УДК 515.12

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HYPERSPACES WITH THE ATTOUCH-WETS TOPOLOGY HOMEOMORPHIC TO ℓ_2

R. I. Voytsitskyy. *Hyperspaces with the Attouch-Wets topology homeomorphic to* ℓ_2, Matematychni Studii, **29** (2008) 207–214.

It is shown that the hyperspace of all nonempty closed subsets $\mathrm{Cld}_{AW}(X)$ of a separable metric space (X,d) endowed with the Attouch-Wets topology is homeomorphic to ℓ_2 if and only if the completion of X is proper, locally connected and contains no bounded connected component, X is topologically complete and not locally compact at infinity.

Р. И. Войцицкий. *Гиперпространства с топологией Аттуша-Ветса, гомеоморфные* l_2 // Математичні Студії. – 2008. – Т.29, №2. – С.207–214.

Доказано, что гиперпространство всех непустых замкнутых подмножеств $\mathrm{Cld}_{AW}(X)$ сепарабельного метрического пространства (X,d), наделенное топологией Аттуша-Ветса гомеоморфно ℓ_2 тогда и только тогда, когда пополнение пространства X – совершенное, локально связное и не содержит ограниченных связных компонент, а пространство X – топологически полное и не локально компактное в бесконечности.

1. Introduction. For a metric space $X=(X,d)$, let $C(X)$ be the set of all continuous real valued functions of X and $\mathrm{Cld}(X)$ be the set of all nonempty closed subsets of X. By identifying each $A\in\mathrm{Cld}(X)$ with the continuous function $X\ni x\mapsto d(x,A)\in\mathbb{R}$, we can embed $\mathrm{Cld}(X)$ into the function space $C(X)$.

The function space $C(X)$ carries at least three natural topologies: of point-wise convergence, of uniform convergence and of uniform convergence on bounded subsets of X. Those three topologies of $C(X)$ induce three topologies on the hyperspace $\mathrm{Cld}(X)$: the *Wijsman* topology, the *metric Hausdorff* topology and the *Attouch-Wets* topology. The hyperspace $\mathrm{Cld}(X)$ endowed with one of these topologies is denoted by $\mathrm{Cld}_W(X)$, $\mathrm{Cld}_H(X)$, and $\mathrm{Cld}_{AW}(X)$, respectively. The Wijsman topology coincides with the Attouch-Wets topology if and only if closed and bounded subsets of X are totally bounded [2, Theorem 3.1.4]. On the other hand the Attouch-Wets topology coincides with the Hausdorff metric topology if and only if (X,d) is a bounded metric space [2, Exercise 3.2.2]. The Hausdorff metric topology on $\mathrm{Cld}_H(X)$ is generated by the Hausdorff metric $d_H(A,B)=\sup_{x\in X}|d(x,A)-d(x,B)|$, where $A,B\in\mathrm{Cld}(X)$.

In [1] it is proved that for an infinite-dimensional Banach space X of weight $w(X)$, the hyperspace $\mathrm{Cld}_{AW}(X)$ is homeomorphic to ($\cong$) the Hilbert space of weight $2^{w(X)}$[1, Theorem 5.3]. In particular, for an infinite-dimensional separable Banach space X, the hyperspace $\mathrm{Cld}_{AW}(X)$ is homeomorphic to $\ell_2(2^{\aleph_0})$. On the other hand, for each finite-dimensional normed linear space X, since every bounded closed set in X is compact, the Attouch-Wets

2000 *Mathematics Subject Classification:* 54B20, 57N20.

topology on $\mathrm{Cld}(X)$ agrees with the Fell topology [2, p.144]. Then, by the result of [12], we have $\mathrm{Cld}_{AW}(X) \cong Q \setminus \{0\}$. Thus, for a Banach space X the hyperspace $\mathrm{Cld}_{AW}(X)$ is either locally compact or non-separable. In [1] the authors asked: does there exist an unbounded metric space X such that $\mathrm{Cld}_{AW}(X) \cong \ell_2$? And, more generally, what are the necessary and sufficient conditions under which the hyperspace $\mathrm{Cld}_{AW}(X)$ is homeomorphic to ℓ_2? We give an answer to these questions. The main result of this paper is the following characterizing theorem.

Theorem 1. *The hyperspace $\mathrm{Cld}_{AW}(X)$ of a metric space X is homeomorphic to ℓ_2 if and only if the completion $\overline{X}$ of X is proper, locally connected and contains no bounded connected component, X is topologically complete and is not locally compact at infinity.*

A metric space X is defined to be

- *proper* if each closed bounded subset of X is compact;
- *not locally compact at infinity* if no bounded subset of X has locally compact complement.

