rangle + \mu_1 \langle Ap, p\rangle = 0, \quad -6\lambda_1 \langle p, r\rangle - \lambda_2(3\langle A\gamma, \delta\rangle + \langle Ap, p\rangle) + 2\mu_1 \langle \delta, r\rangle = 0.$$

The eigenvector (u_3, v_3) has the following form: $u_3 = \lambda_2 \gamma + \lambda_3 \delta$, $v_3 = -\lambda_3 p$, λ_2, λ_3 are connected by the relations: $\lambda_2(\langle Ap, p\rangle + 4\langle A\gamma, \delta\rangle) + 2\lambda_3(2\langle A\delta, \delta\rangle) + \langle \delta, r\rangle) = 0$.

The proof of Theorem 3 can be fuond in [8].

Now we can give the exact formulation of the main result of the paper.

Theorem 4. *The cases of Euler ([10]), Hess ([13]), Bobylyov–Steklov ([16], [17]), Steklov ([18]), Grioli ([12]) and solutions with β–points having the integer λ_0 form a complete collection of the single-valued solutions of the Euler–Poisson equations.*

2. On the presentation the single-valued solutions of the Euler–Poisson equations with zero significance of the parameters $\beta_0,\ \beta^0$ of the β-points. One can see that the solution $p(t), \gamma(t)$ is single-valued in the neighborhood of the $\beta-$point if $\beta_0 = \beta^0 = 0$. Hence the solution is single-valued if this condition is true for all β-points and α-points. We want to find a complete collection of such solutions. This is the main result of the available paper. In this case we have no restrictions for the parameters of the solid body, but we can get the suitable presentation for such solutions.

Theorem 5. *Let the solution $p(t), \gamma(t)$ of the Euler–Poisson equations (0.1) have only β-points with zero values of the parameters $\beta_0,\ \beta^0$. Then this solution has the following representation*

$$\begin{cases} p(t) = \sum_{k\in M} \overset{\sim 0}{p}_k f(t-t_k) + p_0, \\ \gamma(t) = -\sum_{k\in M} \overset{\sim 0}{\gamma}_k f'(t-t_k) + \gamma_0, \end{cases} \tag{2.1}$$

where $f(t)$ is t^{-1}, $ctg(t)$, $cth(t)$ or $\zeta(t)$ is the Weierstrass function, $\{(\overset{\sim 0}{p}_k, \overset{\sim 0}{\gamma}_k)\}$ are the $\beta-$solutions of the characteristic system (1.1), M is the subset of indices from 1 to 8 which numbers $\beta-$solutions of the characteristic system; $p_0,\ \gamma_0$ are some constants.

Proof. Let us substitute asymptotic (1.2) with zero values $\beta_0 = \beta^0 = 0$ into the first integrals of the Euler–Poisson equations. We obtaine

$\mathcal{H} = \beta_2(\langle \overset{\sim 0}{p}, u_2\rangle + \langle v_2, r\rangle),\quad \mathcal{M} = \beta_3(\langle A\overset{\sim 0}{p}, v_3\rangle + \langle u_3, A\overset{\sim 0}{\gamma}\rangle),\quad \mathcal{T} = \beta_2^2\langle v_2, v_2\rangle + \beta_4\langle v_4, \overset{\sim 0}{\gamma}\rangle.$

We see that for the identical root $\varrho_k,\ k\in M = \{1, ..., 8\}$ all singular $\beta(\varrho_k)$-points have the identical asymptotics $p(t) = \overset{\sim 0}{p}_k t^{-1} + u_{2k}t + ...,\ \gamma(t) = \overset{\sim 0}{\gamma}_k t^{-2} + v_{2k} + ...$.

Let us consider the solution $p(t), \gamma(t)$ with the finite collection of the singular $\beta(\varrho)$-points with different asymptotic behaviour. Then this solution has the following form

$$\begin{cases} p(t) = \sum_{k\in M} \overset{\sim 0}{p}_k (t-t_k)^{-1} + \widehat{p}(t), \\ \gamma(t) = \sum_{k\in M} \overset{\sim 0}{\gamma}_k (t-t_k)^{-2} + \widehat{\gamma}(t), \end{cases} \tag{2.2}$$

where the functions $\widehat{p}(t), \widehat{\gamma}(t)$ are entire.

Now we shall prove that the entire functions $\widehat{p}(t), \widehat{\gamma}(t)$ are constant. Let X be a bounded closed set containing the singular points (2.2).

