УДК 519.512

I. V. Protasov

BALLEANS OF BOUNDED GEOMETRY AND G-SPACES

I. V. Protasov. *Balleans of bounded geometry and G-spaces*, Matematychni Studii, **30** (2008) 61–66.

A ballean (or a coarse structure) is a set endowed with some family of subsets which are called the balls. The properties of the family of balls are postulated in such a way that a ballean can be considered as an asymptotical counterpart of a uniform topological space. We prove that every ballean of bounded geometry is coarsely equivalent to a ballean on some set X determined by some group of permutations of X.

И. В. Протасов. *Болеаны ограниченной геометрии и G-пространства* // Математичні Студії. – 2008. – Т.30, №1. – C.61–66.

Болеан (или эквивалентная структура) — множество оснащенное некоторым семейством подмножеств, называемых шарами. Свойства семейства шаров постулируются таким образом, что болеан можно рассматривать как асимптотический аналог равномерно топологического пространства. Доказывается, что каждый болеан ограниченной геометрии грубо эквивалентен болеану некоторого множества X, определяемого с помощью некоторой группы перестановок X.

1. Ball structures and balleans. A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X, \alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii*. Given any $x \in X, A \subseteq X, \alpha \in P$ we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},\ B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,
$$B^*(x, \alpha') \subseteq B(x, \alpha),\ B(x, \beta') \subseteq B^*(x, \beta);$$
- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,
$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$
- *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$
- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

2000 *Mathematics Subject Classification:* 37B05, 54E15.

© I. V. Protasov, 2008

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ball structure. Then the family

$$\Big\{ \bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P \Big\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x, y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

We say that a ball structure $\mathcal{B}$ is a *ballean* if $\mathcal{B}$ is upper symmetric and upper multiplicative. In this paper we follow terminology from [6, 7]. A structure on X, equivalent to a ballean, can also be defined in terminology of entourages. In this case it is called a coarse structure [8] or a uniformly bounded space [5]. For motivations to study balleans see also [1, 2, 4].

2. Morphisms. Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a *$\prec$-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

A bijection $f : X_1 \to X_2$ is called an *asymorphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$ if f and f^{-1} are $\prec$-mappings.

Let $\mathcal{B} = (X, P, B)$ be a ballean, S be a set. Two mappings $f, f' : S \to X$ are called *close* if there exists $\alpha \in P$ such that $f'(s) \in B(f(s), \alpha)$ for every $s \in S$.

Two balleans $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ are called *coarsely equivalent* if there exist the $\prec$-mappings $f_1 : X_1 \to X_2$, $f_2 : X_2 \to X_1$ such that $f_1 \circ f_2$, $f_2 \circ f_1$ are close to the identity mappings id_{X_1}, id_{X_2}.

Let $\mathcal{B} = (X, P, B)$ be a ballean. Every non-empty subset $Y \subseteq X$ determines the subballean $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) = B(Y, \alpha) \cap Y$, $y \in Y$, $\alpha \in P$. A subset Y is called *large* if there exists $\gamma \in P$ such that $B(Y, \gamma) = X$. If Y is large, then $\mathcal{B}_Y$ and $\mathcal{B}$ are coarsely equivalent. We shall use also the following observations. Two balleans $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ are coarsely equivalent if and only if there exist the large subsets $Y_1 \subseteq X_1$,$Y_2 \subseteq X_2$ such that the subballeans $\mathcal{B}_{Y_1}$ and $\mathcal{B}_{Y_2}$ are asymorphic.

3. Density and capacity. Let $\mathcal{B} = (X, P, B)$ be a ballean, $Y \subseteq X$, $S \subseteq Y$, $\alpha \in P$. We say that a subset S is *α-dense* in Y if $Y \subseteq B(S, \alpha)$. An *α-density* of Y is the cardinal

$$den_\alpha(Y) = min\{|S| : S \; is \; an \; \alpha - dense \; subset \; of \; Y\}.$$

A subset S of X is called *α-separated* if $B(x, \alpha) \cap B(y, \alpha) = \varnothing$ for all distinct $x, y \in S$. An *α-capacity* of Y is the cardinal

$$cap_\alpha(Y) = sup\{|S| : S \; is \; an \; \alpha - separated \; subset \; of \; Y\}.$$

Let $\mathcal{B} = (X, P, B)$ be an arbitrary ballean. Replacing every ball $B(x, \alpha)$ to $B'(x, \alpha) = B(x, \alpha) \cap B^*(x, \alpha)$, we get the asymorphic ballean $\mathcal{B}' = (X, P, B')$ with $(B')^* = B'$. Thus, in what follows we may suppose that $B^*(x, \alpha) = B(x, \alpha)$ for all $x \in X$, $\alpha \in P$.