Observe that under the conditions of Theorem 1 the Attouch-Wets topology coincides with the Wijsman topology (cf. [2, Theorem 3.1.4]). So, for free, we obtain the following

Corollary 1. *For a metric space (X, d) the hyperspace $\mathrm{Cld}_W(X)$ is homeomorphic to ℓ_2 if the completion $\overline{X}$ of X is proper, locally connected and contains no bounded component, X is topologically complete and not locally compact at infinity.*

Applying Theorem 1 and Corollary 1 to the space $\mathbb{P}$ of irrational numbers of the real line we obtain

Corollary 2. $\mathrm{Cld}_{AW}(\mathbb{P}) = \mathrm{Cld}_W(\mathbb{P}) \cong \ell_2$.

As a by-product of the proof of Theorem 1 we obtain the following characterization of metric spaces whose hyperspaces with the Attouch-Wets topology are separable absolute retracts.

Theorem 2. *The hyperspace $\mathrm{Cld}_{AW}(X)$ of a metric space X is a separable absolute retract if and only if the completion $\overline{X}$ of X is proper, locally connected and contains no bounded connected components.*

Our Theorems 1 and 2 are "Attouch-Wets" counterparts of the following two results from [5].

Theorem 3. *The hyperspace $\mathrm{Cld}_H(X)$ of a metric space (X, d) is homeomorphic to ℓ_2 if and only if X is a topologically complete nowhere locally compact space and the completion $\overline{X}$ of X is compact, connected, and locally connected.*

Theorem 4. *The hyperspace $\mathrm{Cld}_H(X)$ of a metric space X is a separable absolute retract if and only if the completion $\overline{X}$ of X is compact, connected and locally connected.*

2. Topology of Lawson semilattices and some auxiliary facts. Theorem 2 will be derived from a more general result concerning Lawson semilattices. By a *topological semilattice* we understand a pair $(L, \vee)$ consisting of a topological space L and a continuous associative commutative idempotent operation $\vee\colon L \times L \to L$. A topological semilattice $(L, \vee)$

is a *Lawson semilattice* if open subsemilattices form a base of the topology of L. A typical example of a Lawson semilattice is the hyperspace $\mathrm{Cld}_H(X)$ endowed with the operation of union $\cup$.

Each semilattice $(L,\vee)$ carries a natural partial order: $x\le y$ iff $x\vee y=y$. A semilattice $(L,\vee)$ is called *complete* if each subset $A\subset L$ has the smallest upper bound $\sup A\in L$. It is well-known (and can be easily proved) that each compact topological semilattice is complete.

Lemma 1. *If L is a locally compact Lawson semilattice, then each compact subset $K\subset L$ has the smallest upper bound $\sup K\in L$. Moreover, the map $\sup\colon \mathrm{Comp}(L)\to L$, $\sup\colon K\mapsto \sup K$, is a continuous semilattice homomorphism. Also for every subset $A\subset L$ with compact closure $\overline{A}$ we have $\sup A=\sup\overline{A}$.*

This lemma easily follows from its compact version proved by J. Lawson in [11].

In Lawson semilattices many geometric questions reduce to the one-dimensional level. The following fact illustrating this phenomenon is proved in [10].

Lemma 2. *Let X be a dense subsemilattice of a metrizable Lawson semilattice L. If X is relatively LC^0 in L (and X is path-connected), then X and L are ANRs (ARs) and X is homotopy dense in L.*

A subset $Y\subset X$ is defined to be *relatively LC^0* in X if for every $x\in X$, each neighborhood U of x in X contains a smaller neighborhood V of x such that every two points of $V\cap Y$ can be joined by a path in $U\cap Y$.

Under a suitable completeness condition, the density of a subsemilattice is equivalent to the homotopical density. A subset Y of a topological space X is *homotopy dense* in X if there is a homotopy $(h_t)_{t\in[0,1]}\colon X\to X$ such that $h_0=id$ and $h_t(X)\subset Y$ for every $t>0$.

A subsemilattice X of semilattice L is defined to be *relatively complete* in L if for any subset $A\subset X$ having the smallest upper bound $\sup A$ in L this bound belongs to X.

