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LOGARITHMIC DERIVATIVE ESTIMATES FOR SUBHARMONIC FUNCTIONS

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We obtain sharp estimates of the mean values of analogues logarithmic derivatives for subharmonic functions outside of sets of small linear density.

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Получены точные оценки интегральных средних аналогов логарифмических производных для субгармонической функции вне множеств малой линейной плотности.

1. Introduction and main results. We assume that the reader is familiar with the fundamental results and the standard notation of the theory of subharmonic functions in $\mathbb{C}$ ([1]). We denote by $C(a,t)=\{\zeta : |\zeta-a|\le t\}$ the closed disk with the center $a\in\mathbb{C}$ and radius $t>0$.

For a measurable set $E\subset[0,\infty)$ the upper and lower densities are defined as

$$\overline{D}(E)=\limsup_{r\to\infty}\frac{m(E\cap[0,r])}{r},\quad \underline{D}(E)=\liminf_{r\to\infty}\frac{m(E\cap[0,r])}{r},$$

where $m(E)=\int_E dr$ is the Lebesgue measure of the set E.

For any subharmonic function u we have the following Poisson-Jensen formula in the disk $C(0,R)$ ([1, p. 139], [2])

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}u\left(Re^{i\varphi}\right)\operatorname{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi+\int_{C(0,R)}\ln\left|\frac{R(z-\zeta)}{R^2-z\bar{\zeta}}\right|d\mu(\zeta),$$

where $\mu=\mu_u$ there is the Riesz measure of the function u. Let us consider the function

$$q(z)=\frac{1}{2\pi}\int_0^{2\pi}u\left(Re^{i\varphi}\right)\left(\frac{d}{dz}\right)\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi+\int_{C(0,R)}\left(\frac{d}{dz}\right)\ln\frac{R(z-\zeta)}{R^2-z\bar{\zeta}}d\mu(\zeta). \qquad (1)$$

The function $q(z)$ does not depend on a choice of the number R and is locally summable ([2]). Hence q is defined almost everywhere. It is known ([3, Lemma 1.6]) that the function

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$u(z)$, $z = re^{i\theta} \in \mathbb{C}$ is absolutely continuous as a function of r for almost all $\theta \in [0, 2\pi]$, and for almost all z : $\dfrac{\partial}{\partial r} u(re^{i\theta}) = \mathrm{Re}(q(re^{i\theta})e^{i\theta})$.

If $u = \ln|f|$, where f is an entire function, then $q = \frac{f'}{f}$. That is $q(z)$, is an analogue of the logarithmic derivative.

We shall consider Nevanlinna's characteristics for a subharmonic function u, $u(0) \neq -\infty$, in the complex plane

$$T(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u^+\left(re^{i\theta}\right) d\theta, \quad m(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u^-\left(re^{i\theta}\right) d\theta, \quad N(r,u) = \int_0^r \frac{n(t,u)}{t} dt, \quad (2)$$

where $u^+(z) = \max\{u(z), 0\}$, $u^-(z) = -\inf\{u(z), 0\}$, $n(t,u) = \mu_u(C(0,t))$ is the Riesz mass of the disk $C(0,t)$. For subharmonic functions first fundamental Nevanlinna's theorem is true ([1])

$$T(r,u) = m(r,u) + N(r,u) + u(0), \quad (3)$$

which is equivalent to Jensen-Privalov's formula.

We denote $w(z) = zq(z)$. The quantity $\mathrm{Re}\, w(z)$ was investigated in papers [2] and [4] (in the case $u = \ln|f|$).

In [5] an estimate for a normal derivative of a subharmonic function in $\mathbb{R}^{\mathbb{N}}$ is proved. In the case of the complex plane this estimate in terms of Nevanlinna's characteristics becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |q(re^{i\theta})| d\theta \le \beta \log\ T(r,u) + \log\log r,$$

for all $r > 0$ outside an exceptional set $E \subset [0, \infty)$ of finite Lebesgue measure $m(E) \le 6$, $\beta > \frac{3}{2}$. The question on sharpness of the constant $3/2$ remains open.

