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CONCENTRATION OF ZEROS AND POLES, h-MEASURES, AND UNIFORM LOGARITHMIC DERIVATIVE ESTIMATES OF MEROMORPHIC FUNCTIONS

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We prove new uniform estimates of the logarithmic derivative $f'(re^{i\theta})/f(re^{i\theta})$ of a meromorphic function f in terms of the Nevanlinna characteristic $T(r,f)$ and a number of poles and zeros in some neighborhood of z outside an exceptional set of finite h-measure. Obtained results are sharp when f has finite order of the growth and improve known results in the general case.

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Получены новые оценки логарифмической производной $f'(re^{i\theta})/f(re^{i\theta})$ произвольной мероморфной функции f в терминах характеристики Неванлинны $T(r,f)$ и числа полюсов и нулей в некоторой окрестности z вне исключительного множества конечной h-меры. Полученные результаты являются точными, когда f имеет конечный порядок роста, и уточняют известные результаты в общем случае.

1. Introduction and main results. We assume that the reader is familiar with standard notation and fundamental results of the theory of meromorphic functions in $\mathbb{C}$ ([1], [2]). We denote by $\operatorname{mes} E$ the Lebesgue measure of a measurable set $E \subset \mathbb{R}$. The symbol C with indices stands for some positive constants that depend on variables going in braces. We write $a \asymp b$ if $C_1 a(r) \le b \le C_2 a(r)$ for some positive constants C_1 and C_2, and $a(r) \sim b(r)$ if $\lim_{r\to 1} a(r)/b(r) = 1$.

Let f be a meromorphic function in a domain $D(0,R)$, $0 < R \le +\infty$, where $D(z,r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$. Estimates of the so-called logarithmic derivative $f'(z)/f(z)$ or, more general, of the expression $f^{(k)}(z)/f(z)$, $k \in \mathbb{N}$, play an important role in the Nevanlinna theory ([1, 2]), complex differential equations ([3], [4], [5], [6]), and difference quotients ([7]). Estimates of $f'(z)/f(z)$ in a neighbourhood of a maximum modulus point z_0, i.e. $|f(z_0)| = M(|z_0|, f)$ play important role in Macintyre's approach to the Wiman-Valiron theory ([8, 3]). A sharp upper estimate for $m(r, f'/f)$ via $T(\rho, f)$ where f is meromorphic in $D(0,R)$, $0 < r < \rho < R$ is proved by A. Gol'dberg and V. Grinshtein in [9] (see also [10]). If we estimate $m(r, f'/f)$ via $T(r,f)$, then an exceptional set can appear. Sharp estimates of $m(r, f'/f)$ outside exceptional sets were obtained by J. Miles ([11]) and M. Jankowski ([12]).

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But in differential equations we need also uniform estimates of the logarithmic derivative. There are always exceptional sets, except trivial cases, because the logarithmic derivative of f has simple poles at the zeros and poles of f. For meromorphic functions of finite order in $\mathbb{C}$ sharp estimates were proved in [4] and for meromorphic functions in the unit disk in [5] (see also [6]).

In [13] the author improved Gundersen's and Strelitz' results in the case when an exceptional set has finite logarithmic measure and proved the following theorem.

Theorem A. *Let f be a non-constant meromorphic function in $\mathbb{C}$. Let $\alpha > 1$ be a constant, ψ a positive non-decreasing function on $(0, +\infty)$ satisfying $\int_0^\infty \frac{dt}{\psi(t)} < +\infty$. Then there exists a sequence of disks $D_j = D(z_j, r_j)$ ($|z_j| > 0$) such that $\sum_j r_j/|z_j| < +\infty$ and*

$$\Big|\frac{f'(z)}{f(z)}\Big| \le C(\alpha)\Big(\frac{T(\alpha r, f)}{r} + \psi(\log(\alpha r))\frac{n(\alpha r, 0, \infty, f)}{r}\sqrt{\log^+ n(\alpha r, 0, \infty, f)}\Big),\ z \notin \bigcup_j D_j, \tag{1.1}$$

where $C(\alpha) = \frac{4\alpha}{(\sqrt{\alpha}-1)^2}$.