Proposition 1. *Let L be a locally compact locally connected Lawson semilattice. Each dense relatively complete subsemilattice $X\subset L$ is homotopy dense in L.*

Proof. According to Lemma 2 it suffices to check that X is relatively LC^0 in L. Given a point $x_0\in L$ and a neighborhood $U\subset L$ of x_0, consider the canonical retraction $\sup\colon\mathrm{Comp}(L)\to L$. Using the ANR-property of L and continuity of sup, find a path-connected neighborhood $V\subset L$ of x_0 such that $\sup(\mathrm{Comp}(\overline{V}))\subset U$. We claim that any two points $x,y\in X\cap V$ can be connected by a path in $X\cap U$. First we construct a path $\gamma\colon[0,1]\to\overline{V}$ such that $\gamma(0)=x$, $\gamma(1)=y$ and $\gamma^{-1}(X)$ is dense in $[0,1]$. Let $\{q_n\colon n\in\omega\}$ be a countable dense subset in $[0,1]$ with $q_0=0$ and $q_1=1$. The space L, being locally compact, admits a complete metric ρ. The path-connectedness of V implies the existence of a continuous map $\gamma_0\colon[0,1]\to V$ such that $\gamma_0(0)=x$ and $\gamma_0(1)=y$. Using the local path-connectedness of L we can construct inductively a sequence of functions $\gamma_n\colon[0,1]\to V$ such that:

(i) $\gamma_n(q_k)=\gamma_{n-1}(q_k)$ for all $k\le n$; (ii) $\gamma_n(q_{n+1})\in X$; (iii) $\sup_{t\in[0,1]}\rho(\gamma_n(t),\gamma_{n-1}(t))<2^{-n}$. Then the map $\gamma=\lim_{n\to\infty}\gamma_n\colon[0,1]\to\overline{V}$ is continuous and has the desired properties: $\gamma(0)=x$, $\gamma(1)=y$ and $\gamma(q_n)\in X$ for all $n\in\omega$.

For every $t\in[0,1]$ consider the set $\Gamma(t)=\{\gamma(s)\colon |t-s|\le\mathrm{dist}(t,\{0,1\})\}$. Is is clear that the map $\Gamma\colon[0,1]\to\mathrm{Comp}(L)$ is continuous and so is the composition $\sup\circ\Gamma\colon[0,1]\to L$. Observe that $\sup\circ\Gamma(0)=\sup\{\gamma(0)\}=\gamma(0)=x$, $\sup\circ\Gamma(1)=y$, and $\sup\circ\Gamma([0,1])\subset\sup(\mathrm{Comp}(\overline{V}))\subset U$. Since for every $t\in(0,1)$ the set $\Gamma(t)=\overline{\Gamma(t)\cap X}$, we get $\sup\Gamma(t)=$

$\sup(\Gamma(t) \cap X) \in X$ by the relative completeness of X in L. Thus $\sup \circ \Gamma\colon [0,1] \to U \cap X$ is a path connecting x and y in U. □

For a metric space X by $\mathrm{Fin}(X)$ we denote the subspace of $\mathrm{Comp}(X)$ consisting of non-empty finite subspaces of X.

Lemma 3. *If Y is a subset of a locally path-connected space X, then the subset $L = \mathrm{Fin}(X) \setminus \mathrm{Fin}(Y)$ is relatively LC^0 in $\mathrm{Comp}(X)$.*

Proof. By the argument of [6] we can show that $\mathrm{Fin}(X)$ is relatively LC^0 in $\mathrm{Comp}(X)$. Consequently, for every compact set $K \in \mathrm{Comp}(X)$ and a neighborhood $U \subset \mathrm{Comp}(X)$ of K there is a neighborhood $V \subset \mathrm{Comp}(X)$ of K such that any two points $A, B \in \mathrm{Fin}(X) \cap V$ can be linked by a path in $\mathrm{Fin}(X) \cap U$. Since $\mathrm{Comp}(X)$ is a Lawson semilattice, we may assume that U and V are subsemilattices of $\mathrm{Comp}(X)$. We claim that any two points $A, B \in L \cap V$ can be connected by a path in $L \cap U$. Since $L \subset \mathrm{Fin}(X)$, there is a path $\gamma\colon [0,1] \to U \cap \mathrm{Fin}(X)$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Define a new path $\gamma'\colon [0,1] \to U \cap \mathrm{Fin}(X)$ letting $\gamma'(t) = \gamma(\max\{0, 2t-1\}) \cup \gamma(\min\{2t, 1\})$. Observe that $A \subset \gamma'(t)$ if $t \le 1/2$ and $B \subset \gamma'(t)$ if $t \ge 1/2$. Since $A, B \notin \mathrm{Fin}(Y)$, we conclude that $\gamma'([0,1]) \subset L \cap U$. □

We also need the following nontrivial fact from [4, Corollary 2].

