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INVERSE PROBLEM FOR SEMILINEAR PARABOLIC EQUATION

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In this paper we consider the problem of finding the pair $(a(t), u(x,t))$ from the equation $u_t = a(t)u_{xx} + b(x,t,u,u_x)$, an initial condition, Dirichlet's boundary condition and given heat flux as overdetermination condition. Existence and uniqueness of a smooth solution are proved.

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В работе рассматривается проблемма нахождения пары $(a(t), u(x,t))$ из уравнения $u_t = a(t)u_{xx} + b(x,t,u,u_x)$, начальных условий, краевого условия Дирихле и заданного теплового потока в качестве условия переопределения. Доказаны существование и единственность гладкого решения.

The first attempts to study coefficient inverse problems for quasilinear parabolic equations have been undertaken not so far. One can mention investigations on the uniqueness of solution for the quasilinear heat equation with unknown coefficients of conductivity and heat capacity made by N.V. Muzyliov ([1-4]). The stability of solution of inverse problem for the equations

$$u_t - b(t)c(u)u_{xx} = 0, \quad x > 0, \quad 0 < t < T,$$

with unknown coefficient $b(t)$ and

$$u_t - [v(u(x,t))u_x]_x = f(x,t), \quad 0 < x < l, \quad 0 < t < T,$$

with unknown coefficient $v(x,t)$ was established by A.Lorenzi ([5,6]). Some papers of J.R.Cannon and Y.Lin [7-10] were devoted to studying of existence and uniqueness conditions for inverse problems for the equation with unknown coefficient $q(t)$,

$$u_t = (a(x,t,u,u_x))_x + q(t)u + F(x,t,u,u_x),$$

and some particular cases of this equation.

In this paper, we establish the possibility to determine uniquely a time-dependent major coefficient in a semilinear heat equation.

In the domain $Q_T = \{(x,t)\colon 0 < x < h < \infty, 0 < t < T < \infty\}$ consider the parabolic equation

$$u_t = a(t)u_{xx} + b(x,t,u,u_x) \tag{1}$$

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with unknown coefficient $a(t) > 0$ subject to the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,h], \tag{2}$$

boundary conditions

$$u(0,t) = \mu_1(t), \quad u(h,t) = \mu_2(t), \quad t \in [0,T], \tag{3}$$

and overdetermination condition

$$a(t)u_x(0,t) = \mu_3(t), \quad t \in [0,T]. \tag{4}$$

Definition 1. A pair of functions $(a,u) \in C[0,T] \times C^{2,1}(\overline{Q}_T), a(t) > 0, t \in [0,T]$, is called a *solution of problem (1)–(4)* if conditions (1)–(4) are satisfied by these functions.

Suppose that the following assumptions hold:

(A1) $\varphi(x) \in C^2[0,h], \mu_i(t) \in C^1[0,T], i \in \{1,2\}, \mu_3(t) \in C[0,T], b(x,t,u,v), b_x(x,t,u,v), b_u(x,t,u,v), b_v(x,t,u,v) \in C(\overline{Q}_T \times \mathbb{R}^2)$;

(A2) $|b(x,t,u,v)| \le \nu(|u|)(1+|v|^2), (x,t) \in \overline{Q}_T, u \in \mathbb{R}, v \in \mathbb{R}; b_u(x,t,u,0) \le -\nu_0 < 0, (x,t) \in \overline{Q}_T, u \in \mathbb{R}$, where $\nu(s) > 0$ is a non-decreasing continuous function on $[0,\infty)$, $\nu_0 = \text{const}$;

(A3) $\varphi'(x) > 0, x \in [0,h], \mu_3(t) > 0, t \in [0,T]$;

(A4) $\varphi(0) = \mu_1(0), \varphi(h) = \mu_2(0), \mu_1'(0) = \dfrac{\mu_3(0)}{\varphi'(0)}\varphi''(0) + b(0,0,\varphi(0),\varphi'(0)), \mu_2'(0) = \dfrac{\mu_3(0)}{\varphi'(0)}\varphi''(h) + b(h,0,\varphi(h),\varphi'(h))$.