First we suppose that the functions $\widehat{p}(t),\ \widehat{\gamma}(t)$ are not constant. Then they are the entire functions and there is a trajectory $\Gamma(t)$ in the domain $\mathbb{C}\backslash X$ such that $\|\widehat{p}(\Gamma(t)), \widehat{\gamma}(\Gamma(t))\| \to \infty$ as $t \to t_* \in \overline{\mathbb{C}}$.

Let the coordinate $\widehat{p}_i$ of the function $\widehat{p}(t), \widehat{\gamma}(t)$ be unlimited in the domain $\mathbb{C}\setminus X$. In this case by the maximum principle we can find a curve $\Gamma(t)$ such that the modulus $|\widehat{p}_i(t)|$ strictly increases along this curve.

Let us choose the initial point of the curve $\Gamma(t_0)$ such that $|\widehat{p}_i(t_0)| > \max\limits_{\partial X} |p_i|$.

Then we project the curve $L(\Gamma(t)) = (p(\Gamma(t)), \gamma(\Gamma(t)))$ onto the foliation $\mathcal{F}$ and consider its closure $\overline{\pi(L)}$. For any point $g \in \overline{\pi(L)}$ one can take the curve $\Gamma_g(\tau), \tau \in [\tau_0, \tau_g]$, $\pi(L(\Gamma_g(\tau_0))) = g$ such that $2|p_i(\Gamma_g(\tau_0))| < |p_i(\Gamma_g(\tau_g))|$. It is clear that there exists a neighbourhood U_g such that this inequality is true for the any point $h \in U_g$ and respectively the curve $\Gamma_g(\tau),\ \tau \in [\tau_0, \tau_g],\ \pi(L(\Gamma_g(\tau_0))) = h$. The neighbourhoods U_g, where $g \in \overline{\pi(L)}$ form an open covering of $\overline{\pi(L)}$. By the compactness of P^5_*, we can take a finite sub-covering of $\overline{\pi(L)}$.

Then we construct the curve $\widehat{\Gamma} = \bigcup_i \Gamma_{g_i}(\tau)$, when $g_{i+1} = \pi(L(\Gamma_{g_i}(\tau_{g_i}))$. According to our construction the point $\tau_{g_\infty} \in \mathbb{C}$ is singular. It gives a contradiction. Hence the functions $\widehat{p}(t), \widehat{\gamma}(t)$ are constant and presentation (2.1) is true.

Now let us consider the case when the solution $p(t), \gamma(t)$ of the Euler –Poisson equations has two or more singular points $\beta(\varrho_k)$ with equal k. The asymptotics of this point is identical, consequently the solution is periodic, for example with period T. The number $\overline{T}$ is a period as well. It follows from the fact that the solution is real. Moreover these periods are real or imaginary. Otherwise, the solution is doubly-periodic.

In these cases we have the following representation for the solution:

$$\begin{cases} p(t) = \sum_{k\in\mathfrak{M}} \overset{\sim 0}{p_k} \mathfrak{f}(t-t_k) + \widehat{p}(t), \\ \gamma(t) = -\sum_{k\in\mathfrak{M}} \overset{\sim 0}{\gamma_k} \mathfrak{f}'(t-t_k) + \widehat{\gamma}(t), \end{cases} \tag{2.3}$$

where $\mathfrak{f} = \text{ctg}, \text{cth}$ or ζ, and $\widehat{p}(t)\ \widehat{\gamma}(t)$ are entire functions.

Then we can prove that the functions $\widehat{p}(t),\ \widehat{\gamma}(t)$ are constant in the periodic case as above. If the solution is doubly-periodic this assertion is a well-known classic fact (see [23]). □

3. On the algebraic connection between coordinates of the singular points periodic and doubly-periodic solutions with β-points.

Proposition 1. *Let the solution $p(t)$ of the Euler –Poisson equations have the following form: $p(t) = \sum\limits_{k\in\mathfrak{M}} \overset{\sim 0}{p_k} \mathfrak{f}(t-t_k) + p_0$, where $\mathfrak{f}(z)$ is equal to $z^{-1}, ctg(z), cth(z)$ and $\zeta(z)$, correspondingly, then*

$$\sum_{i\in\mathfrak{M}, i\neq j} \overset{\sim 0}{p_i} \mathfrak{f}(t_j - t_i) + p_0 = 0,$$

Proof. The proposition follows from the fact that the asymptotic behaviour of the functions z^{-1}, ctg, cth и $\zeta(z)$, at zero has the form $f_{-1}\, t^{-1} + f_1\, t + ...$ and the same behaviour has (see asymptotics (1.2) of $\beta-$point) the solution $p(t)$. □

Now we consider the important partial case when $\sum_k \overset{\sim 0}{p_k} = 0$.