Lemma 1. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $Y \subseteq X$, $\alpha, \beta \in P$ and $B(B(x, \alpha)) \subseteq B(x, \beta)$ for every $x \in X$. Then the following statements hold*

(i) $den_\beta(Y) \leqslant cap_\alpha(Y) \leqslant den_\alpha(Y)$;

(ii) if $Z \subseteq X$ and $Y \subseteq B(Z, \alpha)$, then $den_\beta(Y) \leqslant |Z|$.

Proof. (i) Let S be an α-separated subset of Y, D be an α-dense subset of Y. Then every ball $B(x, \alpha)$, $x \in D$ has at most one point of S. Since $S \subseteq Y \subseteq \bigcup_{x \in D} B(x, \alpha)$, we have $|S| \leqslant |D|$, so $cap_\alpha(Y) \leqslant den_\alpha(Y)$.

Let S be a maximal by inclusion α-separated subset of Y. Then every ball $B(x,\alpha)$, $x \in Y$ meets at least one ball $B(y,\alpha)$, $y \in S$. It follows that $Y \subseteq \bigcup_{x\in S} B(x,\beta)$, so S is β-dense in Y and $den_\beta(Y) \leqslant cap_\alpha(Y)$.

(ii) We put $Z' = \{z \in Z : B(z,\alpha) \cap Y \neq \varnothing\}$ and, for every $z \in Z'$, pick some point $y_z \in B(z,\alpha) \cap Y$. Then the subset $\{y_z : z \in Z'\}$ of Y is β-dense in Y, so $den_\beta(Y) \leqslant |Z'| \leqslant |Z|$. □

4. Locally finite balleans. A ballean $\mathcal{B} = (X, P, B)$ is called *locally finite* if every ball $B(x,\alpha)$, $x \in X$, $\alpha \in P$ is finite.

Let $\mathcal{B} = (X, P, B)$, $\mathcal{B}' = (X', P', B')$ be balleans, $f : X \to X'$ be an injective $\prec$-mapping. If $\mathcal{B}'$ is locally finite then $\mathcal{B}$ is locally finite. In particular, every ballean asymorphic to a locally finite ballean is locally finite.

We say that a ballean $\mathcal{B}$ is *coarsely locally finite* if $\mathcal{B}$ is coarsely equivalent to some locally finite ballean.

Proposition 1. *A ballean $\mathcal{B} = (X, P, B)$ is coarsely locally finite if and only if there exists $\beta \in P$ such that β-capacity of every ball $B(x,\gamma)$,$x \in X$, $\gamma \in P$ is finite.*

Proof. Let $\mathcal{B}' = (X', P', B')$ be a locally finite ballean coarsely equivalent to $\mathcal{B}$. Then there exist the large subsets $Y \subseteq X$, $Y' \subseteq X'$ such that the subballeans $\mathcal{B}_Y$ and $\mathcal{B}_{Y'}$ are asymorphic. We choose $\alpha \in P$ such that $B(Y,\alpha) = X$ and take an arbitrary $x \in X$, $\gamma \in P$. Since $\mathcal{B}_Y$ is locally finite then the subset $Z = B(B(x,\gamma),\alpha) \cap Y$ is finite. Since $B(x,\gamma) \subseteq B(Z,\alpha)$, by Lemma 1 (ii), $den_\beta(B(x,\gamma)) \leqslant |Z|$. Since Z is finite, by Lemma 1 (i), β-capacity of $B(x,\gamma)$ is finite.

On the other hand, let β-capacity of every ball $B(x,\gamma)$ is finite. We choose a maximal by inclusion β-separated subset Y of X. Clearly, Y is large in X, so $\mathcal{B}_Y$ is coarsely equivalent to $\mathcal{B}$. Since $cap_\beta B(x,\gamma)$ is finite, then $B(x,\gamma) \cap Y$ is finite. Hence, $\mathcal{B}_Y$ is locally finite. □

Every metric space (X,d) determines the metric ballean $\mathcal{B}(X,d) = (X, \mathbb{R}^+, B_d)$, where $B_d(x,r) = \{y \in X : d(x,y) \leqslant r\}$. For criterion of metrizability of balleans see [7, Theorem 2.1.1]. A metric space is called *proper* if every ball $B_d(x,r)$ is compact.