Both addends in the right-hand side of estimate (1.1) are sharp up to a constant factor ([13]). For the non-vanishing entire function $g(z) = \exp\{z^\rho\}$, $\rho \in \mathbb{N}$ we have $T(r, g) \asymp r^\rho$, and $\Big|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\Big| \asymp r^{\rho-1}$ as $r \to \infty$. It shows that the first addend in the right-hand side of estimate (1.1) is the best possible.

A defect of (1.1) and many other mentioned results is that they do not give the best possible estimate when z is far from the zero set of f, in particular in a neighborhood of a maximum modulus point. The main idea of the present paper is to obtain estimates of $f'(z)/f(z)$ in terms of number of zeros and poles in some neighbourhood of z instead of $n(r, 0, \infty, f)$, and outside exceptional sets of finite h-measure (cf. [18]). We prove sharp estimates in case of functions of finite order. For functions of an arbitrary growth, our estimates improve similar results by Sh. Strelitz ([14]). Let

$$n_z(h) = n_z(h, 0, \infty, f) = \mathrm{card}\{a_n : |a_n - z| \le h|z|\},$$

where a_n are zeros of a meromorphic function f.

Theorem 1. *Let f be a non-constant meromorphic function in $\mathbb{C}$. Let $\beta > 1$ be a constant, ψ a positive non-decreasing function on $(0, +\infty)$ satisfying $\int_0^\infty \frac{dt}{\psi(t)} < +\infty$, $h(r)$ be a positive increasing unbounded differentiable function on $(0, +\infty)$ with $h'(r)r\psi(\log r) \to +\infty$ and $h'(2r) \asymp h'(r)$ $(r \to +\infty)$. Then there exists a sequence of disks $D_j = D(z_j, r_j)$ such that $\sum_j r_j h'(|z_j|) < +\infty$ and*

$$\Big|\frac{f'(z)}{f(z)}\Big| \le C(\beta)\Big(\frac{T(\beta r, f)}{r} + \psi(\log(\beta r))h'(r)n_z((\beta-1), 0, \infty, f)\sqrt{\log^+ n_z((\beta-1), 0, \infty, f) + 1}\Big),\ z \notin \bigcup_j D_j, \tag{1.2}$$

where $C(\beta) = O((\beta-1)^{-2})$ as $\beta \to 1$.

Corollary 1. *Suppose that the assumptions of Theorem 1 are satisfied and all zeros and poles, except possibly a finite number, lay on finite number of rays emanating from the origin. Then*

$$\Big|\frac{f'(z)}{f(z)}\Big| \le C(\beta)\frac{T(\beta r, f)}{r}, \quad z \notin \mathcal{A}, \beta > 1 \tag{1.3}$$

where $\mathcal{A}$ is a union of open angles of opening $\beta - 1$ containing the rays with zeros and poles.

Theorem 2 shows that the description of an exceptional set in (1.2) is best possible.

Theorem 2. *For an arbitrary $\rho \in (0, +\infty)$, an arbitrary positive non-decreasing function ψ satisfying $\int_0^\infty \frac{dt}{\psi(t)} = +\infty$, and an arbitrary positive increasing unbounded differentiable function $h(r)$ on $(0, +\infty)$ with $h'(r)r\psi(\log r) \ge 1$ and $h'(2r) \asymp h'(r)$ $(r \to +\infty)$ there exists an entire function such that $n(r, 0, g) \asymp T(r, g) \asymp r^\rho$ as $r \to +\infty$, and for any covering $\{D_j\}$, $D_j = D(z_j, r_j)$, $z_j \ne 0$ of the set*

$$\Big\{z \in \mathbb{C} : \Big|\frac{g'(z)}{g(z)}\Big| \ge \frac{n_z(0.01, 0, g)}{r}\sqrt{\log^+ n_z(0.01, 0, g)}h'(r)\psi(\log^+ r)\Big\} \tag{1.4}$$

we have $\sum_j r_j h'(|z_j|) = +\infty$.