Thus, due to Proposition 1, $\sum_{i\in\mathfrak{M}, i\neq j} \overset{\sim 0}{p_i} \mathfrak{f}(t_j - t_i) + p_0 = 0$. Let us number the singular points from 1 to n and obtain $\sum_{i=1,i\neq j}^{n-1} \mathfrak{f}(t_j - t_i)\ \overset{\sim 0}{p_i} + \mathfrak{f}(t_j - t_n)\ \overset{\sim 0}{p_n} + p_0 = 0$.

If we suppose that $\sum_k \overset{\sim 0}{p_k} = 0$, then

$$\sum_{i=1,i\neq j}^{n-1} \mathfrak{f}(t_j - t_i)\ \overset{\sim 0}{p_i} - \mathfrak{f}(t_j - t_n) \sum_{i=1,i\neq j}^{n-1} \overset{\sim 0}{p_i} + p_0 = 0,$$

$$\sum_{i=1,i\neq j}^{n-1} (\mathfrak{f}(t_j - t_i) - \mathfrak{f}(t_j - t_n) - \mathfrak{f}(t_n - t_i))\ \overset{\sim 0}{p_i} + \sum_{i=1}^{n-1} \mathfrak{f}(t_n - t_i)\ \overset{\sim 0}{p_i} + p_0 = 0,$$

$$\sum_{i=1,i\neq j}^{n-1} (\mathfrak{f}(t_j - t_i) - \mathfrak{f}(t_j - t_n) - \mathfrak{f}(t_n - t_i))\ \overset{\sim 0}{p_i} = 0. \tag{3.1}$$

Proposition 2. *The condition $\sum_{k\in\mathfrak{M}} \overset{\sim 0}{p_k} = 0$ is necessary for the solution of the Euler–Poisson equations (0.1) having form (2.1), where $\mathfrak{f}(t) = t^{-1}$,* ctg, cth *to exist.*

Proof. Let us denote $\sum_{k\in\mathfrak{M}} \overset{\sim 0}{p}_k = S_p$, $\sum_{k\in\mathfrak{M}} \overset{\sim 0}{\gamma}_k = S_\gamma$. Suppose that $\mathfrak{f}(t) = t^{-1}$, then

$$\begin{cases} p(t) = p_0 + S_p\, t^{-1} + ..., \\ \gamma(t) = \gamma_0 + S_\gamma\, t^{-2} + ..., \end{cases} \tag{3.2}$$

is the asymptotics of the solution of the Euler–Poisson equations (0.1) at infinity.

When substituting representation (3.2) into (0.1) we get the system

$$\begin{cases} Ap_0 \times p_0 + \gamma_0 \times r = 0, \\ \gamma_0 \times p_0 = 0, \end{cases}$$

which has the simple solution. It has the trivial solution $p_0 = \gamma_0 = 0$, and in addition to this, a)$p_0 = 0, \quad \gamma_0 = \nu r$; and b)$p_0 = (A - \mu E)^{-1}\nu r, \quad \gamma_0 = \nu p_0$.

For the trivial solution we obtain that $\mathcal{T} \to 0,\ t \to \infty$, but it is impossible for real motion.

In the case a) we substitute (3.2) into the trivial integral $\mathcal{T}$ and get $\langle S_\gamma, r\rangle = 0$, and then substitute (3.2) into the energy integral $\mathcal{H}$ and get $\langle AS_p, S_p\rangle = 0$. This takes place only when $S_p = 0$.

In the case b) first we substitute (3.2) into the second equation of the system (0.1). We get $S_p \times \gamma_0 = 0$. Then we substitute (3.2) into the first equation of the system (0.1) and get $AS_p \times p_0 + Ap_0 \times S_p = 0 = AS_p \times S_p$ because $p_0 \sim \gamma_0$. Finally substituting (3.1) into the energy integral we get $\langle AS_p, p_0\rangle = 0 = \langle AS_p, S_p\rangle$ that is true only if $S_p = 0$.