Theorem 1. *Under assumptions (A1)–(A4) there exists a solution of problem (1)–(4) defined for $x \in [0,h], t \in [0,T_0]$, where the number $T_0, 0 < T_0 \le T$, depends on the given data.*

Proof. First, we introduce the equation

$$a(t) = \frac{\mu_3(t)}{u_x(0,t)}, \quad t \in [0,T], \tag{5}$$

supposing for instant that $u_x(0,t) > 0$. To find the solution of direct problem and its derivatives we use the Green functions

$$G_k(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(\theta(t)-\theta(\tau))}} \sum_{n=-\infty}^{\infty} \left(\exp\left(-\frac{(x-\xi+2nh)^2}{4(\theta(t)-\theta(\tau))} \right) + (-1)^k \exp\left(-\frac{(x+\xi+2nh)^2}{4(\theta(t)-\theta(\tau))} \right) \right), \quad k \in \{1,2\}, \quad \theta(t) = \int_0^t a(\tau)d\tau,$$

of the first (k=1) and second (k=2) initial boundary value problems for the heat equation

$$u_t = a(t)u_{xx}. \tag{6}$$

Thus, if the coefficient $a(t) > 0$ is known we can reduce the direct problem (1)–(3) to equivalent system of integral equations

$$u(x,t) = u_0(x,t) + \int_0^t \int_0^h G_1(x,t,\xi,\tau) b(\xi,\tau,u(\xi,\tau),v(\xi,\tau)) d\xi d\tau, \tag{7}$$

$$v(x,t) = u_{0x}(x,t) + \int_0^t \int_0^h G_{1x}(x,t,\xi,\tau) b(\xi,\tau,u(\xi,\tau),v(\xi,\tau)) d\xi d\tau, \quad (x,t) \in \overline{Q}_T, \tag{8}$$

where $u_0(x,t)$ is a solution of problem (6), (2), (3) which can be found by the formula

$$u_0(x,t) = \int_0^h G_1(x,t,\xi,0)\varphi(\xi)d\xi + \int_0^t G_{1\xi}(x,t,0,\tau)a(\tau)\mu_1(\tau)d\tau - \int_0^t G_{1\xi}(x,t,h,\tau)a(\tau)\mu_2(\tau)d\tau. \tag{9}$$

Differentiating (9) with respect to x we calculate the derivative $u_{0x}(x,t)$:

$$u_{0x}(x,t) = \int_0^h G_2(x,t,\xi,0)\varphi'(\xi)d\xi - \int_0^t G_2(x,t,0,\tau)\mu_1'(\tau)d\tau + \int_0^t G_2(x,t,h,\tau)\mu_2'(\tau)d\tau. \tag{10}$$

Now we can affirm that the inverse problem (1)–(4) is reduced to the equivalent system of equations (5), (7), (8) in the meaning that, if $(a(t), u(x,t))$ is a solution of problem (1)–(4) accordingly to the Definition, then $(a(t), u(x,t), v(x,t) \equiv u_x(x,t)) \in C[0,T] \times (C(\overline{Q}_T))^2$ will be a solution of the system (5), (7), (8). And, vice versa, if $(a(t), u(x,t), v(x,t)) \in C[0,T] \times (C(\overline{Q}_T))^2$ is a solution of system (5), (7), (8), then $(a(t), u(x,t))$ will belong to the class $C[0,T] \times C^{2,1}(\overline{Q}_T)$ and will verify conditions ((1)–(4).

To prove the existence of solution of the system (5), (7), (8) we shall apply the Schauder fixed-point theorem and we start by a priori estimations of solutions of system (5), (7), (8). To begin, we suppose for instant that the coefficient $a(t) > 0$ is known. Under assumptions (A1) it is easy to reduce equation (1) to the following form

$$u_t = a(t)u_{xx} + b_0(x,t,u,u_x)u_x + b_1(x,t,u)u + b(x,t,0,0). \tag{11}$$

From (A2) it follows that $b_1(x,t,u,0) \le -\nu_0 < 0$ for $(x,t) \in \overline{Q}_T, u \in \mathbb{R}$. Applying to problem (11), (2), (3) the maximum principle [11] we derive the estimate