Lemma 4. *Let X be a dense subset of a metric space M. Then the hyperspace $\mathrm{Cld}_H(X)$ is an ANR if and only if so is the hyperspace $\mathrm{Cld}_H(M)$.*

The proof of Theorem 1 and Theorem 4 relies on the next lemma due to D. Curtis [9].

Lemma 5. *A homotopy dense G_δ-subset $X \subset Q$ with homotopy dense complement in the Hilbert cube Q is homeomorphic to ℓ_2.*

3. The metrics d_{AW} and d_H on $\mathrm{Cld}(X)$.

Let $X = (X, d)$ be a metric space. The ε-neighborhood of $x \in X$ (i.e., the open ball centered at x with radius ε) is denoted by $B(x, \varepsilon)$. Let A and B be nonempty subsets of a metric space (X, d). The *excess* of A over B with respect to d is defined by the formula $e_d(A, B) = \sup\{d(a, B) \mid a \in A\}$. We regard $e_d(A, \varnothing) = +\infty$. For the Hausdorff metric we have the following

$$d_H(A, B) = \max\{e_d(A, B), e_d(B, A)\} = \sup_{x \in X} |d(x, A) - d(x, B)|.$$

Now we define the metric d_{AW}. Fix $x_0 \in X$ and let $X_i = \{x \in X \mid d(x, x_0) \le i\}$. The following metric d_{AW} on $\mathrm{Cld}(X)$ generates the Attouch-Wets topology[1]

$$d_{AW}(A, B) = \sup_{i \in \mathbb{N}} \min\Big\{1/i,\ \sup_{x \in X_i} |d(x, A) - d(x, B)|\Big\}.$$

It should be noticed that

$$d_{AW}(A, B) \le d_H(A, B) \quad \text{for every } A, B \in \mathrm{Cld}(X).$$

We need the following fact for the Attouch-Wets convergence in terms of excess, see [2, Theorem 3.1.7].

Proposition 2. *Let (X, d) be a metric space, and $A, A_1, A_2, \ldots$ be nonempty closed subsets of X, $x_0 \in X$ be fixed. The following are equivalent:*

1. $\lim_{n \to \infty} d_{AW}(A_n, A) = 0$;

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[1] In [2], the following metric is adopted $d_{AW}(A, B) = \sum_{i \in \mathbb{N}} 2^{-i} \min\Big\{1,\ \sup_{x \in X_i} |d(x, A) - d(x, B)|\Big\}$.

2. *For each* $i \in \mathbb{N}$, *we have both* $\lim_{n\to\infty} e_d(A\cap X_i, A_n) = 0$ *and* $\lim_{n\to\infty} e_d(A_n\cap X_i, A) = 0$.

Recall that the Attouch-Wets topology depends on the metric for X, that is, the space $\mathrm{Cld}_{AW}(X)$ is not a topological invariant for X. Concerning conditions that two metrics for X induce the same topology, see [2, Theorem 3.3.3].

4. Embedding $\mathrm{Cld}_{AW}(X)$ **in** $\mathrm{Cld}_H(\alpha X)$**.** The main idea in proving Theorems 1, 2 is the following: we reduce the Attouch-Wets topology on $\mathrm{Cld}(X)$ to the Hausdorff metric topology on $\mathrm{Cld}(\alpha X)$ for a suitable one-point extension αX of X. The metric space $(\alpha X, \rho)$ is obtained by adding the infinity point ∞ to the space X. More precisely, we endow the space αX with the metric

$$\rho(x,y) = \begin{cases} \min\Big\{d(x,y), \dfrac{1}{1+d(x,x_0)} + \dfrac{1}{1+d(y,x_0)}\Big\}, & \text{if } x,y \in X \\ \dfrac{1}{1+d(x,x_0)}, & \text{if } x \in X,\ y = \infty. \end{cases}$$

Here, $x_0 \in X$ is a fixed point. Note, that (X,d) is homeomorphic to $(\alpha X \setminus \{\infty\}, \rho)$ and $\mathrm{diam}(\alpha X) < 2$.

Remark 1. We can obtain the space $(\alpha X, \rho)$ in the following way: embed X in $X \times [0,1)$ by the formula $x \mapsto (x, \dfrac{d(x,x_0)}{1+d(x,x_0)})$ and consider the cone metric on this space (induced by the suitable metrization of the quotient space $X \times [0,1]/X \times \{1\}$).