Let $\mathfrak{f}(t) = \mathrm{ctg}(t)$ and $t \to +i\infty$ then $\mathrm{ctg}(t) \to -i$ and $p(t) \to -S_p\, i + p_0 = p^+_\infty, \quad \gamma(t) \to -S_\gamma + \gamma_0 = \gamma^+_\infty$.

If $t \to -i\infty$ we have $\mathrm{ctg}(t) \to i$ and $p(t) \to S_p\, i + p_0 = p^-_\infty, \quad \gamma(t) \to -S_\gamma + \gamma_0 = \gamma^-_\infty$.

At the singular points of the Euler–Poisson equations the vectors p^+_∞, and γ^+_∞, p^-_∞ and γ^+_∞ are pairwise collinear. Moreover the vectors $\gamma^+_\infty = \gamma^-_\infty$ are nonzero because the trivial integral is strictly positive for complex time as well as for real time.

So we see that the vectors p^+_∞ and p^-_∞ are collinear. Obviously, that is possible only if $S_p = 0$ or $p_0 = 0$.

Let us consider the case when $p_0 = 0$. In this case $p^+_\infty = \lambda\gamma^+_\infty$ и $p^-_\infty = -\lambda\gamma^+_\infty$, where λ is some imaginary number because $S_p \in \mathbb{R}$, $S_\gamma \in \mathbb{R}$).

The integral of torque moment is equal to zero because

$$\mathcal{M} = \langle Ap^+_\infty, \gamma^+_\infty\rangle = \lambda\,\langle A\gamma^+_\infty, \gamma^+_\infty\rangle = \langle Ap^-_\infty, \gamma^+_\infty\rangle = -\lambda\,\langle A\gamma^+_\infty, \gamma^+_\infty\rangle\,.$$

The operator inertia A is positively defined consequently we get $S_p = 0$.

Finally let $\mathfrak{f}(t) = cth(t)$. Similarly to above we consider the limits if the conditions $t \to \pm\infty$ are true. We obtain $p^+_\infty = S_p + p_0$, $p^-_\infty = -S_p + p_0$, $\gamma^+_\infty = \gamma^-_\infty = S_\gamma + \gamma_0$, $S_p \sim p_0 \sim \gamma^+_\infty$.

Considering the torque moment in infinity we get

$$\mathcal{M} = \left\langle A(S_p + p_0), \gamma^+_\infty\right\rangle = \left\langle A(-S_p + p_0), \gamma^+_\infty\right\rangle \Rightarrow \left\langle AS_p, \gamma^+_\infty\right\rangle = 0 = \langle AS_p, S_p\rangle\,.$$

□

Remark 2. One can solve ([19], [20]) the equations

$$\sum_{i=1, i\neq j}^{n-1} \left(\mathfrak{f}\,(t_j - t_i) - \mathfrak{f}\,(t_j - t_n) - \mathfrak{f}\,(t_n - t_i)\right)\ \overset{\sim 0}{p}_i = 0$$

and $\sum\limits_{k\in\mathfrak{M}} \overset{\sim 0}{p}_k = 0$ for the cases of 2, 4 and 6 different singular points of the solutions of the Euler–Poisson equations.

4. The absence of the solutions of the Euler–Poisson equations with 8 types of the β-points. Now we consider the possibility of existence of the solutions (2.1) having 8 singular points. In this case the matrix $\overline{\mathfrak{F}}_8$ has the dimensions 7×7 and rank 7-3=4. These conditions are not sufficiently restricted.

Therefore we use a new approach which gives us an opportunity to solve our problem. We shall find 3 polynomials $\mathcal{F}_i(p,\gamma)$, $i=1,2,3$, of the third power such that the functions $\mathcal{F}_i(p(t),\gamma(t))$, $i=1,2,3$ have no singular points if the functions $p(t)$, $\gamma(t)$ have some singular points.

According to properties of the functions $p(t)$, $\gamma(t)$ (2.1) it follows that the functions $\mathcal{F}_i(p(t),\gamma(t))$ are bounded in $\mathbb{C}$ and consequently they are constants. Then the functions $\mathcal{F}_i(p,\ \gamma)$, $i=1,2,3$ are the integrals of solid body problem.