$$|u(x,t)| \le M_0 < \infty, \quad (x,t) \in \overline{Q}_T.$$

Taking into account assumption (A3) we obtain from (8) and (10)

$$v(0,t) \ge \min_{[0,h]} \varphi'(x) \int_0^h G_2(0,t,\xi,0)d\xi - \int_0^t G_2(0,t,0,\tau)\mu_1'(\tau)d\tau$$
$$+ \int_0^t G_2(0,t,h,\tau)\mu_2'(\tau)d\tau + \int_0^t \int_0^h G_{1x}(0,t,\xi,\tau) b(\xi,\tau,u(\xi,\tau),v(\xi,\tau))d\xi d\tau.$$

It is easy to verify that

$$\int_0^h G_2(x,t,\xi,0)d\xi = 1.$$

Since $\min\limits_{[0,h]} \varphi'(x) > 0$ and the last three integrals in the previous formula are equal to 0 when $t = 0$, there exists a segment $[0,t_0], 0 < t_1 \le T$, such that the following inequality holds:

$$v(x,t) \ge \frac{1}{2} \min_{[0,h]} \varphi'(x) \equiv M_1 > 0, \quad (x,t) \in [0,h] \times [0,t_1]; \tag{12}$$

the value t_1 will be estimated later. Hence, from (12) we get

$$a(t) \le A_1 < \infty, \quad t \in [0, T_0], \tag{13}$$

where $T_0 = \min\{t_0, t_1\}$.

To evaluate $v(x,t)$ we use the estimates of the Green functions [12],

$$G_k(x,t,\xi,\tau) \le C_1 + \frac{C_2}{\sqrt{\theta(t)-\theta(\tau)}}, \quad k \in \{1,2\},$$

and the assumption $|b(x,t,u,v)| \le \nu(M_0)(1+|v|^2)$. This leads us to the inequality

$$V(t) \le C_3 + C_4 \int_0^t \frac{d\tau}{\sqrt{\theta(t)-\theta(\tau)}} + C_5 \int_0^t \frac{(1+V^2(\tau))d\tau}{\sqrt{\theta(t)-\theta(\tau)}}, \tag{14}$$

where $V(t) \equiv \max_{x\in[0,h]} |v(x,t)|$. Taking into account that from (5) we have $a(t) \ge \dfrac{C_6}{V(t)}$, we transform (14) as follows:

$$V(t) \le C_3 + C_7 \int_0^t \frac{a(\tau)V(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}} + C_8 \int_0^t \frac{(1+V^2(\tau))a(\tau)V(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}}.$$

Denote $V_1(t) = V(t) + 1$. Thus, we arrive to the inequality

$$V_1(t) \le C_9 + C_{10} \int_0^t \frac{a(\tau)V_1^3(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}}, \quad t \in [0,T]. \tag{15}$$

Cubing the both sides of this inequality we have

$$V_1^3(t) \le C_{11} + C_{12} \left(\int_0^t \frac{a(\tau)V_1^3(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}} \right)^3.$$

Apply the Hölder inequality:

$$V_1^3(t) \le C_{11} + C_{12} \left(\int_0^t \frac{a^{\frac{3}{2}}(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}} \right)^{\frac{2}{3}} \int_0^t \frac{V_1^9(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}}.$$

Taking into account (13) we obtain

$$V_1^3(t) \le C_{11} + C_{13} \int_0^t \frac{V_1^9(\tau)d\tau}{\sqrt{\theta(t)-\theta(\tau)}}.$$

Put $t=\sigma$, multiply by $\frac{a(\sigma)}{\sqrt{\theta(t)-\theta(\sigma)}}$ and integrate with respect to σ from 0 to t:

$$\int_0^t \frac{a(\sigma)V_1^3(\sigma)d\sigma}{\sqrt{\theta(t)-\theta(\sigma)}} \le C_{14} + C_{15} \int_0^t V_1^9(\tau)d\tau.$$

Substitute this inequality into (15):

$$V_1(t) \le C_{16} + C_{17} \int_0^t V_1^9(\tau)d\tau, \quad t \in [0,T]. \tag{16}$$