Theorem 1 yields a 'good' estimate for the logarithmic derivative of a meromorphic functions of finite order. If f has infinite order, then $T(\alpha r, f)$ cannot be compared with $T(r, f)$ as $r \to +\infty$. Here we prove the following result.

Theorem 3. *Let f be a meromorphic function in $\mathbb{C}$, ψ a positive function on $(0, +\infty)$ such that $\varepsilon(t) = \psi(t)/t$ is non-decreasing and $\int_0^\infty \frac{dt}{\psi(t)} < +\infty$, $h(r)$ a positive increasing unbounded differentiable function on $(0, +\infty)$ with*

$$\frac{1}{B}h'(r) \le h'(\rho) \le Bh'(r), \quad r \le \rho \le r + \frac{1}{h'(r)\varepsilon(T(r, f))}, r \to +\infty, \tag{1.5}$$

$$h'(r)r\varepsilon(T(r, f)) \to +\infty, r \to +\infty, \tag{1.6}$$

Then there exists a measurable set $F \subset \mathbb{R}_+$ such that $\int_F dh(r) < +\infty$ and for $|z| \notin F$ we have

$$\Big|\frac{f'(z)}{f(z)}\Big| \le C_1\Big(rT(r, f)(h'(r)\varepsilon(T(r, f)))^2 +$$

$$+h'(r)\psi\Big(\varepsilon\big(T(r, f)\big)h(r)\Big)n_z\Big(\frac{1}{rh'(r)\varepsilon(T(r, f))}, f\Big)\sqrt{\log^+ n_z\Big(\frac{1}{rh'(r)\varepsilon(T(r, f))}, f\Big) + 1}, \tag{1.7}$$

where $n_z\Big(\tau, f\Big) = n_z\Big(\tau, 0, \infty, f\Big)$, C_1 is a constant.

Corollary 2. *Suppose that the assumptions of Theorem 3 are satisfied. Then there exists a measurable set $F \subset \mathbb{R}_+$ such that $\int_F dh(r) < +\infty$ and for $|z| \notin F$ we have*

$$\Big|\frac{f'(z)}{f(z)}\Big| \le C_2\Big(T(r, f)\varepsilon^2(T(r, f))rh(r)\varepsilon\Big(\varepsilon\big(T(r, f)h(r)\big)\Big)(h'(r))^2 \times$$

$$\times\Big(\sqrt{\log^+ T(r, f) + \log(rh'(r)\varepsilon(T(r, f))}\Big), \tag{1.8}$$

where $\varepsilon(t) = \psi(t)/t$, C is a constant.

Corollary 3. *Let f be a meromorphic function in $\mathbb{C}$ of positive lower order, i.e. $T(r,f)\ge r^\eta$ for some $\eta>0$ as $r\to+\infty$, ψ a positive function on $(0,+\infty)$ such that $\varepsilon(t)=\psi(t)/t$ is non-decreasing and $\int_0^\infty \frac{dt}{\psi(t)}<+\infty$, $h(r)$ a positive increasing unbounded differentiable function on $(0,+\infty)$ such that $1/Bh'(r)\le h'((1+o(1))r)\le Bh'(r)$ for some constant $B>1$ and $h'(r)r\log r\ge\delta$, $\delta>0$ as $r\to+\infty$. Then there exists a set F of finite h-measure such that (1.7) holds outside F.*

Remark 1. We note that the choice $\varepsilon(x)=x^p$, $p>0$ or $\varepsilon(x)=(\log(x+3))^q$, or $\varepsilon(x)=\log(x+3)(\log\log(x+3))^q$, $q>1$ is always possible.

Remark 2. The estimate given by Corollary 2 improves an analogous estimate from [14, p.133].