Now let us substitute $p(t)$, $\gamma(t)$ from representation (1.2) into the function

$$\mathcal{F}(p(t),\gamma(t)) = a_0p_1p_2p_3 + \textstyle\sum_{i,j} a_{ij}p_i^2p_j + \sum_{i,j} b_{ij}p_ip_j.$$

We obtain

$$\mathcal{F}(p(t),\gamma(t)) = \Big(a_0\, \overset{\sim 0}{p}_1\overset{\sim 0}{p}_2\overset{\sim 0}{p}_3 + \sum_{i,j} a_{ij}(\overset{\sim 0}{p}_i)^2\, \overset{\sim 0}{p}_j + \sum_{i,j} b_{ij}\, \overset{\sim 0}{p}_i\overset{\sim 0}{\gamma}_j\Big)t^{-3} + ... \tag{4.1}$$

From the formulae of Theorem 1 we express $\overset{\sim 0}{p}_i, \overset{\sim 0}{\gamma}$ by ξ and ξ_0. Then getting rid of the residue $\xi_0=\sqrt{-1-\xi^2}$ we obtain an equation of 9th power $\mathcal{E}_8(\xi,a_0,a_{ij},b_{ij})\xi=0$ along ξ without free term.

Further we suppose that $\prod_\sigma(B_{12}C_{12}r_1)\neq 0$. In this case $\xi\neq 0$, because in another case we have $C_{21}B_{31}+C_{31}B_{12}=-A_1B_{23}=0$ from $\mathcal{Q}(\xi)=0$.

For a convenient coefficient k the polynomial $\mathcal{E}_8-k\ \mathcal{Q}$ is the one of 7th power and has 8 roots. This is possible if all coefficients of the polynomial $\mathcal{E}_8-k\ \mathcal{Q}$ are equal to zero. So we get 8 linear equations for a_0,a_{ij},b_{ij}.

There exists a simple algorithm to solve the linear systems but our system is more then awkward therefore we give only solutions of this system. Let note that the rank of the system is maximal and equals 7. The coefficient $a_0=0$ and the other will be presented after the following remark.

Remark 3. The Euler–Poisson system (0.1) is invariant under circle permutation of indices. If we permute two indices then all right sides of the equations of system (0.1) change the sign. Hence all integrals are invariant under all permutations of indices. It is natural that this property is true for the coefficients of the polynomials $\mathcal{E}_8-k\ \mathcal{Q}$ and for representations of a_0,a_{ij},b_{ij} as the functions of $\overset{\sim 0}{p}_i, \overset{\sim 0}{\gamma}$.

Definition 2. Let us note the following symmetries:

$$s_1\colon 2\leftrightarrow 3,\quad \sigma,$$
$$S=\{s_1,s_2,s_3,\sigma\}.$$

Using this notations we give the presentation of the coefficients a_0,a_{ij},b_{ij}:

$$a_{12} = -\frac{B_{12}C_{13}r_2a_{11}}{A_1B_{31}r_1} + \frac{A_1a_{22}}{A_2} + \frac{A_1B_{12}r_2a_{33}}{B_{31}A_3r_3} + \frac{B_{12}C_{13}b_{12}}{2B_{31}r_1} - \frac{A_1B_{12}b_{32}}{2B_{31}r_3}, S, \tag{4.2}$$

$$\sum_\sigma\Big(\frac{B_{23}}{r_1}(r_1b_{11}+r_2b_{12}+r_3b_{13})\Big) - 2\sum_\sigma\frac{B_{23}a_{11}}{A_1r_1}\sum_\sigma r_1^2 = 0. \tag{4.3}$$

If relations (4.2), (4.3) are true, the asymptotics of the function $\mathcal{F}(p(t),\ \gamma(t))$ at the singular points take the following form:

$$\sum_{ij}\left(a_{ij}((\overset{\sim 0}{p}_i)^2 u_j + 2\,\overset{\sim 0}{p}_i\overset{\sim 0}{p}_j\,u_i) + b_{ij}(\overset{\sim 0}{p}_i\,v_j + u_i\,\overset{\sim 0}{\gamma}_j)\right)t^{-1} + ..., \tag{4.4}$$

here $u_i, v_i,\ i = 1,2,3$, are coordinates of the vectors $\beta_2 u_2,\ \beta_2 v_2$ (see (1.2)).

Now we can express u_i, v_i, by means of the variable ξ, but the linear system appears to be very awkward. Therefore we take the root of the polynomial $\mathcal{P}(\varrho)$ as a variable.