Denote

$$w(t) = C_{16} + C_{17} \int_0^t V_1^9(\tau)d\tau.$$

From (16) we obtain

$$w'(t) \le C_{17} w^9(t).$$

By integrating we receive

$$w(t) \le \frac{C_{16}}{(1 - 8C_{16}^8 C_{17} t)^{1/8}}. \tag{17}$$

Thus, we obtain from (17) the estimate

$$|v(x,t)| \le M_1, \quad (x,t) \in [0,h] \times [0,t_0], \tag{18}$$

where the number $t_0, 0 < t_0 \le T$, is such that $1 - 8C_{16}^8 C_{17} t_0 > 0$. Estimate (18) implies the lower bound to $a(t)$:

$$a(t) \ge A_0 > 0, \quad t \in [0,t_0]. \tag{19}$$

Thus, we have established estimates (), (18), (19), (13) of the solution of system (5), (7), (8). After this, we can apply the Schauder fixed-point theorem to system (5), (7), (8).

Denote $\mathcal{N} = \{(a,u,v) \in C[0,T_0] \times C(\overline{Q}_{T_0}) \times C(\overline{Q}_{T_0})\colon A_0 \le a(t) \le A_1, |u(x,t)| \le M_0, M_2 \le v(x,t) \le M_1\}$. Let consider system (5), (7), (8) as an equation

$$\omega = P\omega, \tag{20}$$

where $\omega = (a,u,v)$ and the operator P is defined by the right hand parts of (5), (7), (8). It is evident that P maps $\mathcal{N}$ into $\mathcal{N}$. The compactness of the operator P is proved in [14]. Thus, the Schauder fixed-point theorem may be applied to equation (20) and, consequently, there exists a solution of equation (20). Because of the equivalence of system (5), (7), (8) and problem (1)–(4), the proof of Theorem 1 is complete.

Theorem 1 establishes the local existence of a solution with respect to the time variable t. A possible diminution of time interval appears twice: during estimation (12) of $u_x(x,t)$ from below and resolution of nonlinear inequality (15). Inequality (15) was solved on the interval $[0,t_0]$ where the number t_0 depends only on the constants C_{16}, C_{17}.

To determine the number t_1, the inequality

$$\Bigg| -\int_0^t G_2(0,t,0,\tau)\mu_1'(\tau)d\tau + \int_0^t G_2(0,t,h,\tau)\mu_2'(\tau)d\tau + \int_0^t \int_0^h G_{1x}(0,t,\xi,\tau) b(\xi,\tau,u(\xi,\tau),v(\xi,\tau))d\xi d\tau \Bigg| \le \frac{1}{2} \min_{[0,h]} \varphi'(x) \tag{21}$$

must be satisfied on $[0,t_1]$. Inequality (21) may be replaced by the following one:

$$\left|\int_0^t G_2(0,t,0,\tau)\mu_1'(\tau)d\tau - \int_0^t G_2(0,t,h,\tau)\mu_2'(\tau)d\tau\right|$$
$$+\nu(M_0)\int_0^t\int_0^h G_{1x}(0,t,\xi,\tau)(1+|v(\xi,\tau)|)d\xi d\tau \le \frac{1}{2}\min_{[0,h]}\varphi'(x). \tag{22}$$

Taking into account estimates (18), (19) we obtain from (22):

$$C_{11}\sqrt{t}\left(\max_{[0,T]}|\mu_1'(t)| + \max_{[0,T]}|\mu_2'(t)|\right) + \nu(M_0)(1+M_1)\int_0^t\int_0^h G_{1x}(0,t,\xi,\tau)d\xi d\tau \le \frac{1}{2}\min_{[0,h]}\varphi'(x).$$

Now it is evident that the choice of number t_1 depends only on the known constants. □

The conditions of uniqueness of solution of problem (1)–(4) are given by the following theorem.

Theorem 2. *Suppose that the following assumptions hold:*
(B1) $b(x,t,u,v), b_u(x,t,u,v), b_v(x,t,u,v) \in C(\overline{Q}_T\times\mathbb{R}^2)$;
(B2) $\mu_3(t)\ne 0, t\in[0,T]$.
Then the solution of problem (1)–(4) is unique.