A question about finding of exact upper estimates of integral mean of $q(z)$ and $\mathrm{Re}\, w(z)$, which are determined by a subharmonic function u in $\mathbb{C}$ arises.

The mentioned estimates of W. Hayman and J. Miles (Lemma 5 [4]) is extended on subharmonic functions by a slight modification of the original proof. We obtain

Theorem 1. *Suppose that u is subharmonic function in $\mathbb{C}$, and $u(0) \neq -\infty$. If $1 < r < R$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Re}\, w\left(re^{i\theta}\right)| d\theta \le \frac{2T(R,u) - u(0)}{\log(R/r)}. \quad (4)$$

If we want to obtain an estimate of the mean value of $q(z)$, that is to omit the operator of the real part in the left-hand side, then exceptional sets appear.

Theorem 2. *Let $0 < \delta < 1$ and $k > 1$, u be a subharmonic function in $\mathbb{C}$, $u(0) = 0$. Then there exist a measurable set $E \subset [0, \infty)$ with $\overline{D}(E) < \delta$ and constant $C = C(\delta) > 0$ such that*

$$\int_0^{2\pi} |q(re^{i\theta})| d\theta \le C \frac{T(kr,u)}{r}, \quad r \notin E. \quad (5)$$

Let us take the logarithms of the both sides of estimate (5) and apply Jensen's inequality (Lemma 1.1 from [6, p. 116]). Then we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|q\left(re^{i\theta}\right)\right| d\theta \le \log^+ T(kr,u) + \log^+ C - \log^+ r, \quad r \notin E,$$

where C is an absolute constant, E is a set of small linear density.

The estimate of an exceptional set in (5) is sharp. It follows from Theorem 3 for $u = \ln|f|$.

Theorem 3. *For any $\rho \in (1,2)$ there exist an entire function f, constants $C > 0$, $\eta > 0$ and a set $F \subset [0,\infty)$ such that $\underline{D}(F) > \eta$, $T(r,f) \asymp r^\rho$, $r \to \infty$ and*

$$\int_0^{2\pi} \left|\operatorname{Im}\left(\frac{f'(re^{i\varphi})}{f(re^{i\varphi})} re^{i\varphi}\right)\right| d\varphi \geq CT(r,f), \qquad r \in F.$$

Prof. Skaskiv has drown our attention to the fact that it is possible to reduce exceptional set E in Theorem 2. In this case we obtain a weaker estimate than (5). This fact is illustrated in the following theorem.

Theorem 4. *Let $k > 1$, u be a subharmonic function in $\mathbb{C}$, $u(0) = 0$. Then for any function $\psi(x) \to +\infty$ $(x \to +\infty)$ there exists a measurable set $E \subset [0,\infty)$ with $\overline{D}(E) = 0$ such that*

$$\int_0^{2\pi} |q(re^{i\theta})|\, d\theta \leq \frac{T(kr,u)}{r}\psi(r), \qquad r \notin E.$$

2. Proofs of the theorems.

Proof of Theorem 1. Denote $z = re^{i\theta}$. We have

$$w(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi}) \frac{2Rze^{i\varphi}}{(Re^{i\varphi} - z)^2} d\varphi + \int_{C(0,R)} z\frac{d}{dz}\ln\frac{R(z-\zeta)}{R^2 - z\overline{\zeta}} d\mu(\zeta),$$

where $\mu = \mu_u$ there is the Riesz measure of the function u.