Our proofs of the theorems are based on the recent deep results by J.M. Anderson and V.Ya.Eiderman ([15], [16]), lemmas of Borel-Nevanlinna's type and, in the case of Theorem 3, on a special construction of an exhaustion of the complex plane.

Theorem B ([15, 16])**.** *Let $\mathcal{Z}=\{z_1,z_2,\dots,z_N\}\subset\mathbb{C}$, $N>1$. There is an absolute constant c such that for every $P>0$ there exists a finite set of disks $D_j=D(w_j,r_j)$ with the properties:*

(1) $\left|\sum\limits_{k=1}^{N}\dfrac{1}{z-z_k}\right|<P,\quad z\in\mathbb{C}\setminus\bigcup_j D_j;$

(2) $\sum_j r_j<\frac{c}{P}N\sqrt{\log N};$

(3) $(\forall j)$ $D_j\cap\mathcal{Z}\ne\varnothing.$

I think that similar results can be obtained for meromorphic functions in the unit disc as well as estimates of $f^{(l)}/f^{(k)}(z)$, $l>k$, $l,k\in\mathbb{Z}_+$. In the last case we should take into account zeros and poles of all derivatives of order $j\in[k,l]$.

2. Preliminaries. Let f be a meromorphic function in $D(0,R_0)$, $0<R_0\le\infty$, $\{a_\mu\}$ and $\{b_\nu\}$ denote the sequences of all zeros and poles of f, respectively. From the differentiated Nevanlinna's formula [1, Theorem 2.4, p.17] ($|z|=r<R$) we obtain

$$\Big|\frac{f'(z)}{f(z)}\Big|\le\frac{R}{\pi}\int_0^{2\pi}\frac{|\log|f(Re^{i\theta})||\,d\theta}{|Re^{i\theta}-z|^2}+\Big|\sum_{|a_\mu|<R}\frac{1}{z-a_\mu}\Big|+$$
$$+\Big|\sum_{|b_\nu|<R}\frac{1}{z-b_\nu}\Big|+\Big|\sum_{|a_\mu|<R}\frac{1}{z-\frac{R^2}{\overline{a}_\mu}}\Big|+\Big|\sum_{|b_\nu|<R}\frac{1}{z-\frac{R^2}{\overline{b}_\nu}}\Big|\equiv I+\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4. \tag{2.1}$$

The estimate of the integral I is standard:

$$I\le\frac{R}{\pi(R-r)^2}\int_0^{2\pi}\Big(\log^+|f(Re^{i\theta})|+\log^+\Big|\frac{1}{f(Re^{i\theta})}\Big|\Big)\,d\theta\le$$
$$\le\frac{2R}{(R-r)^2}(m(R,f)+m(R,1/f))\le\frac{4R}{(R-r)^2}(T(R,f)+O(1)),\quad R\uparrow R_0. \tag{2.2}$$

We are going to deduce a general estimates for the sums from (2.1). Let (α_ν) be a sequence of positive numbers increasing to $R_0\in(0,+\infty]$. We write $A_\nu=\{\zeta:\alpha_{\nu-1}<|\zeta|\le\alpha_\nu\}$.

Without loss of generality, we may assume that $z\in A_\nu$, $\arg z=0$. Let

$$A_{s\mu}=\Big\{\zeta\in A_s:\frac{(\mu-1)\pi}{N_s}\le\arg\zeta<\frac{\mu\pi}{N_s}\Big\},$$

$$\mu \in I_s = \{-N_s+1, \dots, 0, 1, \dots, N_s\}, \quad s \in \mathbb{N}, \quad N_\nu = 2\Big[\frac{\pi\alpha_\nu}{\alpha_\nu - \alpha_{\nu-1}}\Big].$$

Let $I_s^* = \begin{cases} I_s, & 1 \le s \le \nu-2, \\ I_s \setminus \{0,1\}, & \nu-1 \le s \le \nu+1. \end{cases}$