Proof. Suppose that there exist two solutions $(a_i(t), u_i(x,t)), i\in\{1,2\}$, of problem (1)–(4). Denote $a(t)=a_1(t)-a_2(t), u(x,t)=u_1(x,t)-u_2(x,t)$. The functions $(a(t),u(x,t))$ satisfy the conditions

$$u_t = a_1(t)u_{xx} + b_1(x,t)u_x + b_2(x,t)u + a(t)u_{2xx}(x,t), \quad (x,t)\in Q_T, \tag{23}$$
$$u(x,0)=0,\ x\in[0,h], \quad u(0,t)=u(h,t)=0,\ t\in[0,T], \tag{24}$$
$$a_1(t)u_x(0,t) = -a(t)u_{2x}(0,t),\ t\in[0,T], \tag{25}$$

where the coefficients $b_i(x,t), i\in\{1,2\}$, are evidently determined by $b_u(x,t,u,v), b_v(x,t,u,v)$.

Denote by $G^*(x,t,\xi,\tau)$ the Green function of problem (23), (24). With the aid of the Green function, we find the solution of problem (23), (24):

$$u(x,t)=\int_0^t\int_0^h G^*(x,t,\xi,\tau)a(\tau)u_{2\xi\xi}(\xi,\tau)d\xi d\tau. \tag{26}$$

After substituting (26) into (25) we obtain the homogeneous Volterra integral equation

$$a(t)u_{2x}(0,t) = -a_1(t)\int_0^t\int_0^h G_x^*(0,t,\xi,\tau)a(\tau)u_{2\xi\xi}(\xi,\tau)d\xi d\tau, \quad t\in[0,T]. \tag{27}$$

Since $u_2(x,t)$ satisfies condition (4) and by assumption (B2) $\mu_3(t)\ne 0$, we have $u_{2x}(0,t)\ne 0, t\in[0,T]$. It means that equation (27) is of the second kind and has the unique solution $a(t)\equiv 0, t\in[0,T]$. Then $u(x,t)\equiv 0, (x,t)\in\overline{Q}_T$, as a solution of a homogeneous initial boundary value problem for parabolic equation [11]. The proof is complete. □

Now we establish the continuous dependence of solution of problem (1)–(4) on the given data.

Theorem 3. *Suppose that the following assumptions hold:*

(C1) $(a_i(t), u_i(x,t)) \in C[0,T] \times C^{2,1}(\overline{Q}_T), i \in \{1,2\}$, *are solutions of problems*

$$u_{it} = a_i(t)u_{ixx} + b(x,t,u_i,u_{ix}), \quad (x,t) \in Q_T, \tag{28}$$

$$u_i(x,0) = \varphi(x),\ x \in [0,h], \quad u_i(0,t) = \mu_1(t),\ u_i(h,t) = \mu_2(t), t \in [0,T], \tag{29}$$

$$a_i(t)u_{ix}(0,t) = \mu_3^{(i)}(t),\ t \in [0,T],\ i \in \{1,2\}; \tag{30}$$

(C2) $b(x,t,u,v), b_u(x,t,u,v), b_v(x,t,u,v) \in C(\overline{Q}_T \times \mathbb{R}^2)$;

(C3) $\mu_3^{(i)}(t) > 0, t \in [0,T], i \in \{1,2\}$;

(C4) functions $a_i(t), i \in \{1,2\}$, *verify estimates (19), (13).*

Then for every $\varepsilon > 0$ *there exists* $\delta > 0$ *such that the inequality* $|\mu_3^{(1)}(t) - \mu_3^{(2)}(t)| < \delta, t \in [0,T]$, *implies* $|a_1(t) - a_2(t)| < \varepsilon, t \in [0,T], |u_1(x,t) - u_2(x,t)| < \varepsilon, |u_{1x}(x,t) - u_{2x}(x,t)| < \varepsilon, (x,t) \in \overline{Q}_T$.