Corollary 1. *Let (X,d) be a proper metric space. Then the metric ballean $\mathcal{B}(X,d)$ is coarsely locally finite.*

Proof. It suffices to note that an 1-capacity of every ball in (X,d) is finite, and apply Proposition 1. □

5. Uniformly locally finite balleans. A ballean $\mathcal{B} = (X, P, B)$ is called *uniformly locally finite* if there exists a function $h : P \to \omega$ such that $|B(x,\alpha)| \leqslant h(\alpha)$ for all $x \in X$, $\alpha \in P$.

Let $\mathcal{B} = (X, P, B)$, $\mathcal{B}' = (X', P', B')$ be balleans, $f : X \to X'$ be an injective $\prec$-mapping. If $\mathcal{B}'$ is uniformly locally finite then so is $\mathcal{B}$. In particular, every ballean asymorphic to an uniformly locally finite ballean is uniformly locally finite.

We say that a ballean $\mathcal{B} = (X, P, B)$ has *bounded geometry* if there exist $\beta \in P$ and a function $h : P \to \omega$ such that $cap_\beta B(x,\alpha) \leqslant h(\alpha)$ for all $x \in X$, $\alpha \in P$.

Repeating the arguments proving Proposition 1 we get the following statements.

Proposition 2. *A ballean $\mathcal{B} = (X, P, B)$ has bounded geometry if and only if $\mathcal{B}$ is coarsely equivalent to some uniformly locally finite ballean.*

Example 1. Let $\Gamma(V,E)$ be a connected graph with the set of vertices V and the set of edges E. Given any $u,v \in V$, we denote by $d(u,v)$ the length of a shortest path between u and v. Then we get the metric space (V,d) associated with $\Gamma(V,E)$ and the metric ballean $\mathcal{B}(V,d)$. Clearly, $\mathcal{B}(V,d)$ is uniformly locally finite if and only if there exists a natural number r such that $|B_d(v,1)| \leqslant r$ for every $v \in V$.

Example 2. Let G be a finitely generated group with the identity e, F be a symmetric ($F = F^{-1}$) set of generators of G such that $e \notin F$. The Cayley graph $Cay(G,F)$ is a graph with the set of vertices G and set of edges $\{\{u,v\} : uv^{-1} \in F\}$. Let d_F be a path metric on $Cay(G,F)$. Then the metric ballean $\mathcal{B}(G,d_F)$ is uniformly locally finite.

Example 3. Let G be an arbitrary group, $\mathcal{F}_e$ the family of all symmetric subsets of G containing e. Then we get a ballean $\mathcal{B}(G) = (G, \mathcal{F}_e, B)$, where $B(g,F) = Fg$. Clearly, $\mathcal{B}(G)$ is uniformly locally finite and in the case G is finitely generated, $\mathcal{B}(G)$ is asymorphic to the ballean $\mathcal{B}(G,d_F)$ determined in Example 2.

Example 4. Let G be a group and X be a G-space with the action of G on X defined by $(g,x) \mapsto g(x)$. We denote by $\mathcal{F}_e$ the family of all finite symmetric subsets of G containing e. Then we get the ballean $\mathcal{B}(G,X) = (X, \mathcal{F}_e, B)$, where $B(x,F) = \{g(x) : g \in F\}$, $x \in X$, $F \in \mathcal{F}_e$. Clearly, $\mathcal{B}(G,X)$ is uniformly locally finite.

Example 5. Let G be a gruppoid (=inverse semigroup) of partial bijections of a set X, $\mathcal{F}$ be a family of all finite subsets of G such that $F = F^{-1}$ for every $F \in \mathcal{F}$. Given any $x \in X$ and $F \in \mathcal{F}$, we put $B(x,F) = \{x\} \cup \{g(x) : g \in F\}$ and get the uniformly locally finite ballean $\mathcal{B}(G,X)$.

Example 6. Let G be a locally compact topological group, C be the family of all compact symmetric subsets of G containing e. Then, by Proposition 2, the ballean $\mathcal{B}(G) = (G, C, B)$, where $B(x,C) = Cx$, is of bounded geometry.

