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A HOMOGENEOUS FIRST-COUNTABLE ZERO-DIMENSIONAL COMPACTUM FAILING TO BE A LEFT-TOPOLOGICAL GROUP

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Answering a question from [2] we observe that the homogeneous first-countable zero-dimensional compactum constructed by E. van Douwen [4] fails to be a left-topological group.

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Отвечая на вопрос из [2], мы показываем, что однородный нульмерный компакт с первой аксиомой счетности, построенный Е. ван Дауэном [4] не гомеоморфен лево-топологической группе.

Answering a question from [2] we shall construct a first-countable zero-dimensional homogeneous compactum homeomorphic to no left-topological group. By a *left-topological group* we understand a group G endowed with a topology τ making all the left shifts $l_g\colon x\mapsto g*x$ of G continuous. If, moreover, the group operation $*\colon G\times G\to G$ is separately continuous (resp. jointly continuous), then (G,τ) is called a *semi-topological group* (resp. *paratopological group*).

The question on the existence of compatible group structures on topological spaces is classical in topological algebra. The necessary condition for this is the homogeneity of a given space X (we call a topological space X *homogeneous* if for any points $x,y\in X$ there is a homeomorphism h of X such that $h(X)=y$). However, not all homogeneous spaces admit a structure of a (left) topological group. An easy example is the Hilbert cube $Q=[-1,1]^\omega$ (which has the fixed point property) or the two-dimensional sphere, see [3, 10.2] or [9]. A more challenging exercise is to find a *zero-dimensional* homogeneous space which is not a topological group. Many different examples of such spaces are described in [3, §10]. One of them is the homogeneous countable k_ω-space constructed by Arkhangelski and Franklin in [1] and the other is the classical Aleksandrov "two-arrows" space. In fact, the "two-arrows" space admits no compatible structure of a semi-topological group (since each compact semi-topological group is a topological group [5]). In contrast, the Arkhangelski-Franklin space does admit such a structure (in spite of the fact that it admits no compatible structure of a paratopological group). This follows from a remarkable recent result of E.Zelenyuk [9] stating that for any regular countable topological space X and any countable group G there is a left-topological group homeomorphic to X and algebraically isomorphic to G. To some our surprise, we invented that the Aleksandrow "two-arrows" space also has a compatible

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structure of a left-topological group (algebraically isomorphic to the semidirect product $\mathbb{T} \ltimes \mathbb{Z}_2$ of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the group $\mathbb{Z}_2 = \{0,1\}$). Moreover, according to [8], each homogeneous space X is homeomorphic to the quotient space of some left-topological group G (isomorphic to the homeomorphism group of X) by a closed discrete subgroup of G. In this situation it was natural to ask if any homogeneous first-countable zero-dimensional compactum is a left-topological group.

In this note we answer this question in negative. Namely, using the method proposed by E. van Douwen [4] and developed by D.B. Motorov [7] we shall construct a homogeneous first-countable zero-dimensional compactum admitting no compatible structure of a left-topological group. To produce such a compactum we glue some “jumps” in the Aleksandrov “two-arrows” space.

More precisely, given a dense subset $B \subset (0,1)$ let $A = [0,1] \setminus B$ and consider the space

$$\mathrm{vD}_B = (A \times \{0\}) \cup (B \times \{-1,1\})$$

endowed with the interval topology with respect to the lexicographic order inherited from the product $[0,1] \times \{-1,0,1\}$. Observe that the space vD_B projects onto the closed interval $I = [0,1]$ so that the preimage $\mathrm{pr}^{-1}(t)$ of each point $t \in I$ under the projection $\mathrm{pr}\colon \mathrm{vD}_B \to I$ contains at most 2 points. Moreover, vD_B has a unique Borel regular probability measure μ that projects onto the standard Lebesque measure of I.

Varying the set B it is possible to produce examples of first-countable homogeneous spaces possessing various pathological properties, see [4], [7]. On this way, D.B.Motorov [7] has constructed an infinite homogeneous compactum X which is not homeomorphic to $X \times \{0,1\}$ while E. van Douwen [4] produced a homogeneous compactum which is not h-homogeneous.

We remind that a topological space X is called

- *h-homogeneous* if each non-empty open-and-closed subset of X is homeomorphic to X;
- *n-homogeneous*, $n \geq 1$, if any bijection between n-element subsets of X entends to a homeomorphism of X;
- *densely $\aleph_0$-homogeneous* if for any countable dense subsets $Q_1, Q_2 \subset X$ there is a homeomorphism h of X with $h(Q_1) \subset Q_2$.

The common feature of the examples from [4] and [7] is the choice of the set $B \subset (0,1)$ possessing the following property firstly cristallized by van Mill in [6]:

(vM) $B = (B + \mathbb{Q}) \cap [0,1]$ *and* $h(G \cap B) \not\subset B$ *for any homeomorphism* $h\colon G \to G$ *of a Borel subset* G *of* I *such that* $|\{x \in G\colon h(x) - x \notin \mathbb{Q}\}| = 2^{\aleph_0}$.

Here $\mathbb{Q} \subset \mathbb{R}$ stands for the set of rational numbers. A subset B with the property (vM) can be easily constructed by transfinite induction, see [6], [4] or [7]. Repeating the arguments of [4] and [7] it can be shown that for any subset $B \subset (0,1)$ possessing the property (vM) the space vD_B has the following features:

1. vD_B *is a first-countable zero-dimensional separable linearly ordered compactum;*
2. vD_B *is n-homogeneous for every $n \geq 1$;*
3. vD_B *is not h-homogeneous;*
4. vD_B *is not densely $\aleph_0$-homogeneous;*

5. *for any homeomorphism h of* vD_B *the* G_δ*-subset* $\{x \in \mathrm{vD}_B \colon h(x) \notin \mathrm{pr}^{-1}(\mathrm{pr}(x) + \mathbb{Q})\}$ *is at most countable;*
6. $h(\mathrm{pr}^{-1}(\mathbb{Q})) \cap pr^{-1}(\mathbb{Q}) \neq \varnothing$;
7. $\mu(h(S)) = \mu(S)$ *for any Borel subset* $S \subset \mathrm{vD}_B$ *and any homeomorphism* h *of* vD_B*;*
8. $\mu(U) = \mu(V)$ *for open-and-closed subsets* $U, V \subset D_B$ *if and only if there is a homeomorphism* h *of* vD_B *with* $h(U) = V$*.*

We add to this list another pathological property of vD_B.

Theorem 1. If the set $B \subset (0,1)$ has the property (vM), then the space vD_B is homeomorphic to no left-topological group.

Proof. Assume that vD_B admits a group operation $*$ turning vD_B into a left-topological group. Consider the countable subsets $Q = \mathrm{pr}^{-1}(\mathbb{Q})$, $Q * Q^{-1} = \{x * y^{-1} \colon x, y \in Q\}$ of vD_B and choose a point $g \in \mathrm{vD}_B \setminus Q * Q^{-1}$. Then for the left shift $l_g \colon x \mapsto g * x$ of G we get $l_g(Q) \cap Q = \varnothing$ which contradicts to the property (6) of vD_B. □

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