Proposition 3. *The function* $e\colon \mathrm{Cld}_{AW}(X) \to \mathrm{Cld}_H(\alpha X)$ *defined by the formula* $e(A) = A \cup \{\infty\}$ *is an embedding.*

Proof. Let $\lim_{n\to\infty} d_{AW}(A_n, A) = 0$. Assume to the contrary that $\lim_{n\to\infty} \rho_H(e(A_n), e(A)) \neq 0$. This means that there exists some $\varepsilon_0 > 0$ such that we can find either a sequence $x_{n_k} \in A_{n_k}$, $k \in \mathbb{N}$, with $\rho_H(x_{n_k}, A \cup \{\infty\}) \geq \varepsilon_0$, or there exists a sequence $(y_k) \subset A$ with $\rho_H(A_{n_k} \cup \{\infty\}, y_k) \geq \varepsilon_0$ for all $k \in \mathbb{N}$.
In the former case, since $\infty \in e(A)$, we have $\rho(x_{n_k}, \infty) = \frac{1}{1+d(x_{n_k},x_0)} \geq \varepsilon_0$ for each $k \in \mathbb{N}$. Hence, there exists some $i_0 \in \mathbb{N}$ with $(x_{n_k}) \subset X_{i_0}$, $k \in \mathbb{N}$. For every $y \in A$ and $k \in \mathbb{N}$ $\rho(x_{n_k}, y) \geq \varepsilon_0$, and so $d(x_{n_k}, y) \geq \varepsilon_0$. This implies that for each $k \in \mathbb{N}$ $\sup_{x\in X_{i_0}} |d(x, A_{n_k}) - d(x,A)| \geq d(x_{n_k}, A) \geq \varepsilon_0$. Combining this with the definition of the Attouch-Wets metric we get $d_{AW}(A_{n_k}, A) \geq \min\{\frac{1}{i_0}, \varepsilon_0\}$ for all $k \in \mathbb{N}$. This is a contradiction. In the latter case, similar to the above, we have for each $k \in \mathbb{N}$ $\sup_{x\in X_{i_0}} |d(x, A_{n_k}) - d(x,A)| \geq d(y_k, A_{n_k}) \geq \varepsilon_0$. Whence, $d_{AW}(A_{n_k}, A) \geq \min\{\frac{1}{i_0}, \varepsilon_0\}$ for all $k \in \mathbb{N}$.

Conversely, let $\lim_{n\to\infty} \rho_H(e(A_n), e(A)) = 0$. Assume to the contrary that $\lim_{n\to\infty} d_{AW}(A_n, A) \neq 0$. This means that there exists a subsequence $(A_{n_k}) \subset (A_n)$ with $d_{AW}(A_{n_k}, A) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. Then, by Proposition 2, there exists $i_0 \in \mathbb{N}$ such that either $e_d(A_{n_k} \cap X_{i_0}, A) \geq \varepsilon_0$ or $e_d(A \cap X_{i_0}, A_{n_k}) \geq \varepsilon_0$. Remark, that we can take i_0 so large that $A_n \cap X_{i_0} \neq \varnothing$ for all $n \in \mathbb{N}$. Consequently, in the former case we can find a sequence $x_k \in A_{n_k} \cap X_{i_0}$ with $d(x_k, y) \geq \varepsilon_0$ for each $y \in A$ and $k \in \mathbb{N}$. Hence,

$$\rho(x_k, y) = \min\{d(x_k,y), \tfrac{1}{1+d(x_k,x_0)} + \tfrac{1}{1+d(y,x_0)}\} \geq \min\{\varepsilon_0, \tfrac{1}{1+i_0}\}$$

for all $y \in A$ and $k \in \mathbb{N}$. Since $\rho(x_k, \infty) = \frac{1}{1+d(x_k,x_0)} \geq \frac{1}{1+i_0}$, it follows that

$$\lim_{k\to\infty} \rho_H(A_{n_k} \cup \{\infty\}, A \cup \{\infty\}) \geq \min\{\varepsilon_0, \tfrac{1}{1+i_0}\} > 0,$$

and we have a contradiction. In the latter case, there exists a sequence $y_k \in A \cap X_{i_0}$ with $d(y_k, x) \geq \varepsilon_0$ for all $x \in A_{n_k}$ and $k \in \mathbb{N}$. Whence, for every $k \in \mathbb{N}$ we have $\rho(y_k, x) \geq \min\{\varepsilon_0, \frac{1}{1+d(y_k,x_0)}\} \geq \min\{\varepsilon_0, \frac{1}{1+i_0}\}$ and $\rho(y_k, \infty) = \frac{1}{1+d(y_k,x_0)} \geq \frac{1}{1+i_0}$. This violates that $\lim_{n\to\infty} \rho_H(e(A_n), e(A)) = 0$. □