Lemma 1. *Let $\nu \ge 2$, $\nu+1 \ge s \ge 2$, $\mu \in I_s^*$, $\zeta \in A_{s\mu}$. Then*

$$|\zeta - z| \ge \begin{cases} \frac{(\mu-1)\alpha_{s-1}}{\pi\alpha_s}(\alpha_s - \alpha_{s-1}), & \mu > 1, \\ \frac{|\mu|\alpha_{s-1}}{\pi\alpha_s}(\alpha_s - \alpha_{s-1}), & \mu \le -1, \\ \alpha_{\nu-1} - \alpha_s, & \mu \in \{0,1\}. \end{cases}$$

Proof of Lemma 1. Suppose first that $|\arg\zeta| \le \frac{\pi}{2}$. Since $\zeta \in A_{s\mu}$ and $\arg z = 0$, we have

$$|z-\zeta| \ge |\zeta|\sin(\arg\zeta) \ge \alpha_{s-1}\sin\Big(\frac{(\mu-1)}{N_s}\pi\Big) \ge \alpha_{s-1}\frac{2(\mu-1)}{N_s} \ge$$
$$\ge \frac{(\mu-1)\alpha_{s-1}}{\pi\alpha_s}(\alpha_s - \alpha_{s-1}), \quad \mu > 0.$$

Similarly, $|\zeta - z| \ge \frac{|\mu|\alpha_{s-1}}{\pi\alpha_s}(\alpha_s - \alpha_{s-1})$ if $\mu \le 0$. Hence,

$$|\zeta - z| \ge \begin{cases} \frac{(\mu-1)\alpha_{s-1}}{\pi\alpha_s}(\alpha_s - \alpha_{s-1}), & \mu > 1, \\ \frac{|\mu|\alpha_{s-1}}{\pi\alpha_s}(\alpha_s - \alpha_{s-1}), & \mu \le -1. \end{cases} \tag{2.3}$$

If $1 \le s \le \nu - 2$, we have in addition that

$$|\zeta - z| \ge |z| - \operatorname{Re}\zeta \ge \alpha_{\nu-1} - \alpha_s \ge \alpha_{\nu-1} - \alpha_{\nu-2}. \tag{2.4}$$

Applying (2.3) for $\mu \notin \{0,1\}$ and (2.4) for $\mu \in \{0,1\}$ we obtain the statement of Lemma 1 in the case $|\arg\zeta| \le \pi/2$.

Suppose then that $\frac{\pi}{2} \le |\arg\zeta| \le \pi$. In this case either $N_s/2 \le \mu \le N_s$ or $-N_s+1 \le \mu \le -N_s/2+1$. The statement of Lemma 1 now follows by the inequality $|z-\zeta| \ge \alpha_{\nu-1}$. □

In order to estimate Σ_j, $j \in \{1,2,3,4\}$ we use Theorem B, which plays a key role in our proof.

Lemma 2. *Let (α_ν) be a sequence of positive numbers increasing to $R_0 \in (0,+\infty]$, φ a non-decreasing positive function, (c_ν) a sequence of complex numbers in $D(0,R_0)$ without accumulation points in $D(0,R_0)$ listed according the multiplicities and ordered by increasing moduli and let $n(r)$ denote the counting function of the sequence (c_ν). Then for every $\nu \in \mathbb{N}$, $\alpha_{\nu-1} < |z| \le \alpha_\nu$, and $z \notin \bigcup_j D(z_{\nu j}, r_{\nu j})$ we have*

$$\Big|\sum_{|c_\mu|\le\alpha_{\nu+1}} \frac{1}{z-c_\mu}\Big| + \Big|\sum_{|c_\mu|\le\alpha_{\nu+1}} \frac{1}{z-\frac{\alpha_{\nu+1}^2}{\bar c_\mu}}\Big| \le \max_{\nu-1\le s\le\nu+1} \frac{2\pi\alpha_s}{\alpha_{s-1}(\alpha_s-\alpha_{s-1})} n(\alpha_{\nu+1}) +$$
$$+\frac{2\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n_z\Big(\frac{2(\alpha_{\nu+1}-\alpha_{\nu-2})}{|z|}\Big)\sqrt{\log^+ n_z\Big(\frac{2(\alpha_{\nu+1}-\alpha_{\nu-2})}{|z|}\Big)+1}, \tag{2.5}$$