We use the scheme of the proof from [4]. We take the real parts of the previous inequality and estimate the terms on the right-hand side. Thus

$$|\operatorname{Re} w(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |u(Re^{i\varphi})| \frac{2rR}{|Re^{i\varphi} - z|^2} d\varphi + \left|\operatorname{Re}\int_{C(0,R)} z\frac{d}{dz}\left(\ln\frac{R(z-\zeta)}{R^2 - z\overline{\zeta}}\right) d\mu(\zeta)\right|.$$

Therefore, $|\operatorname{Re} w(z)| \leq \frac{1}{2\pi}\int\limits_0^{2\pi} |u(Re^{i\varphi})| \frac{2rR}{|Re^{i\varphi} - z|^2} d\varphi + \int\limits_{C(0,R)} \left|\frac{d\arg\frac{R(z-\zeta)}{R^2 - z\overline{\zeta}}}{d\theta}\right| d\mu(\zeta)$. We next integrate each term on the right-hand side of the previous inequality with respect to θ.

$$\frac{1}{2\pi}\int_0^{2\pi} |\operatorname{Re} w(z)|\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} |u(Re^{i\varphi})| \frac{2rR}{|Re^{i\varphi} - z|^2} d\varphi\right) d\theta +$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\left(\int_{C(0,R)} \left|\frac{d\arg\left(R(z-\zeta)/\left(R^2 - z\overline{\zeta}\right)\right)}{d\theta}\right| d\mu(\zeta)\right) d\theta = I_1 + I_2.$$

By a calculation based on the Poisson kernel, using Fubini's theorem and (2), it follows that

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|u\left(Re^{i\varphi}\right)\right|\frac{2rR}{\left|Re^{i\varphi}-z\right|^2}d\varphi\right)d\theta = \frac{2rR}{R^2-r^2}\frac{1}{2\pi}\int_0^{2\pi}\left|u\left(Re^{i\varphi}\right)\right|d\varphi =$$
$$= \frac{2rR}{R^2-r^2}\frac{1}{2\pi}\int_0^{2\pi}\left(u^+\left(Re^{i\varphi}\right)+u^-\left(Re^{i\varphi}\right)\right)d\varphi = \frac{2rR}{R^2-r^2}\left(T\left(R,u\right)+m\left(R,u\right)\right) \leq$$
$$\leq \frac{T\left(R,u\right)+m\left(R,u\right)}{\log(R/r)}.$$

Using Fubini's theorem, we obtain

$$I_2 = \int_{C(0,R)}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d\arg\left(R\left(z-\zeta\right)/\left(R^2-z\overline{\zeta}\right)\right)}{d\theta}\right|d\theta\right)d\mu\left(\zeta\right).$$

In [4, p. 256] the following estimate is proved

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d\arg\left(R\left(z-\zeta\right)/\left(R^2-z\overline{\zeta}\right)\right)}{d\theta}\right|d\theta < \frac{\log\left(R/|\zeta|\right)}{\log\left(R/r\right)}.$$

Therefore from the latter relationships we have

$$I_2 < \frac{1}{\log\left(R/r\right)}\int_{C(0,R)}\log\left(R/|\zeta|\right)d\mu\left(\zeta\right) = \frac{N\left(R,u\right)}{\log\left(R/r\right)}.$$

Finally, by (3)

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\operatorname{Re} w\left(z\right)\right|d\theta \leq \frac{\left(T\left(R,u\right)+m\left(R,u\right)\right)}{\log\left(R/r\right)} + \frac{N\left(R,u\right)}{\log\left(R/r\right)} = \frac{2T\left(R,u\right)-u(0)}{\log\left(R/r\right)}.$$

□

We need the following elementary lemma.