Let us give some notation. We mean $\mathrm{Cld}_H(\alpha X)_\infty = \{F \in \mathrm{Cld}_H(\alpha X) \mid \infty \in F\} = e(\mathrm{Cld}_{AW}(X)) \cup \{\infty\}$ and $\mathrm{Cld}_H(\overline{\alpha X})_\infty = \{F \in \mathrm{Cld}_H(\overline{\alpha X}) \mid \infty \in F\}$. Similar, we define $\mathrm{Fin}_H(\alpha X)_\infty$ and $\mathrm{Fin}_H(\overline{\alpha X})_\infty$. Note, that $\mathrm{Cld}_H(\alpha X)_\infty$ is homeomorphic to $\mathrm{Cld}_H(\alpha X)_\infty \setminus \{\infty\}$ which is homeomorphic to $\mathrm{Cld}_{AW}(X)$. Further, without loss of generality, we consider $\mathrm{Cld}_H(\alpha X)_\infty$ instead $\mathrm{Cld}_{AW}(X)$.

Proposition 4. *Let $X = (X, d)$ be a metric space with a fixed point $x_0 \in X$. Then the hyperspace $\mathrm{Cld}_H(X|\{x_0\})$ is an ANR (an AR) if and only if so is the hyperspace $\mathrm{Cld}_H(X)$, where $\mathrm{Cld}_H(X|\{x_0\}) = \{F \in \mathrm{Cld}_H(X) \mid x_0 \in F\}$.*

Proof. The "if" part is obvious, since there is a natural retraction $r\colon \mathrm{Cld}_H(X) \to \mathrm{Cld}_H(X|\{x_0\})$ defined by $r(F) = F \cup \{x_0\}$.

To prove the "only if" part, assume that $\mathrm{Cld}_H(X|\{x_0\})$ is an ANR. By [1, Proposition 3.2] we have to check the local path-connectedness of $\mathrm{Cld}_H(X)$. Choose an arbitrary $F \in \mathrm{Cld}_H(X)$ and consider two cases: $x_0 \notin F$ and otherwise. Let us show the first case. Since the hyperspace $\mathrm{Cld}_H(X|\{x_0\})$ is locally path-connected at $F \cup \{x_0\}$, for arbitrary $\varepsilon > 0$ find $\delta(\varepsilon) > 0$ such that each two points in $B(F \cup \{x_0\}, \delta) \cap \mathrm{Cld}_H(X|\{x_0\})$ can be connected by a path $f\colon [0,1] \to \mathrm{Cld}_H(X|\{x_0\})$ with diameter $< \varepsilon$. Naturally, we can assume that the point $x_0 \in X$ is isolated in $f(t)$ for each $t \in [0,1]$. Choose an arbitrary $G \in B(F, \delta)$ and observe that the path $f'\colon [0,1] \to \mathrm{Cld}_H(X)$ defined by $f' = f \setminus \{x_0\}$ is as required, i.e., we obtain a path from G to F with diameter $< 2\varepsilon$.

The other case: since $F \in \mathrm{Cld}_H(X|\{x_0\})$, for arbitrary $\varepsilon > 0$ find $\delta > 0$ such that each two points in $B(F, \delta) \cap \mathrm{Cld}_H(X|\{x_0\})$ can be connected by a path in $\mathrm{Cld}_H(X|\{x_0\})$ with diameter $< \varepsilon$. Choose an arbitrary point $G \in B(F, \delta) \subset \mathrm{Cld}_H(X)$. If $x_0 \in G$, then there exists a path in $\mathrm{Cld}_H(X|\{x_0\})$ connecting G and F with diameter $< \varepsilon$. Otherwise, let $G' = G \cup \{x_0\}$. Then, G' and F can be connected by a path $f\colon [0,1] \to \mathrm{Cld}_H(X|\{x_0\})$ with diameter $< \varepsilon$. Observe, that the path $g\colon [0,1] \to \mathrm{Cld}_H(X)$ defined as $g(t) = f(t) \setminus \{x_0\}$ if x_0 is an isolated point in $f(t)$ and $g(t) = f(t)$ otherwise, $t \in [0,1]$, is as required, i.e., we obtain a path from G to F with diameter $< 2\varepsilon$. □

Corollary 3. *The hyperspace $\mathrm{Cld}_H(\alpha X)_\infty$ is an ANR (an AR) if and only if so is the hyperspace $\mathrm{Cld}_H(\alpha X)$.*

Corollary 3 implies the following fact about Attouch-Wets hyperspace topology having separate interest.