where

$$\alpha_{\nu-2} - \frac{c\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})} \le |z_{\nu j}| \le \frac{\alpha_{\nu+1}^2}{\alpha_{\nu-2}} + \frac{c\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}, \tag{2.6}$$

c is an absolute constant, $\sum_j r_{\nu j} \le \frac{2c\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}$.

Proof of the lemma. Estimate the first sum in (2.5). Let $A^*_\nu = \bigcup_{s=\nu-1}^{\nu+1}\bigcup_{\mu=0}^{1} A_{s\mu}$. First, we suppose that $n(A^*_\nu) > 1$. We set

$$P_\nu = \frac{\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(A^*_\nu)\sqrt{\log^+ n(A^*_\nu)}. \tag{2.7}$$

Applying Theorem B to the set of $\{c_\mu\}$ that are contained in A^*_ν we conclude that there exist disks $D_{\nu j} = D(z_{\nu j}, r_{\nu j})$, $1 \le j \le j_\nu$ such that

$$\Big|\sum_{c_\mu\in A^*_\nu} \frac{1}{z-c_\mu}\Big| \le P_\nu, \quad z\in A^*_\nu \setminus \bigcup_j D_{\nu j} \tag{2.8}$$

$$\sum_{j=1}^{j_\nu} r_{\nu j} \le c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}, \tag{2.9}$$

$$\alpha_{\nu-2} - c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})} \le |z_{\nu j}| \le \alpha_{\nu+1} + c\frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}. \tag{2.10}$$

According to Lemma 1 we have

$$\sum_{c_\mu\in \overline{D(0,\alpha_{\nu+1})}\setminus A^*_\nu} \frac{1}{|z-c_\mu|} = \Big(\sum_{s=\nu-1}^{\nu+1}\sum_{\mu\in I^*_s} + \sum_{|c_\mu|\le\alpha_{\nu-2}}\Big)\frac{1}{|z-c_\mu|} \le$$
$$\le \sum_{s=\nu-1}^{\nu+1} \frac{\pi\alpha_s(n(\alpha_s)-n(\alpha_{s-1}))}{\alpha_{s-1}(\alpha_s-\alpha_{s-1})} + \frac{n(\alpha_{\nu-2})}{\alpha_{\nu-1}-\alpha_{\nu-2}} \le \pi \max_{\nu-1\le s\le\nu+1} \frac{\alpha_s}{\alpha_{s-1}(\alpha_s-\alpha_{s-1})} n(\alpha_{\nu+1}). \tag{2.11}$$

Hence, using (2.8) and (2.11) we obtain

$$\Big|\sum_{|c_\mu|\le\alpha_{\nu+1}} \frac{1}{z-c_\mu}\Big| \le \frac{\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n(A^*_\nu)\sqrt{\log^+ n(A^*_\nu)} + \max_{\nu-1\le s\le\nu+1} \frac{\pi\alpha_s}{\alpha_{s-1}(\alpha_s-\alpha_{s-1})} n(\alpha_{\nu+1}). \tag{2.12}$$

Then, we show that $A^*_\nu \subset D(z, 2(\alpha_{\nu+1}-\alpha_{\nu-2}))$. Recall that, $z\in[\alpha_{\nu-1},\alpha_\nu]$. Since $\frac{\pi\alpha_s}{\alpha_s-\alpha_{s-1}} \ge 3$, we have

$$N_s = 2\Big[\frac{\pi\alpha_s}{\alpha_s-\alpha_{s-1}}\Big] \ge \frac{3\pi\alpha_s}{2(\alpha_s-\alpha_{s-1})}.$$