Lemma 1 ([6], Lemma 7.1)**.** *Let $R' > R > 1$. Then*

$$n\left(R,u\right) \leq \frac{N\left(R',u\right)}{\ln(R'/R)} \leq \frac{R'}{R'-R}N\left(R',u\right).$$

Proof of Theorem 2. From (1) we have

$$|q(z)| \leq \frac{1}{2\pi}\int_0^{2\pi}\left|u\left(Re^{i\varphi}\right)\right|\left|\frac{2Re^{i\varphi}}{\left(Re^{i\varphi}-z\right)^2}\right|d\varphi + \int_{C(0,R)}\left|\frac{1}{z-\zeta}+\frac{\overline{\zeta}}{R^2-z\overline{\zeta}}\right|d\mu\left(\zeta\right).$$

We integrate the previous inequality with respect to θ and obtain

$$\begin{aligned}\int_0^{2\pi}\left|q\left(z\right)\right|d\theta \leq \int_0^{2\pi}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|u\left(Re^{i\varphi}\right)\right|\frac{2R}{\left|Re^{i\varphi}-re^{i\theta}\right|^2}d\varphi\right\}d\theta+\\ +\int_0^{2\pi}\left\{\int_{C(0,R)}\left|\frac{1}{re^{i\theta}-\zeta}+\frac{\overline{\zeta}}{R^2-re^{i\theta}\overline{\zeta}}\right|d\mu\left(\zeta\right)\right\}d\theta.\end{aligned} \tag{6}$$

In order to prove the theorem we need the following lemma.

Lemma 2. *Let u be a subharmonic function in $\mathbb{C}$, k be a fixed number $\left(1<k<\frac{3}{2}\right)$, $r_2=\left(1+\frac{k-1}{4}\right)r_1$. Then if $r'<r_1<r_2<R<\infty$, we have*

$$\int\limits_{r_1}^{r_2}\int\limits_{0}^{2\pi}\left|q\left(re^{i\theta}\right)\right|rdrd\theta\le K_0 r_1 T\left(kr_1,u\right),$$

where K_0 is an absolute constant.

For subharmonic functions the proof of Lemma 2 similar to the proof of Lemma 2 from [7]. Analogous estimate for the logarithmic derivative of a meromorphic function is proved in [8, Theorem 7].

Proof of Lemma 2. For a measurable set $G\subset\mathbb{R}_+$ we set $\mu(G)=\int_G rdr$. Using inequality (6), we have $(z=re^{i\theta}, R=\left(1+\frac{k-1}{2}\right)r_1)$

$$\int\limits_{r_1}^{r_2}\int\limits_{0}^{2\pi}|q(z)|rdrd\theta\le\int\limits_{r_1}^{r_2}r\left\{\int\limits_{0}^{2\pi}\left[\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|u\left(Re^{i\varphi}\right)\right|\frac{2R}{\left|Re^{i\varphi}-re^{i\theta}\right|^2}d\varphi\right]d\theta\right\}dr+$$
$$+\int\limits_{r_1}^{r_2}r\left\{\int\limits_{0}^{2\pi}\left[\int\limits_{C(0,R)}\left|\frac{1}{re^{i\theta}-\zeta}+\frac{\overline{\zeta}}{R^2-re^{i\theta}\overline{\zeta}}\right|d\mu(\zeta)\right]d\theta\right\}dr\equiv I_1+I_2.$$

By a calculation based on the Poisson kernel, using Fubini's theorem, (2) and (3), it follows that

$$I_1=\int\limits_{r_1}^{r_2}r\left\{\int\limits_{0}^{2\pi}\left[\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|u\left(Re^{i\varphi}\right)\right|\frac{2R}{\left|Re^{i\varphi}-re^{i\theta}\right|^2}d\theta\right]d\varphi\right\}dr=$$
$$=\int\limits_{r_1}^{r_2}r\left\{\int\limits_{0}^{2\pi}\left|u\left(Re^{i\varphi}\right)\right|\frac{2R}{R^2-r^2}d\varphi\right\}dr=$$
$$=\int\limits_{r_1}^{r_2}\frac{4\pi Rr}{R^2-r^2}\left\{T(R,u)+m(R,u)\right\}dr\le 8R\pi T(R,u)\int\limits_{r_1}^{r_2}\frac{r}{R^2-r^2}dr=$$
$$=4R\pi T(R,u)\ln\frac{R^2-r_1^2}{R^2-r_2^2}=$$
$$=4\left(1+\frac{k-1}{2}\right)r_1\pi T(R,u)\ln\frac{4(k+3)}{3k+5}\le Kr_1T(R,u),\quad r\to\infty,$$