Lemma 6. *Let X be a dense subset of a metric space M. Then, the hyperspace $\mathrm{Cld}_{AW}(X)$ is an absolute neighborhood retract (an absolute retract) if and only if so is the hyperspace $\mathrm{Cld}_{AW}(M)$.*

Proof. It follows from the Proposition 3, Corollary 3 and Lemma 4. □

5. The completion $\overline{\alpha X}$ is a Peano continuum. We need the following lemma proved in [12, Lemma 2].

Lemma 7. *If X is a locally connected, locally compact separable metrizable space with no compact components, then its Alexandroff one-point compactification αX is a Peano continuum.*

Using the previous lemma we can easily obtain

Lemma 8. *Suppose that the completion $\overline{X}$ of a metric space X is a proper locally connected space with no bounded connected components. Then, $\overline{\alpha X}$ is a Peano continuum.*

Proof. Note, that the completion $\overline{X}$ of X satisfies the conditions of Lemma 7 and $\overline{\alpha X}$ (the completion of αX) coincides with the Alexandroff one-point compactification of $\overline{X}$. □

Then, by the Curtis-Schori Hyperspace Theorem [8], $\mathrm{Cld}_H(\overline{\alpha X}) = \mathrm{Comp}(\overline{\alpha X})$ is homeomorphic to the Hilbert cube Q.

Lemma 9. *Let X satisfy the conditions of Theorem 1. Then the hyperspace $\mathrm{Cld}_H(\overline{\alpha X})_\infty$ is homeomorphic to the Hilbert cube Q.*

Proof. Observe that $\mathrm{Cld}_H(\overline{\alpha X})_\infty$ is a retract of $\mathrm{Cld}_H(\overline{\alpha X})$, and thus is a compact absolute retract. Then, we use the Characterization Theorem for the Hilbert cube, see [3, Theorem 1.1.23]. By this theorem we have to check that for each $\varepsilon > 0$, every $n \in \mathbb{N}$, and each maps $f_1, f_2 \colon I^n \to \mathrm{Cld}_H(\overline{\alpha X})_\infty$ there are maps $f_1', f_2' \colon I^n \to \mathrm{Cld}_H(\overline{\alpha X})_\infty$ such that $d(f_i, f_i') < \varepsilon$, $i = 1, 2$, and $f_1'(I^n) \cap f_2'(I^n) = \varnothing$. Fix $\varepsilon > 0$, $n \in \mathbb{N}$, and maps $f_1, f_2 \colon I^n \to \mathrm{Cld}_H(\overline{\alpha X})_\infty$. By the argument of [6] we can show that $\mathrm{Fin}_H(\overline{\alpha X})$ is homotopy dense in $\mathrm{Cld}_H(\overline{\alpha X})$. Therefore, we can find an $\varepsilon/2$-close to f_i map $g_i \colon I^n \to \mathrm{Fin}_H(\overline{\alpha X})$, $i = 1, 2$, respectively, see [3, Ex. 1.2.10]. Observe, that $d(f_i, g_i \cup \{\infty\}) < \varepsilon/2$, $i = 1, 2$. Then, it is easily seen that maps $f_1' = g_1 \cup \{\infty\}$ and $f_2' = g_2 \cup B(\infty, \varepsilon/2)$ are as required. □

6. Proof of Theorem 2. To prove the "only if" part, assume that $\mathrm{Cld}_{AW}(X)$ is a separable absolute retract. The separability of $\mathrm{Cld}_{AW}(X)$ implies that each bounded subset of X is totally bounded [1, Theorem 5.2], which is equivalent to the properness of the completion $\overline{X}$ of X. By Lemma 6, the hyperspace $\mathrm{Cld}_{AW}(\overline{X})$ is a separable absolute retract too. In this case $\mathrm{Cld}_{AW}(\overline{X}) = \mathrm{Cld}_F(\overline{X})$ (by $\mathrm{Cld}_F(X)$ we denote the hyperspace $\mathrm{Cld}(X)$ endowed with the Fell topology, see [2, Theorem 5.1.10]) is an absolute retract, and we can apply [12, Propositions 1, 2] to conclude that the locally compact space $\overline{X}$ is locally connected and contains no bounded (=compact) connected component.

