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ABSORBING SPACES IN HARTMAN-MYCIELSKI'S CONSTRUCTION

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Absorbing spaces D_β related to C-compacta were constructed in [5]. We obtain these spaces as Hartman-Mycielski constructions of some compacta.

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Поглощающие пространства D_β для некоторых классов C-компактов построеные в [5]. Мы строем такие пространства в конструкции Хартмана-Мицельского над некоторыми компактами.

0. All spaces under the discussion are assumed metrizable and separable, all ordinals are assumed countable. The set of all countable ordinals will be denoted by ω_1.

Let $\mathcal{K}$ be a class of spaces. The class σ-$\mathcal{K}$ consists of all spaces which can be represented as the countable unions of their closed subspaces from the class $\mathcal{K}$. The notion of $\mathcal{K}$-absorbing spaces was introduced by Bestvina and Mogilski in [2]. They constructed absorbing spaces for Borel classes and proved the characterization theorem for absorbing spaces.

Dobrowolski and Mogilski asked to find more absorbing spaces ([3]). They proved that for uncountable many of ordinals β there exist pre-Hilbert spaces E_β which are absorbing for the classes of compacta X with trind $< \beta$ ([4]). All these spaces are countable-dimensional. In [5] it was shown that for each ordinal $\alpha \in \omega_1$ there exists an ordinal $\beta \geq \alpha$ and a pre-Hilbert space D_β which is $\mathcal{D}(\beta)$-absorbing where $\mathcal{D}(\beta)$ is the class of compacta with $\dim_C$ less than β ($\dim_C$ is a transfinite extension of covering dimension ([1]) which classifies C-compacta). All spaces D_β beginning from some $\beta_0 \in \omega_1$ are not countable-dimensional, hence different from E_β.

We obtain in this paper the absorbing spaces D_β as the Hartman-Mycielski construction of some compacta.

The paper is organized as follows: in Section 1 we give some necessary definitions and results about $\dim_C$, absorbing spaces and recall the construction of D_β, in Section 2 we obtain the main results.

1. Recall briefly some necessary definitions of the theory of absorbing spaces (see [2] or [6] for details).

Two maps $f, g\colon X \to Y$ are said to be *$\mathcal{U}$-close*, where $\mathcal{U}$ is a cover of Y, if for each $x \in X$ the set $\{f(x), g(x)\}$ is contained in an element of $\mathcal{U}$.

A closed subset X of Y is called *Z-set* if for every open cover $\mathcal{U}$ of Y there exists a map $f\colon Y \to Y$ which is $\mathcal{U}$-close to Id_Y and $f(Y) \cap X = \varnothing$. If additionally $f(Y)$ is closed in Y

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we say that X is *strong* Z-set in Y. Let us remark that a closed subset X of Y is strong Z-set in Y if there exists a homotopy $H\colon Y\times[0,1]\to Y$ such that $H(Y\times(0,1])\cap X=\varnothing$ and $H(y,0)=y$ for each $y\in Y$. An embedding into Y is called *Z-embedding* if its image is a Z-set in Y.

Let $\mathcal{K}$ be a class of spaces. A space X is *strongly $\mathcal{K}$-universal* if for every map $f\colon C\to X$ from a space $C\in\mathcal{K}$, for every closed subset $D\subset C$ such that $f|D\colon D\to X$ is a Z-embedding and for every $\mathcal{U}\in\operatorname{cov}(X)$, there exists a Z-embedding $h\colon C\to X$ such that $h|D=f|D$ and h is $\mathcal{U}$-close to f.

A space $X\in AR$ is called *$\mathcal{K}$-absorbing* space if $X=\bigcup_{i=1}^\infty X_i$ where each X_i is a strong Z-set in X, $X_i\in\mathcal{K}$, and X is strongly $\mathcal{K}$-universal. We say that a $\mathcal{K}$-absorbing space X is *representable* in l_2 if X is homeomorphic to a subset X_0 in l_2 such that $l_2\setminus X_0$ is *locally homotopy negligible* in l_2 (i.e. for every open set $U\subset l_2$ the inclusion $U\setminus X_0$ induces an isomorphism of thr homotopy groups). In [2] it is proved that if there exists a $\mathcal{K}$-absorbing space X representable in l_2 then every $\mathcal{K}$-absorbing space is homeomorphic to X.

Let L be an arbitrary set. By $\operatorname{Fin}L$ we shall denote the collection of all finite, non-empty subsets of L. Let M be a subset of $\operatorname{Fin}L$. For $\sigma\in\{\varnothing\}\cup\operatorname{Fin}L$ we put

$$M^\sigma=\{\tau\in\operatorname{Fin}L\colon\ \sigma\cup\tau\in M\text{ and }\sigma\cap\tau=\varnothing\}.$$

Let M^a abbreviate $M^{\{a\}}$.