where K is an absolute constant. So

$$I_1\le Kr_1T(R,u),\quad r\to\infty. \tag{7}$$

From [7, Lemma 2, estimate (14)] in $|\zeta|\le R$ we have

$$\int\limits_{r_1}^{r_2}\int\limits_{0}^{2\pi}\left|\frac{1}{re^{i\theta}-\zeta}-\frac{1}{re^{i\theta}-R^2/\overline{\zeta}}\right|rdrd\theta\le K\left(r_2-r_1\right).$$

Let $R_1=\left(1+\frac{3}{4}(k-1)\right)r_1$. Then from the previous inequality, Fubini's theorem and Lemma 1 it follows that

$$\begin{aligned}I_2&\le K\left(r_2-r_1\right)n(R,u)\le K\left(r_2-r_1\right)\frac{R_1}{R_1-R}N\left(R_1,u\right)\le\\&\le Kr_1\left[T\left(R_1,u\right)+O(1)\right],\quad r\to\infty.\end{aligned}\tag{8}$$

Summarizing estimates (7) and (8) we obtain the estimate

$$\int_{r_1}^{r_2}\int_0^{2\pi} \left|q\left(re^{i\theta}\right)\right| rdrd\theta \leq Kr_1\left[T\left(R_1,u\right)+O\left(1\right)\right] \leq K_0 r_1 T\left(kr_1,u\right), \; r\to\infty.$$

$\square$

Denote $k=\sqrt{k^*}$, $b_n=\alpha^n$, where $\alpha=\sqrt{k^*}>1$, $n\in\mathbb{N}$, then from Lemma 2 it follows that

$$\int_{r_1}^{r_2}\int_0^{2\pi} \left|q\left(re^{i\theta}\right)\right| rdrd\theta \leq \widetilde{K} b_n T\left(kb_{n+1},u\right), \tag{9}$$

where $\widetilde{K}$ is an absolute constant. Let $0<\delta_1<\frac{\delta(\alpha-1)}{\alpha^2}$,

$$G_n=\left\{r\in[b_n,b_{n+1}]: \int_0^{2\pi}\left|q\left(re^{i\theta}\right)\right|d\theta \geq \frac{\widetilde{K}T\left(kb_{n+1},u\right)}{\delta_1 b_n}\right\}.$$

We have

$$\mu\left(G_n\right)\geq b_n m\left(G_n\right), \tag{10}$$

where $m\left(G_n\right)$ is the Lebesgue measure of the set G_n. By (9) and the Chebyshev-Markov inequality it follows that

$$\mu\left(G_n\right)\leq \frac{\delta_1 b_n}{\widetilde{K}T\left(kb_{n+1},u\right)}\int_{b_n}^{b_{n+1}}\int_0^{2\pi}\left|q\left(re^{i\theta}\right)\right| rdrd\theta \leq \delta_1 b_n^2.$$

By (10) it follows that $m\left(G_n\right)\leq \mu\left(G_n\right)/b_n\leq \delta_1 b_n$.

Let $E=\bigcup\limits_{n=1}^{\infty} G_n$, then for $r\in[b_N,b_{N+1}]$ we obtain

$$\frac{1}{r}m\left(E\cap[0,r]\right)\leq \frac{1}{b_N}\sum_{n=1}^{N+1} m\left(G_n\right)\leq \frac{1}{b_N}\delta_1\sum_{n=1}^{N+1} b_n<\delta,$$

which shows that $\overline{D}\left(E\right)<\delta$.