So we give the necessary formulae. For convenience, we shall write $p_i,\ \gamma_j$ instead of $\overset{\sim 0}{p}_i, \overset{\sim 0}{\gamma}_j$. According to Theorem 1

$$p_1^2 = (2A_2 - \varrho)(2A_3 - \varrho)(B_{12}B_{31})^{-1},\ \sigma, \tag{4.5}$$

and we obtain $Ap \times p = \pm(\varrho - 2A)p$.

Further from the first formula of the characteristic system (1.1) and Theorem 1 it follows that $\langle Ap \times p, r\rangle + \langle Ap, r\rangle = 0, \quad \langle Ap, r\rangle = \varrho\langle p, r\rangle$ and we have

$$Ap \times p = (\varrho - 2A)p \ \Leftrightarrow\ B_{23}p_2p_3 = (\varrho - 2A)p_1,\ \sigma. \tag{4.6}$$

It follows from (4.6) and Theorem 1 that

$$\gamma = (2A - \varrho)\langle p, r\rangle^{-1}p. \tag{4.7}$$

By means of (4.5), (4.6), one can replace any polynomials of $p_i,\ \gamma_i$ by linear function of p_i with polynomial coefficients of ϱ.

Multiplying (4.6) by p_i we obtain the following system

$$B_{31}(\varrho - A_2)(\varrho - 2A_3)r_2p_3 + B_{12}(\varrho - A_3)(\varrho - 2A_2)r_3p_2$$

$$+(\varrho - A_1)(\varrho - 2A_2)(\varrho - 2A_3)r_1 = 0,\ \sigma.$$

The roots of this system have the following form:

$$p_1 = \frac{B_{23}r_1^2(\varrho - 2A_3)(\varrho - 2A_2)(\varrho - A_1)^2}{2B_{12}B_{31}r_2r_3(\varrho - A_2)(\varrho - A_3)(\varrho - 2A_1)} -$$

$$-\frac{r_2(\varrho - A_2)}{2B_{12}r_3(\varrho - A_3)(\varrho - 2A_3)} - \frac{r_3(\varrho - A_3)(\varrho - 2A_2)}{2B_{31}r_2(\varrho - A_2)},\ \sigma. \tag{4.8}$$

To express the vectors $u,\ v$, by ϱ we use the relations from Theorem 3. These relations are $u = \lambda_1 p + \lambda_2\gamma$, $v = \mu_1 p - \lambda_1\gamma$ and $\lambda_2\langle A\gamma, \gamma\rangle + \mu_1\langle Ap, p\rangle = 0$, $-6\lambda_1\langle p, r\rangle - \lambda_2(3\langle A\gamma, \delta\rangle + \langle Ap, p\rangle) + 2\mu_1\langle\delta, r\rangle = 0$.

Moreover from Theorem 1 we take $\langle Ap, p\rangle = -2\varrho$, and from (4.7) we obtain

$$\langle A\gamma, \gamma\rangle = -2(2A_1 - \varrho)(2A_2 - \varrho)(2A_3 - \varrho)\langle p, r\rangle^{-2}.$$

Now we prove that $\langle A\gamma, \delta\rangle = A_1 + A_2 + A_3 - \varrho/2$, $\langle\delta, r\rangle = (\langle p, r\rangle^2 + 4\langle r, r\rangle)/(4\varrho)$.

The vector

$$\delta = \frac{1}{4\gamma_3}(p_1p_3 - 2p_2, 2p_1 + p_2p_3, 4 + p_3^2) \tag{4.9}$$

can be found from the simple linear system (see Theorem 1):

$$p \times \delta = -2\delta, \quad \langle \gamma, \delta \rangle = 2, \tag{4.10}$$

and we find $\langle A\gamma, \delta \rangle$. Then we have $Ap \times \delta = -(Ap \times p + \gamma \times r) \times \delta =$

$$= - \langle Ap, \delta \rangle p + \langle p, \delta \rangle Ap - \langle \gamma, \delta \rangle r + \langle \delta, r \rangle \gamma = - \langle p, r \rangle p - 2r + \langle \delta, r \rangle \gamma,$$

$$\langle p, r \rangle^2 = \langle Ap, \delta \rangle^2 = \langle Ap, Ap \rangle \langle \delta, \delta \rangle - \langle Ap \times \delta, Ap \times \delta \rangle =$$

$$= - \langle \langle p, r \rangle p + 2r - \langle \delta, r \rangle \gamma, \langle p, r \rangle p + 2r - \langle \delta, r \rangle \gamma \rangle = -4 \langle r, r \rangle + 4\varrho \langle \delta, r \rangle .$$

Here we have used the relations $\langle Ap, \delta \rangle = \langle p, r \rangle$, $\langle p, p \rangle = -4$, $\langle p, \gamma \rangle = \langle \gamma, \gamma \rangle = 0$, $\langle \gamma, r \rangle = \varrho$.