Definition 1 (Definition 1 [1])**.** Define the *ordinal number* $\operatorname{Ord}M$ inductively as follows:
$\operatorname{Ord}M=0$ iff $M=\varnothing$,
$\operatorname{Ord}M\le\alpha$ iff for every $a\in L$, $\operatorname{Ord}M^a<\alpha$,
$\operatorname{Ord}M=\alpha$ iff $\operatorname{Ord}M\le\alpha$ and $\operatorname{Ord}M<\alpha$ is not true, and
$\operatorname{Ord}M=\infty$ iff $\operatorname{Ord}M>\alpha$ for every ordinal number α.

We say that a family $\mathcal{V}$ *refines* a family $\mathcal{U}$ if for each element $V\in\mathcal{V}$ there exists $U\in\mathcal{U}$ with $V\subset U$. The family $\mathcal{V}$ of subsets of X is called *disjoint* if every two elements of $\mathcal{V}$ are disjoint and is *open* if each element of $\mathcal{V}$ is open.

A finite sequence $\{\alpha_i\}_{i=1}^m$ of finite open covers of a space X is called *inessential* if there are open disjoint families $\beta_i, i=1,\dots,n$ such that β_i refines α_i and $\bigcup_{i=1}^n\beta_i$ covers X. Otherwise it is called *essential.*

We have the following characterization of the classical covering dimension: $\dim X\le n$ if and only if every sequence $\{\alpha_i\}_{i=1}^{n+1}$ of finite covers of X is inessential ([11]).

Let X be a space. Denote by $C(X)$ the set of all finite covers of X and

$$M_{C(X)}=\{\sigma\in\operatorname{Fin}C(X)\colon\ \sigma\text{ is essential in }X\}.$$

Definition 2 (Definition 2 [1])**.** For a space X we set

$$\dim_C X=\operatorname{Ord}M_{C(X)}.$$

Let us denote by $\mathcal{D}(\beta)$ the class of compacta with $\dim_C$ less than β. We say that a space Y is *$\mathcal{D}(\beta)$-universal* if Y contains topologically all compacta from $\mathcal{D}(\beta)$. Let us recall the constructions of $\mathcal{D}(\beta)$-absorbing spaces D_β from [5].

It was proved in [5] that for each ordinal number $\alpha\in\omega_1$ there exists a limit ordinal number $\beta\in\omega_1,\beta\ge\alpha$ and compactum K_β such that $\dim_C K_\beta=\beta,\ K_\beta\in\sigma$-$\mathcal{D}(\beta)$, the class $\mathcal{D}(\beta)$ is closed with respect to finite products and K_β is $\mathcal{D}(\beta)$-universal. Let us remark that K_β is a one-point compactification of the discrete sum of compacta $\bigsqcup_{n=1}^\infty K_n(\beta)$ such that for each compactum L with $\dim_C L<\beta$ there exist $n\in\mathbb{N}$ and an embedding $i\colon L\to K_n(\beta)$. The compactification point will be denoted by a_β. There exists an embedding $\nu\colon K\to l_2$ such that $\nu(K)$ is a linearly independent set ([7]). Put $D_\beta = \mathrm{span}(\nu(K_\beta))$. It was proved in [5] that D_β is a $\mathcal{D}(\beta)$-absorbing space representable in l_2. So, we have the following characterization of the spaces D_β: an AR-space Y is homeomorphic to D_β if and only if Y is strongly $\mathcal{D}(\beta)$-universal and Y can be represented as the countable union of compacta from $\mathcal{D}(\beta)$ which are strong Z-sets in Y ([5]).

2. Recall the definition of the Hartman-Mycielski construction ([8]). Let X be a space and d an admissible metric on X bounded by 1. By $HM(X)$ we shall denote the space of all maps from $[0,1)$ to the space X such that $f|[t_i, t_{i+1}) \equiv const$, for some $0 = t_0 \le \cdots \le t_n = 1$, with respect to the following metric

$$d_{HM}(f,g) = \int_0^1 d(f(t), g(t))dt, \qquad f, g \in HM(X).$$

Let $x \in X$. By $\delta(x)$ we denote the element of HMX defined as follows $\delta(x)(t) = x$ for each $t \in [0,1)$. We need also the following map $e\colon HMX \times HMX \times I \to HMX$ defined by the condition that $e(\alpha_1, \alpha_2, t)(l)$ is equal to $\alpha_1(l)$ if $l < t$ and $\alpha_2(l)$ in the opposite case for $\alpha_1, \alpha_2 \in HMX$, $t \in I$ and $l \in [0,1)$. It is known that e is continuous.

For every compactum Z consider

$$HM_n(Z) = \Big\{ f \in HM(Z)\colon \quad \text{there exist } 0 = t_1 < \cdots < t_{n+1} = 1 \quad \text{with } f\,\Big|_{[t_i, t_{i+1})} \equiv z_i \in Z, i = 1, \ldots, n \Big\}.$$

It is known that $HM_n Z$ is compact.