Question 1. *Let G be a locally compact group. Does there exist a discrete group D such that the balleans $\mathcal{B}(G)$ and $\mathcal{B}(D)$ are coarsely equivalent? This is so if G is Abelian or a connected Lie group.*

6. G-space realization. Let $\mathcal{B}$, $\mathcal{B}'$ be balleans with the same support X. We write $\mathcal{B} \prec \mathcal{B}'$ if the identity mapping $id : X \to X$ is a $\prec$-mapping from $\mathcal{B}$ to $\mathcal{B}'$. If $\mathcal{B} \prec \mathcal{B}'$ and $\mathcal{B}' \prec \mathcal{B}$, we identify $\mathcal{B}$ and $\mathcal{B}'$ and write $\mathcal{B} = \mathcal{B}'$.

Let $\mathcal{B}$ be a uniformly locally finite ballean with the support X. Applying Lemma 4.10 from [8], one can show that there exists a gruppoid G of partial bijections of X such that $\mathcal{B} = \mathcal{B}(G,X)$ where $\mathcal{B}(G,X)$ is a ballean determined in Example 5. Our next result states that instead of the gruppoid G we can take some group of permutations of X.

Theorem 1. *For every uniformly locally finite ballean $\mathcal{B} = (X,P,B)$, there exists a group G of permutations of X such that $\mathcal{B} = \mathcal{B}(G,X)$.*

Proof. We fix an arbitrary $\alpha \in P$ and choose $\beta \in P$ such that $B(B(x,\alpha),\alpha) \subseteq B(x,\beta)$ for each $x \in X$. Then we define the graph Γ_β with the set of vertices X and the set of edges E_β defined by the rule: $\{x,y\} \in E_\beta$ if and only if $x \in B(y,\beta)$. Since $\mathcal{B}$ is uniformly locally finite, there exists a natural number $n(\alpha)$ such that the local degree of every vertex of Γ_β does not exceed $n(\alpha)$. By [3, Corollary 12.2], the chromatic number of Γ_β does not exceed

$n(\alpha)+1$. It follows that we can partition $X = X_1 \cup \ldots \cup X_{n(\alpha)+1}$ so that any two vertices from X_j are non-adjacent, in particular, every subset X_i is α-separated.

Now we fix $i \in \{1,\ldots,n(\alpha)+1\}$ and, for every vertex $x \in X_i$, enumerate the set $B(x,\alpha)\setminus\{x\} = \{x(1),\ldots,x(n_x)\}$, where $n_x \leqslant n(\alpha)$. Then we define the set $S_i(\alpha)$ of $n(\alpha)$ permutations of X as follows. For each $j \in \{1,\ldots,n(\alpha)\}$ and $x \in X_i$, we put $\pi_j(x) = x(j)$,$\pi_j(x_j) = x$ if $j \leqslant n_x$, and $\pi_j(x) = x$ otherwise. Then we extend π to X putting $\pi_j(y) = y$ for all $y \in X \setminus \bigcup_{x\in X_i}\{x, x(j)\}$. Since X_i is α-separated, this definition is correct. Thus, we get the set $S_i(\alpha) = \{\pi_1,\ldots,\pi_{n(\alpha)}\}$ of permutations of X. We put $S(\alpha) = S_1(\alpha)\cup\ldots\cup S_{n(\alpha)+1}(\alpha)$ and denote by G the group of permutations of X generated by $\bigcup_{\alpha\in P} S(\alpha)$.

At last we show that the identity mapping $id : X \to X$ is an asymorphism between $\mathcal{B}$ and the ballean $\mathcal{B}(G,X) = (X,\mathcal{F}_e,B')$ determined in Example 4. Given any $\alpha \in P$ and $x \in X$, we have $B(x,\alpha) \subseteq B'(x,S_\alpha)$. On the other hand, let F be a finite subset of G, $g \in F$. Then there exists $\alpha_1,\ldots,\alpha_m \in P$ and $s(\alpha_1) \in S(\alpha_1),\ldots,s(\alpha_m) \in S(\alpha_m)$ such that $g = s(\alpha_m)\ldots s(\alpha_1)$. We choose $\gamma_g \in P$ such that

$$B(\ldots(B(B(x,\alpha_1),\alpha_2),\ldots),\alpha_m) \subseteq B(x,\gamma_g)$$

for every $x \in X$. Then $B'(x,\{g\}) \subseteq B(x,\gamma_g)$ for every $x \in X$. Since F is finite, there exists $\gamma \in P$ such that, for each $x \in X$, we have $B'(x,F) \subseteq B(x,\gamma)$. □

Sticking together Proposition 2 and Theorem 1 we get the following statement.