Proof. Let $\varepsilon > 0$ be an arbitrary small number and

$$|\mu_3^{(1)}(t) - \mu_3^{(2)}(t)| < \delta, \quad t \in [0,T],$$

with some number $\delta > 0$.

Denote by $G_2^{(i)}(x,t,\xi,\tau)$ the Green function of the equation

$$u_{it} = a_i(t)u_{ixx} \quad i \in \{1,2\},$$

with conditions (29). Reduce problems (28), (29) to the equivalent integro-differential equations with the aid of the Green function

$$u_i(x,t) = \int_0^h G_1^{(i)}(x,t,\xi,0)\varphi(\xi)d\xi + \int_0^t G_{1\xi}^{(i)}(x,t,0,\tau)a_i(\tau)\mu_1(\tau)d\tau$$
$$- \int_0^t G_{1\xi}^{(i)}(x,t,h,\tau)a_i(\tau)\mu_2(\tau)d\tau + \int_0^t\int_0^h G_1^{(i)}(x,t,\xi,\tau)b(\xi,\tau,u_i(\xi,\tau),u_{i\xi}(\xi,\tau))d\xi d\tau, \tag{31}$$

$i \in \{1,2\}$. Find the derivatives of $u_i(x,t)$ with respect to x:

$$u_{ix}(x,t) = \int_0^h G_2^{(i)}(x,t,\xi,0)\varphi'(\xi)d\xi - \int_0^t G_2^{(i)}(x,t,0,\tau)\mu_1'(\tau)d\tau + \int_0^t G_2^{(i)}(x,t,h,\tau)\mu_2'(\tau)d\tau$$
$$+ \int_0^t\int_0^h G_{1x}^{(i)}(x,t,\xi,\tau)b(\xi,\tau,u_i(\xi,\tau),u_{i\xi}(\xi,\tau))d\xi d\tau, \quad i \in \{1,2\}. \tag{32}$$

From (30) we deduce

$$|a_1(t) - a_2(t)| \le \frac{|\mu_3^{(1)}(t) - \mu_3^{(2)}(t)|}{u_{1x}(0,t)} + \frac{|\mu_3^{(2)}(t)||u_{1x}(0,t) - u_{2x}(0,t)|}{u_{1x}(0,t)u_{2x}(0,t)}. \tag{33}$$

Using inequality (12) and assumption (C3) we have

$$u_{1x}(0,t) = \frac{\mu_3(t)}{a_1(t)} > 0, \quad t \in [0,T].$$

Thus we obtain

$$\frac{|\mu_3^{(1)}(t) - \mu_3^{(2)}(t)|}{u_{1x}(0,t)} < \varepsilon, \quad t \in [0,t_1], \tag{34}$$

if $\delta > 0$ is chosen sufficiently small.
From (32) we find

$$|u_{1x}(x,t) - u_{2x}(x,t)| \le \int_0^h |G_2^{(1)}(x,t,\xi,0) - G_2^{(2)}(x,t,\xi,0)|\varphi'(\xi)d\xi +$$
$$+ \int_0^t |G_2^{(1)}(x,t,0,\tau) - G_2^{(2)}(x,t,0,\tau)||\mu_1'(\tau)|d\tau + \int_0^t |G_2^{(1)}(x,t,h,\tau) -$$
$$-G_2^{(2)}(x,t,h,\tau)||\mu_2'(\tau)|d\tau + \int_0^t\int_0^h |G_{1x}^{(1)}(x,t,\xi,\tau)b(\xi,\tau,u_1(\xi,\tau),u_{1\xi}(\xi,\tau)) -$$
$$-G_{1x}^{(2)}(x,t,\xi,\tau)b(\xi,\tau,u_2(\xi,\tau),u_{2\xi}(\xi,\tau))|d\xi d\tau \equiv \sum_{k=1}^4 \Delta_k(x,t).$$

Consider $\Delta_1(x,t)$. Using the properties of Poisson integral, for given $\varepsilon > 0$ one can indicate a number $t_2, 0 < t_2 \le T_0$, such that

$$\left|\int_0^h G_2^{(i)}(x,t,\xi,0)\varphi'(\xi)d\xi - \varphi'(x)\right| < \frac{\varepsilon}{2}, \quad x \in [0,h],\ t \in [0,t_2], \quad i \in \{1,2\}.$$