Next, we prove the "if" part of Theorem 2. Assume that the completion $\overline{X}$ of X is proper, locally connected with no bounded connected components. By Proposition 3, we can identify $\mathrm{Cld}_{AW}(X)$ with the subspace $\mathrm{Cld}_H(\alpha X)_\infty$. Note, that $\mathrm{Cld}_H(\alpha X)_\infty$ is a retract of $\mathrm{Cld}_H(\alpha X)$ under the natural retraction $r(F) = F \cup \{\infty\}$, $F \in \mathrm{Cld}_H(\alpha X)$. Lemma 4 implies that $\mathrm{Cld}_H(\alpha X)$ is an absolute retract if and only if so is the hyperspace $\mathrm{Cld}_H(\overline{\alpha X})$. Finally, since $\mathrm{Cld}_H(\overline{\alpha X})$ is homeomorphic to the Hilbert cube Q by the Curtis-Schori Theorem [8], we have the result.

7. Proof of Theorem 1. The "only if" part. If $\mathrm{Cld}_{AW}(X)$ is homeomorphic to ℓ_2, then X is topologically complete by [7]. The total boundedness of each bounded subset of X follows from [1, Theorem 5.2]. Since ℓ_2 is a separable absolute retract, we may apply Theorem 2 to conclude that the completion $\overline{X}$ of X is locally connected and contains no bounded connected component. It remains to show that X is not locally compact at infinity. Assume the contrary, i.e., there exists a bounded subset $B \subset X$ with locally compact complement in X. Then it is easily seen that the point $\infty \in \alpha X$ has an open neighborhood with compact closure. Whence, we can find a compact neighborhood of $\{\infty\}$ in $\mathrm{Cld}_H(\alpha X)_\infty$. But this is impossible because of the nowhere locally compactness of the Hilbert space ℓ_2. This proves the "only if" part of Theorem 1.

To prove the "if" part, assume that X is topologically complete, not locally compact at infinity and the completion $\overline{X}$ of X is proper, locally connected with no bounded connected components. By Proposition 3, we identify $\mathrm{Cld}_{AW}(X)$ with the subspace $\mathrm{Cld}_H(\alpha X)_\infty$ of $\mathrm{Cld}_H(\alpha X)$. By Lemma 9, the hyperspace $\mathrm{Cld}_H(\overline{\alpha X})_\infty = \mathrm{Comp}(\overline{\alpha X})_\infty$ is homeomorphic to Q. Now consider the map $e\colon \mathrm{Cld}_H(\alpha X)_\infty \to \mathrm{Cld}_H(\overline{\alpha X})_\infty$ assigning to each closed subset $F \subset \alpha X$ its closure $\overline{F}$ in $\overline{\alpha X}$ and note that this map is an isometric embedding, which allows us to identify the hyperspace $\mathrm{Cld}_{AW}(X)$ with the subspace $\{F \in \mathrm{Cld}_H(\overline{\alpha X})_\infty\colon F = \mathrm{cl}(F \cap \alpha X)\}$ of $\mathrm{Cld}_H(\overline{\alpha X})_\infty$. It is easy to check that this subspace is dense and relatively complete in the Lawson semilattice $\mathrm{Cld}_H(\overline{\alpha X})_\infty$. Then it is homotopically dense in $\mathrm{Cld}_H(\overline{\alpha X})_\infty$ by Proposition 1 and Lemma 2. The subset $\mathrm{Cld}_H(\alpha X)_\infty$, being topologically complete, is a G_δ-set in $\mathrm{Cld}_H(\overline{\alpha X})_\infty$. The dense subsemilattice $L = \mathrm{Fin}_H(\overline{\alpha X})_\infty \setminus \mathrm{Fin}_H(\alpha X)_\infty$ is homotopy dense in $\mathrm{Cld}_H(\overline{\alpha X})_\infty$, since X is not locally compact at infinity. Since $L \cap \mathrm{Cld}_H(\alpha X)_\infty = \varnothing$, we get that $\mathrm{Cld}_H(\alpha X)_\infty$ is a homotopy dense G_δ-subset in $\mathrm{Cld}_H(\overline{\alpha X})_\infty$ with homotopy dense complement. Applying Lemma 5 we conclude that the space $\mathrm{Cld}_{AW}(X)$ is homeomorphic to ℓ_2.

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Received 15.02.07