Theorem 2. *Every ballean of bounded geometry is coarsely equivalent to some ballean $\mathcal{B}(G,X)$ of G-space X.*

We conclude our paper with two applications of Theorem 1.

Theorem 3. *Let X be a set, S_X be a group of all permutations of X. Then $\mathcal{B}(S_X,X)$ is the strongest uniformly locally finite ballean on X.*

Proof. Let $\mathcal{B}'$ be a uniformly locally finite ballean on X. Using Theorem 1, we choose a group G of permutations of X such that $\mathcal{B}' = \mathcal{B}(G,X)$. Since G is a subgroup of S_X, we have $\mathcal{B}' \prec \mathcal{B}(S_X,X)$. □

A ballean $\mathcal{B} = (X,P,B)$ is called *connected* if, for any $x,y \in X$, there exists $\alpha \in P$ such that $y \in B(x,\alpha)$. Clearly, a ballean $\mathcal{B}(G,X)$ of a G-space is connected if and only if G acts transitively on X.

Let $\mathcal{B}_1 = (X_1,P_1,B_1)$, $\mathcal{B}_2 = (X_2,P_2,B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a *$\succ$-mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that $B_2(f(x),\beta)) \subseteq f(B_1(x,\alpha))$ for each $x \in X_1$. A bijection $f : X_1 \to X_2$ is a $\succ$-mapping if and only if f^{-1} is a $\prec$-mapping. Thus, $\mathcal{B}_1$ and $\mathcal{B}_2$ are asymorphic if and only if there is a bijection $f : X_1 \to X_2$ which is a $\prec$-mapping and a $\succ$-mapping.

Theorem 4. *For every connected uniformly locally finite ballean $\mathcal{B}$ on a set X, there exist a group G of permutations of X and a surjective mapping $f : G \to X$ which is a $\prec$-mapping and a $\succ$-mapping from $\mathcal{B}(G)$ to $\mathcal{B}$.*

Proof. Applying Theorem 1, we identify $\mathcal{B}$ with $\mathcal{B}(G,X)$ for some group G of permutations of X. Then we fix $x_0 \in X$ and, for every $g \in G$, put $f(g) = g(x_0)$. Since $\mathcal{B}$ is connected, (G,X) is a transitive G-space, so f is surjective. For any finite subset F of G, we have $f(Fg) = Fg(x_0) = F(g(x_0)) = F(f(g))$. It follows that f is a $\prec$-mapping and a $\succ$-mapping. □

Let (G, X) be a transitive G-space, $x_0 \in X$. If $St(x_0) = \{g \in G : g(x_0) = x_0\}$ is finite, applying Theorem 4, it is easy to show that the balleans $\mathcal{B}(G)$ and $\mathcal{B}(G, X)$ are coarsely equivalent.

Question 2. *Let (G, X) be a transitive G-space. How to detect whether the ballean $\mathcal{B}(G, X)$ is asymorphic (coarsely equivalent) to the ballean $\mathcal{B}(H)$ of some group H?*

REFERENCES

1. A. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys, **55**(2000), 1085-1129.
2. M. Gromov, *Asymptotic invariants for infinite groups*, in Geometric Group Theory, vol.**2**, 1-295, Cambridge University Press, 1993.
3. F. Harary, *Graph Theory*, Addison-Wesley Publ. Comp., 1969.
4. P. Harpe, *Topics in Geometrical Group Theory*, University Chicago Press, 2000.
5. V. Nekrashevych, *Uniformly bounded spaces*, Problems in Algebra, **14**, 47-67, Gomel University Press, 1999.
6. I. Protasov, T. Banakh, *Ball Structures and Colorings of Groups and Graphs*, Math. Stud. Monogr. Ser., vol.**11**, VNTL, Lviv, 2003.
7. I. Protasov, M. Zarichnyi, *General Asymptology*, Math. Stud. Monogr. Ser., vol.**12**, VNTL, Lviv, 2007.
8. J. Roe, *Lectures on Coarse Geometry*, University Lecture Series, vol.**31**, American Mathematical Society, Providence, RI, 2003.

Department of Cybernetics,
Kyiv Taras Shevchenko University
Volodimirska str., 64, 01033 Kyiv, Ukraine
protasov@unicyb.kiev.ua

Received 31.10.2007