Thus coefficient at the term t^{-1} from asymptotics (4.4) is the polynomial of p_i, ϱ. By means of (4.5) we decrease the power p_i to one and then we use representation (4.6). As a result we get a polynomial of the 14th power of ϱ with coefficients which are the linear functions of a_{ij}, b_{ij}. It is difficult to decrease the power to 7, therefore we cannot present the necessary calculations and give the only answer. Naturally the following formulae are invariant under the symmetries S :

$$a_{12} = \frac{B_{12}(A_2 - 2B_{31})r_2 a_{33}}{A_3^2 r_3} - \frac{C_{12}(B_{12} - A_3)a_{22}}{A_3 A_2}, \ S, \sigma,$$

$$b_{12} = \frac{(A_2 + 4B_{31})r_2 a_{11}}{A_1 B_{31}} - \frac{(A_2 + 2C_{31})r_1 a_{22}}{A_3 A_2} - \frac{C_{31}(A_2 - 2B_{31})r_1 r_2 a_{33}}{A_3^2 B_{31} r_3} \ S, \sigma,$$

$$b_{11} = \frac{C_{12}C_{32} r_2 a_{22}}{A_2 A_3 B_{23}} + \frac{C_{23}(2B_{12} + A_3) r_3^2 a_{22}}{A_2^2 B_{23} r_2} -$$
$$- \frac{C_{13}C_{23} r_3 a_{33}}{A_2 A_3 B_{23}} - \frac{C_{32}(A_2 - 2B_{31}) r_2^2 a_{33}}{A_3^2 B_{23} r_3}, \ S, \sigma.$$

The system under consideration has 16 equations and 19 unknown quantities consequently we get 3 free parameters a_{11}, a_{22}, a_{33}. It means that we have found three integrals.

Theorem 6. *Let there exist a solution of the solid body problem with the following asymptotics of the singular β-points:*

$$p(t) = \overset{\sim 0}{p} \, t^{-1} + u_2 t + ..., \quad \gamma(t) = \overset{\sim 0}{\gamma} \, t^{-2} + v_2 + ... \, .$$

Then the functions

$$\mathcal{F}_1 = p_1^3 + \frac{C_{21}(A_3 + B_{12})p_1 p_2^2}{A_1 A_3} - \frac{B_{23}(2B_{12} - A_3) r_3 p_2^2 p_3}{A_1^2 r_1} + \frac{C_{31}(A_1 + B_{23}) p_3^2 p_1}{A_1 A_2}$$
$$- \frac{B_{23}(A_2 + 2B_{31}) r_2 p_3^2 p_2}{A_1^2 r_1} + \frac{(A_2 + 4B_{31}) r_2 p_1 \gamma_2}{A_1 B_{31}} - \frac{(A_3 - 4B_{12}) r_3 p_1 \gamma_3}{A_1 B_{12}}$$
$$- \frac{(A_1 + 2C_{32}) r_2 p_2 \gamma_1}{A_1 A_3} - \left(\frac{C_{21}C_{31} r_1}{A_1 A_3 B_{31}} + \frac{C_{13}(A_3 - 2B_{12}) r_3^2}{A_1^2 B_{31} r_1} \right) p_2 \gamma_2$$
$$- \frac{C_{12}(A_3 - 2B_{12}) r_2 r_3 p_2 \gamma_3}{A_1^2 B_{12} r_1} - \frac{(A_1 + 2C_{23}) r_3 p_3 \gamma_1}{A_1 A_2} + \frac{C_{13}(A_2 + 2B_{31}) r_2 r_3 p_3 \gamma_2}{A_1^2 B_{31} r_1}$$
$$+ \left(\frac{C_{21}C_{31} r_1}{A_1 A_2 B_{12}} + \frac{C_{12}(A_2 + 2B_{31}) r_2^2 p_3 \Gamma_3}{A_1^2 B_{12} r_1} \right), \sigma \tag{4.10}$$

are the partial integrals for the mentioned solutions.