If $r\in[b_n,b_{n+1}]\setminus E$, then, by the definition of G_n,

$$\int_0^{2\pi}\left|q\left(re^{i\theta}\right)\right|d\theta\leq \frac{\widetilde{K}T\left(kb_{n+1},u\right)}{\delta_1 b_n}=\frac{\widetilde{K}T\left(k\alpha b_n,u\right)}{\delta_1 b_n}\leq C\frac{T\left(k^*r,u\right)}{r},$$

where $C=\frac{\widetilde{K}}{\delta_1}$, $k^*=k\alpha$. $\square$

Proof of Theorem 3. Let $1<\beta<2$. Let $q_n=\left[2^{\beta n}\right]$, $a_n=2^n$ for $n\in\mathbb{N}$. As usual, the Weierstraß primary factor of genus 1 is defined as $E\left(w,1\right)=\left(1-w\right)\exp\left(w\right)$, $w\in\mathbb{C}$. Since

$$\sum_{n=1}^{\infty} q_n\left|\frac{1}{a_n}\right|^2\leq\sum_{n=1}^{\infty}\left(2^{\beta-2}\right)^n<\infty,$$

then the Weierstraß product $f\left(z\right)=\prod\limits_{n=1}^{\infty}E\left(\frac{z}{a_n},1\right)^{q_n}$, $z\in\mathbb{C}$ represents an entire function. We shall prove that the function f has the required properties.

Denote $z=re^{i\varphi}$, and fix $\nu\in\mathbb{N}$ such that $a_\nu\leq r\leq a_{\nu+1}$. We calculate the counting function of the zeros of the function f

$$n(r,f)=\sum_{|a_n|\le r}q_n=\sum_{n=1}^{\nu+1}q_n\le\sum_{n=1}^{\nu+1}2^{\beta n}=2^{\beta}\frac{2^{\beta(\nu+1)}-1}{2^{\beta}-1}\le \\ \le C_1 2^{\beta\nu}\le C_1 r^{\beta},\quad 2^{\nu}\le r\le 2^{\nu+1}, \tag{11}$$

where $C_1=C_1(\beta)=2^{2\beta}/(2^{\beta}-1)$ is a constant.

Denote $M(r,f)=\max\limits_{|z|=r}|f(z)|$, $\mu(r,f)=\min\limits_{|z|=r}|f(z)|$. Then by [9, p. 53] and (11) we have

$$\log M(r,f)\le 16\left\{r\int_0^r\frac{n(t,f)}{t^2}dt+r^2\int_r^{\infty}\frac{n(t,f)}{t^3}dt\right\}\le 16C_1\left\{r\int_0^r t^{\beta-2}dt+r^2\int_r^{\infty}t^{\beta-3}dt\right\}= \\ =16C_1\left\{\frac{r^{\beta}}{\beta-1}-\frac{r^{\beta}}{\beta-2}\right\}=16C_1\frac{r^{\beta}}{(\beta-1)(2-\beta)}=C_2r^{\beta},\quad 2^{\nu}\le r\le 2^{\nu+1}, \tag{12}$$

where $C_2=C_2(\beta)=\frac{16C_1}{(\beta-1)(2-\beta)}$ is a constant. Let us estimate $n(r,f)$ from below

$$n(r,f)=\sum_{n=1}^{\nu+1}q_n\ge\sum_{n=1}^{\nu+1}\left(2^{\beta n}-1\right)\ge 2^{\beta(\nu+1)}-\nu-1\ge r^{\beta}-\log_2 r-1. \tag{13}$$

Denote $\widetilde{f}(z)=f(z)/(1-z/a_{\nu})^{q_{\nu}}$. Then

$$n\left(r,\widetilde{f}\right)\le n(r,f). \tag{14}$$

Then estimates (12) and (14) yield

$$\log M\left(r,\widetilde{f}\right)\le C_2r^{\beta},\quad 2^{\nu}\le r\le 2^{\nu+1}. \tag{15}$$