The Hilbert cube is denoted by Q, and the following subspace of Q:

$$\big\{ (a_n)_{n=1}^\infty \in Q \mid a_k = 0 \ \text{ for all but finitely many } k \big\}$$

is denoted by σ. Telejko has shown in [9] that for any non-degenerated separable metrizable σ-compact strongly countable-dimensional space X the space HMX is homeomorphic to σ. Let us remark that σ is an absorbing space for the class of finite-dimensional compacta. The next natural step is to find in Hartman-Mycielski construction $\mathcal{D}(\beta)$-absorbing spaces.

The main result of this paper is

Theorem 1. *$HM(K_\beta)$ is homeomorphic to D_β.*

The proof of the next lemma follows from the definition of the metric d_{HM} on the space HMX (we suppose that the metric d on X is bounded by 1).

Lemma 1. $d_{HM}(e(\nu, \mu, a), \mu) \le a$ *for each* ν, $\mu \in HMX$ *and* $a \in [0,1]$.

Lemma 2. *The space HMK_β can be represented as $HMK_\beta = \bigcup_{i=1}^\infty Y_i$ where each Y_i is a strong Z-set and $Y_i \in \sigma$-$\mathcal{D}(\beta)$.*

Proof. We can represent $K_\beta = \cup_{i=1}^\infty S_i$ where S_i are compact proper subsets of K_β, $S_i \subset S_{i+1}$ and $S_i \in \mathcal{D}(\beta)$. Then $HMK_\beta = \cup_{i=1}^\infty HMS_i$. Let us show that the closed set HMS_i is a strong Z-set for each $i \in \mathbb{N}$.

Choose any $x \in K_\beta \setminus S_i$ and consider a homotopy $F\colon HMK_\beta \times I \to HMK_\beta$ defined by the formula $F(\mu, t) = e(\delta(x), \mu, t)$. Then we have $F(HMK_\beta \times (0,1]) \cap HMS_i = \varnothing$ and $F(\mu, 0) = \mu$ for each $\mu \in HMK_\beta$. Hence HMS_i is a strong Z-set.

Let us show that $HMS_i \in \sigma$-$\mathcal{D}(\beta)$. It is enough to prove that $HM_n(S_i) \in \sigma$-$\mathcal{D}(\beta)$ for each i, $n \in \mathbb{N}$. We use induction by n.

If $n = 1$, then $HM_1(S_i)$ is homeomorphic to S_i and the statement holds. Suppose $HM_{k-1}(S_i) \in \sigma$-$\mathcal{D}(\beta)$ and consider the case $n = k$. Using the Basmanov map [10], we see that $HM_k(S_i) \setminus HM_{k-1}(S_i)$ is locally homeomorphic to an open subset of the product $S_i^k \times HM_k(k)$

(by k we denote also k-point space). Thus, we can present $HM_k(S_i)\backslash HM_{k-1}(S_i)$ as countable union of compact subsets each of them is homeomorphic to a compact subset of $S_i^k\times HM_k(k)$. Since $HM_k(k)$ is finite-dimensional and $\dim_C(S_i)<\beta$, we see that $\dim_C(S_i^k\times HM_k(k))<\beta$. Hence $HM_k(S_i)\in\sigma\text{-}\mathcal{D}(\beta)$ and the lemma is proved. □

Lemma 3. *HMK_β is strongly $\mathcal{D}(\beta)$-universal.*

Proof. Consider any compactum $L\in\mathcal{D}(\beta)$ and a map $f\colon L\to HMK_\beta$ such that $f|D$ is a Z-embedding for some closed subset $D\subset L$. Since $\dim_C L<\beta$, there exists an embedding $i\colon L\to K_n(\beta)$.

Since $f(D)$ is a Z-set in HMK_β, we can suppose that $f(L\setminus D)\cap f(D)=\varnothing$. Consider any covering $\mathcal{U}$ of HMK_β. Let $\varepsilon>0$ be the Lebesque number for the covering $\mathcal{U}$ and the compactum $f(L)$. We can suppose that $\varepsilon<1$. Define a function $h\colon L\to HMK_\beta$ by the formula $h(l)=e(\delta(i(l),f(l),\frac{1}{2}\min\{d_{HM}(f(l),f(D)),\varepsilon\})$. It is easy to see that h is an embedding such that $h|D=f|D$. We have that h is $\mathcal{U}$-close to f by Lemma 1.

Let us show that $h(L)$ is a Z-set in HMK_β. Since $h|D=f|D$ is a Z-embedding, it is enough to show that $h(S)$ is a Z-set in HMK_β for each compactum $S\subset L\setminus D$.

Define a homotopy $G\colon HMK_\beta\times I\to HMK_\beta$ by the formula $G(\mu,t)=e(\delta(a_\beta),\mu,t)$. Then we have $G(HMK_\beta\times(0,1])\cap h(S)=\varnothing$ and $G(\mu,0)=\mu$ for each $\mu\in HMK_\beta$. Hence $h(S)$ is a Z-set and the lemma is proved. □

Proof of Theorem 1. HMX is an absolute retract for any space X (see [6,§4.2, Ex.14]). Then Theorem 1 follows from Lemmas 2 and 3. □

Let us remark that using analogous arguments we can obtain spaces E_β as Hartman-Mycielski constructions.

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