Hence, we obtain

$$\Delta_1(x,t) < \varepsilon, \quad x \in [0,h], t \in [0,t_2]. \tag{35}$$

For $t \in [t_2, T_0]$ it is easy to establish the estimate

$$|G_2^{(1)}(x,t,\xi,0) - G_2^{(2)}(x,t,\xi,0)| = |\int_{\theta_1(t)}^{\theta_2(t)} \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{\pi s}}\sum_{n=-\infty}^{\infty}\left(\exp\left(-\frac{(x-\xi+2nh)^2}{4s}\right) +\right.\right.$$
$$\left.\left.+\exp\left(-\frac{(x+\xi+2nh)^2}{4s}\right)\right)\right)ds| \le M_3|\theta_1(t) - \theta_2(t)| \le M_3\int_0^t |a_1(\tau) - a_2(\tau)|d\tau, \tag{36}$$

where

$$M_3 = \max_{x,\xi\in[0,h],t\in[A_0t_2,A_1T]}\left|\frac{\partial}{\partial s}\left(\frac{1}{\sqrt{\pi s}}\sum_{n=-\infty}^{\infty}\left(\exp\left(-\frac{(x-\xi+2nh)^2}{4s}\right) + \exp\left(-\frac{(x+\xi+2nh)^2}{4s}\right)\right)\right)\right|.$$

From (35) and (36) it follows

$$\Delta_1(x,t) \le \varepsilon + C_{11}\int_0^t |a_1(\tau) - a_2(\tau)|d\tau, \quad t \in [0,T]. \tag{37}$$

Using the change $\sigma = \theta_1(t) - \theta_1(\tau)$, we evaluate $\Delta_2(x,t)$:

$$\Delta_2(x,t) \le \frac{1}{\sqrt{\pi}A_0} \max_{[0,T]} |\mu_1'(t)| \int\limits_0^{\theta_1(t)} \Bigg| \frac{1}{\sqrt{\sigma}} \sum_{n=-\infty}^{\infty} \exp\Big(-\frac{(x+2nh)^2}{4\sigma}\Big) -$$
$$- \frac{1}{\sqrt{\theta_2(t) - \theta_2(\theta_1^{-1}(\theta_1(t)-\sigma))}} \sum_{n=-\infty}^{\infty} \exp\Big(-\frac{(x+2nh)^2}{\theta_2(t) - \theta_2(\theta_1^{-1}(\theta_1(t)-\sigma))}\Big)\Bigg| d\sigma,$$

where $\theta_1^{-1}(\sigma)$ is a function inverse to $\theta_1(t)$. Taking into account the inequality ([12])

$$\frac{1}{\sqrt{\sigma}} \sum_{n=-\infty}^{\infty} \exp\Big(-\frac{n^2h^2}{\sigma}\Big) \le \frac{1}{\sqrt{\sigma}} + \frac{\sqrt{\pi}}{h},$$

it is easy to indicate a number $t_3, 0 < t_3 \le T_0$, such that

$$\frac{1}{\sqrt{\pi}A_0} \max_{[0,T]} |\mu_1'(t)| \int\limits_0^{t_3} \Bigg| \frac{1}{\sqrt{\sigma}} \sum_{n=-\infty}^{\infty} \exp\Big(-\frac{n^2h^2}{\sigma}\Big) -$$
$$- \frac{1}{\sqrt{\theta_2(t) - \theta_2(\theta_1^{-1}(\theta_1(t)-\sigma))}} \sum_{n=-\infty}^{\infty} \exp\Big(-\frac{n^2h^2}{\theta_2(t) - \theta_2(\theta_1^{-1}(\theta_1(t)-\sigma))}\Big)\Bigg| d\sigma < \varepsilon. \quad (38)$$