Proof. If we substitute representation (2.1) into $\mathcal{F}_i$ we obtain the functions without singular points. According to Theorem 5 these functions are constants. $\square$

Remark 4. It is natural that $\mathcal{F}_i r_i$ are the integrals for the solutions of the Bobylyov–Steklov ([16], [17]) and Steklov ([18]) cases.

Now we shall prove that there exists no solutions with different 8 singular β-points.

Let us suppose that there exists a solution $p(t)$, $\gamma(t)$, such that

$$\mathcal{F}_i(p(t),\gamma(t)) = f_i,\quad \mathcal{H}(p(t),\gamma(t)) = h,\quad \mathcal{M}(p(t),\gamma(t)) = m,\quad \mathcal{T}(p(t),\gamma(t)) = t_c.$$

Substituting the asymptotics (1.2) into $\mathcal{M}$ we express the parameter β_3 by means of m. Then we substitute the same asymptotics into $\mathcal{F}_i$, and take coefficient of the term t^0. We get the polynomial of the 12th power of ϱ and decrease the power to 7 as we did above.

All the coefficients of this polynomial equals zero, in particular the leading coefficient equals zero too. So we have the following S-symmetrical relation:

$$\sum_\sigma m(8A_1^3A_2A_3 + 3A_1^3A_2^2 + 3A_1^2A_2^3 - 3A_1^4A_2 - 3A_1A_2^4 - 6A_1A_2^2A_3^2) = 0. \tag{4.11}$$

We claim that this equality is true only if $m = 0$. To prove this we make the following replacement: $A_1/A_3 = a + b$, $A_2/A_3 = a - b$. We obtain

$$-3a + 2a^4 + 12ab^4 - 12a^3b^2 + 15ab^2 - b^2 - 12a^2b^2 - 3a^3 + 7a^2 - 14b^4 = 0$$

and then we find the extremum of the left side of this equation. We get the system

$$\begin{cases} -3 + 8a^3 + 12b^4 - 36a^2b^2 + 15b^2 - 24ab^2 - 9a^2 + 14a = 0, \\ -2b(-24ab^2 + 12a^3 - 15a + 1 + 12a^2 + 28b^2) = 0 \end{cases}$$

which is simple. If $b = 0$ then a is the root of the equations $8a^3 - 9a^2 + 14a - 3 = 0$. In another case a is the root of the equation

$$432a^6 - 144a^5 - 600a^4 - 1120a^3 + 2235a^2 - 1099a + 138 = 0.$$

The simple test shows us that there are no solutions, satisfying the inequation (8.15) of a triangle.

If $m = 0$, then from the asymptotics (1.2) we obtain that $\beta_3(\langle A\overset{\sim 0}{p}, v_3\rangle + \langle Au_3, \overset{\sim 0}{\gamma}\rangle) = 0$ or if we express $\overset{\sim 0}{p}, \overset{\sim 0}{\gamma}, u_3, v_3$, via ϱ then we have

$$\beta_3\varrho\prod_\sigma(\varrho - 2A_1)^2\sum_\sigma\left(\frac{\varrho - A_1}{\varrho - 2A_1}(2A_1\varrho + (\varrho - 5A_1)(A_2 + A_3) - 4A_2A_3)r_1^2\right) = 0.$$

It should be mentioned that (see Theorem 3), the eigenvector (u_3, v_3) has the representation $u = \lambda_2\gamma + \lambda_3\delta$, $v = -\lambda_3 p$, and the relation $\lambda_2(\langle Ap, p\rangle + 4\langle A\gamma, \delta\rangle) - 2\mu_1(2\langle\delta,\delta\rangle) + \langle\delta, r\rangle) = 0$ is true.

Now we decrease the power of the polynomial of ϱ to the seventh one using the method, used above. Then we obtain $\beta_3 r_1^4\sum_\sigma((2A_1 + A_2 + A_3)r_1^2) = 0$, σ, that is, obviously, possible only when $\beta_3 = 0$.

That, in turn, means that the doubly-periodic function $p(t)$ is an odd function with resrect to all the eight singular points. The simple test shows that it is impossible.

Thus the proof of the following theorem comes to its end.

Theorem 7. *The solutions of the Euler-Poisson equations (0.1) with β-points with the parameters $\beta_0 = \beta^0 = 0$ exist, being doubly-periodic and take place only in the cases of Bobylyov–Steklov ([16], [17]) and Steklov ([18]).*

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