Next, let $z\in[a_{\nu},a_{\nu}+\eta 2^{\nu}]$, where the constant $\eta\in\left(0,\frac12\right)$ is chosen such that $\log\eta<-2C_2$. Then

$$\log\left|1-\frac{z}{a_{\nu}}\right|=\log\left|\frac{a_{\nu}-z}{a_{\nu}}\right|\le\log\left|\frac{-\eta 2^{\nu}}{a_{\nu}}\right|=\log\eta<-2C_2. \tag{16}$$

By (15) and (16), it now follows that

$$\log|f(z)|\le q_{\nu}\log\left|1-\frac{z}{a_{\nu}}\right|+\log M\left(r,\widetilde{f}\right)\le \\ \le r^{\beta}\log\eta+C_2r^{\beta}\le -2C_2r^{\beta}+C_2r^{\beta}=-C_2r^{\beta}. \tag{17}$$

Since $M(r,\widetilde{f})\ge C_0>0$ as $r\ge a_1$, the estimate (17) yields the existence of a constant $C_3>0$ such that

$$\log M(r,f)-\log\mu(r,f)\ge C_3r^{\beta},\quad r\in F, \tag{18}$$

where $F=\bigcup\limits_{\nu=1}^{\infty}[a_{\nu},a_{\nu}+\eta 2^{\nu}]$. If $r\in[a_N,a_{N+1}]$, $N\in\mathbb{N}$, then

$$\frac1r m(F\cap[0,r])\ge 2^{-(N+1)}\sum_{\nu=1}^{N}\eta 2^{\nu}=2^{-(N+1)}\eta\left(2^{N+1}-2\right)>\eta,$$

which shows that
$$\underline{D}(F) > \eta. \tag{19}$$
Since f is entire, $2^{\nu} \le r \le 2^{\nu+1}$, then by [6, Theorem 7.1, p. 54] and (12) we have
$$T(r,f) \le \log M(r,f) \le C_2 r^{\beta}. \tag{20}$$
Let us estimate $N(r,f)$ from below. By (13) we obtain
$$N(r,f) = \int_0^r \frac{n(t,f)}{t}dt \ge \int_1^r \frac{t^{\beta} - \log_2 t - 1}{t}dt = r^{\beta}\left(\frac{1}{\beta} - \frac{1}{r^{\beta}\beta} - \frac{\log_2^2 r}{r^{\beta} 2\ln 2} - \frac{\ln r}{r^{\beta}}\right) \ge C_4 r^{\beta}$$
as $r \to \infty$, where $C_4 \le \frac{1}{\beta}$ as $r \to \infty$. Since $T(r,f) \ge N(r,f) \ge C_4 r^{\beta}$, then by (20) $C_4 r^{\beta} \le T(r,f) \le C_2 r^{\beta}$. Thus, $T(r,f) \asymp r^{\beta}$, $1 < \beta < 2$, $r \to +\infty$.

Let now $a_{\nu} \le r \le a_{\nu+1}$, so the circle $|\zeta| = r$ contains neither zeros nor poles of the function f. Let z_1 and z_2 be points on the circle $|\zeta| = r$ such that $M(r,f) = |f(z_1)|$ and $\mu(r,f) = |f(z_2)|$. Then
$$\int_0^{2\pi} \left|\operatorname{Im}\left(\frac{f'(re^{i\varphi})}{f(re^{i\varphi})} re^{i\varphi}\right)\right| d\varphi \ge \left|\operatorname{Re} \int_{\overset{\smile}{z_1 z_2}} \frac{f'(z)}{f(z)}dz\right| = \log M(r,f) - \log\mu(r,f). \tag{21}$$
Finally, the estimates (18), (20) and (21) yield
$$\int_0^{2\pi} \left|\operatorname{Re}\left(\frac{f'(re^{i\varphi})}{f(re^{i\varphi})} ire^{i\varphi}\right)\right| d\varphi \ge C_3 r^{\beta} \ge \frac{C_3 T(r,f)}{C_2}, \quad r \in F,$$
where $\underline{D}(F) \ge \eta$ by (19). This shows that the function f has the properties required by Theorem 3. $\square$