For $\sigma \in [t_3, \theta_1(t)]$ we can use the estimation

$$\Bigg| \frac{1}{\sqrt{\theta_2(t) - \theta_2(\theta_1^{-1}(\theta_1(t)-\sigma))}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2h^2}{\sqrt{\theta_2(t) - \theta_2(\theta_1^{-1}(\theta_1(t)-\sigma))}}\right)$$
$$- \frac{1}{\sqrt{\sigma}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2h^2}{\sigma}\right)\Bigg| = \left| \int\limits_{\sigma}^{\theta_2(t)-\theta_2(\theta_1^{-1}(\theta_1(t)-\sigma))} \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{s}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2h^2}{s}\right)\right) ds \right|$$
$$\le M_4 |\theta_2(t) - \theta_2(\theta_1^{-1}(\theta_1(t)-\sigma)) - \sigma|, \quad (39)$$

where $M_4 = \max\limits_{[t_3, A_1T]} \left| \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{s}} \sum\limits_{n=-\infty}^{\infty} \exp\left(-\frac{n^2h^2}{s}\right)\right)\right|$. Using (38) and (39) we obtain

$$\Delta_2(t) < \varepsilon + C_{12} \int\limits_{t_3}^{\theta_1(t)} |\theta_2(t) - \theta_2(\theta_1^{-1}(\theta_1(t)-\sigma)) - \sigma| d\sigma.$$

After the change $\sigma = \theta_1(t) - \theta_1(\tau)$ we establish the estimation

$$\Delta_2(t) < \varepsilon + C_{12} \int\limits_0^{A_1t} |\theta_2(t) - \theta_2(\tau) - \theta_1(t) + \theta_1(\tau)| d\tau \le \varepsilon$$
$$+ C_{12} \int\limits_0^{A_1t} d\tau \int\limits_{\tau}^{t} |a_1(y) - a_2(y)| dy \le \varepsilon + C_{13} \int\limits_0^{A_1t} |a_1(y) - a_2(y)| dy. \quad (40)$$

To evaluate $\Delta_3(t)$, we apply the inequality

$$|G_2^{(1)}(0,t,h,\tau) - G_2^{(2)}(0,t,h,\tau)| \leq C_{14} \int_0^t |a_1(y) - a_2(y)| dy$$

which follows from the estimate

$$\left| \frac{\partial}{\partial s} \left(\frac{1}{\sqrt{\pi s}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(2n+1)^2 h^2}{4s} \right) \right) \right| \leq M_5 < \infty, \quad s \in [0, A_1 T].$$

Then it is easy to verify that

$$\Delta_3(t) \leq C_{15} \int_0^t |a_1(y) - a_2(y)| dy. \tag{41}$$

By the similar arguments we establish that

$$\Delta_4(t) < \varepsilon + C_{16} \int_0^t |a_1(y) - a_2(y)| dy. \tag{42}$$

Now from (37), (40), (41), (42) we obtain

$$|u_{1x}(0,t) - u_{2x}(0,t)| \leq 3\varepsilon + C_{17} \int_0^{A_1 t} |a_1(y) - a_2(y)| dy.$$

From (33) and (34) it follows

$$|a_1(t) - a_2(t)| \leq 4\varepsilon + C_{18} \int_0^{A_1 t} |a_1(y) - a_2(y)| dy.$$

Applying the Gronwall inequality we obtain $|a_1(t) - a_2(t)| < C_{19}\varepsilon, \quad t \in [0, t_1]$. To obtain the estimations

$$|u_1(x,t) - u_2(x,t)| < \varepsilon, \quad |u_{1x}(x,t) - u_{2x}(x,t)| < \varepsilon, \quad (x,t) \in \overline{Q}_{t_1},$$

it is sufficient to repeat the preceding arguments. The proof is complete. □

By the same approach one can investigate an analogous inverse problem for more general equation

$$u_t = a(t) a_0(x) u_{xx} + b(x,t,u,u_x) \tag{43}$$

with given function $a_0(x) > 0$. We state the corresponding theorem.

Theorem 4. *Suppose that, in a addition to assumptions (A1)–(A4), the following assumption holds:*

(A6) $a_0(x) \in C^1[0,h], a_0(x) > 0, x \in [0,h]$.

Then there exists a solution of problem (43), (34), (2)–(4).

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