Proof of Theorem 4. The proof of the theorem is similar to that of Theorem 2. A difference appears at the moment of the definition of the set G_n. Let $\varphi(r) = \inf\{\psi(\rho) : \rho \ge r\}$. Since $\psi(r) \to +\infty$, we have $\varphi(r) \nearrow +\infty$ as $r \to +\infty$. Let $r_0 > 0$ be such that $\varphi(r) \ge 1$ as $r \ge r_0$.

Denote $\widetilde{G}_n = \left\{ r \in [b_n, b_{n+1}] : \int_0^{2\pi} \left|q\left(re^{i\theta}\right)\right| d\theta \ge \varphi(b_n) T(kb_{n+1}, u)/b_{n+1} \right\}$.

We have $\mu(\widetilde{G_n}) \ge b_n m(\widetilde{G_n})$. As in the proof of Theorem 2 we have
$$\mu(\widetilde{G_n}) \le \frac{b_n}{\varphi(b_n)\widetilde{K} T(kb_{n+1}, u)} \int_{b_n}^{b_{n+1}} \int_0^{2\pi} \left|q\left(re^{i\theta}\right)\right| r dr d\theta \le \frac{b_n^2}{\varphi(b_n)}.$$
From the previous inequalities it follows that $m(\widetilde{G_n}) \le \mu(\widetilde{G_n})/b_n \le b_n/\varphi(b_n)$.
Let $E = \bigcup\limits_{n=1}^{\infty} \widetilde{G_n}$, then for $r \in [b_N, b_{N+1}]$ we obtain
$$\frac{1}{r} m(E \cap [0,r]) \le \frac{1}{b_N} \sum_{n=1}^{N+1} m(\widetilde{G_n}) \le \frac{1}{b_N} \sum_{n=1}^{N+1} b_n/\varphi(b_n).$$
Let $\varepsilon > 0$. Then there exists $n_{\varepsilon} \in \mathbb{N}$ such that $(\forall\, n \ge n_{\varepsilon}) : \dfrac{1}{\varphi(b_n)} < \dfrac{\varepsilon(\alpha - 1)}{2\alpha^2}$, then
$$\frac{1}{b_N} \sum_{n=n_{\varepsilon}+1}^{N+1} b_n/\varphi(b_n) < \frac{\varepsilon(\alpha-1)}{2\alpha^2 b_N} \sum_{n=1}^{N+1} b_n < \frac{\varepsilon}{2}.$$

Since $b_N \to \infty$ as $N \to \infty$, then there exists $m_0(n_\varepsilon)$ such that for $N > m_0$ we have
$$\left|\frac{1}{b_N}\sum_{n=1}^{n_\varepsilon} b_n/\varphi(b_n)\right| < \frac{\varepsilon}{2}.$$
So
$$\frac{1}{b_N}\sum_{n=1}^{N+1} b_n/\varphi(b_n) = \frac{1}{b_N}\left(\sum_{n=1}^{n_\varepsilon} b_n/\varphi(b_n) + \sum_{n=n_\varepsilon+1}^{N+1} b_n/\varphi(b_n)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$
which shows that $\overline{D}(E) = 0$.

If $r \in [b_n, b_{n+1}] \setminus E$, then, by the definition of $\widetilde{G_n}$,
$$\int_0^{2\pi} \left|q\left(re^{i\theta}\right)\right| d\theta \le \frac{T(kb_{n+1}, u)}{b_{n+1}}\varphi(b_n) \le \frac{T(k\alpha b_n, u)}{r}\varphi(b_n) \le \frac{T(k^* r, u)}{r}\varphi(r),$$
where $C = \frac{\widetilde{K}}{\delta_1}$, $k^* = k\alpha$. □

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