Hence, using the definition of $A_{s,\mu}$, $0\le\mu\le1$, $\nu-1\le s\le\nu+1$, we obtain

$$\sup_{\zeta\in A^*_\nu} |z-\zeta| \le \max_{\nu\le s\le\nu+1}(\alpha_s-\alpha_{s-2}) + \max_{\nu-1\le s\le\nu+1}\Big\{\alpha_s\frac{\pi}{N_s}\Big\} \le$$
$$\le (\alpha_{\nu+1}-\alpha_{\nu-2}) + \frac{2}{3}\max_{\nu-1\le s\le\nu+1}(\alpha_s-\alpha_{s-1}) \le 2(\alpha_{\nu+1}-\alpha_{\nu-2}). \tag{2.13}$$

Therefore, $A^*_\nu \subset D(z, 2(\alpha_{\nu+1}-\alpha_{\nu-2}))$, and consequently, $n(A^*_\nu) \le n_z(2(\alpha_{\nu+1}-\alpha_{\nu-2})/|z|)$.

The second sum in (2.5) is estimated similarly (see [13, Proof of Lemma 1]). If $n(A^*_\nu) = 0$ inequality (2.8) is trivial, and using (2.11) we obtain (2.5) without exceptional sets. If $n(A^*_\nu)=1$, we have $|z-c_\mu|^{-1} \le \varphi(\alpha_{\nu+1})n(A^*_\nu)/\alpha_{\nu+1}$ and $|z-c^*_\mu|^{-1} \le \varphi(\alpha_{\nu+1})n(A^*_\nu)/\alpha_{\nu+1}$ outside $D(c_\mu, \frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})n(A^*_\nu)}) \cup D(c^*_\mu, \frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})n(A^*_\nu)})$ for the unique $c_\mu\in A^*_\nu$. Therefore (2.8)–(2.10) hold. Consequently we obtain the statement of Lemma 2. □

In order to work with h-measures we need Borel-Nevanlinna's type lemma. A proof of the following lemma bases on a slight generalization of the case when an exceptional set is of finite measure.

Lemma 3. *Let $v\colon [r_0,\infty)\to[v_0,\infty)$ be a non-decreasing unbounded function, $h\colon [r_0,\infty)\to[v_0,\infty)$ be an increasing differentiable unbounded function, $\varepsilon\colon [v_0,\infty)\to[1,\infty)$ be a non-decreasing function such that $\int_{v_0}^{\infty}\frac{dv}{v\varepsilon(v)}<+\infty$. If $h'(r)$ is not a non-increasing function for sufficiently large r, we assume, in addition, that for some $C>1$*

$$h'\Big(h^{-1}\Big(h(r)+\frac{C}{\varepsilon(v(r))}\Big)\Big)\le Ch'(r),\quad r\to+\infty. \tag{2.14}$$

Then

$$v\Big(r+\frac{1}{h'(r)\varepsilon(v(r))}\Big)\le ev(r),\quad r\notin E\subset[r_0,\infty),$$

where $\int_E dh(r)<+\infty$.

Remark 4. Note that inequality (2.14) is a regularity condition, which does not restrict the growth of h. It is easy to check that $h(r)=\exp_k r$, $k\in\mathbb{N}$ where $\exp_k$ is k-iteration of the exponent satisfies (2.14).

3. Proofs of the main results.

Proof of Theorem 1. Let us estimate $\sum_j$, $1\le j\le 4$ from (2.1). We may suppose that $n(r,0,f)$ and $n(r,\infty,f)$ are unbounded. Otherwise, corresponding sums in (2.1) are bounded as $R\to+\infty$.

Given $\beta>1$ we put $\sigma=\sqrt{\beta}-1>0$, and choose $2>\alpha>1$ satisfying $2\Big(\alpha-\frac{1}{\sqrt{\alpha}}\Big)\le\sigma$.

Let $\alpha_\nu=\sqrt{\alpha^\nu}$. Using the restrictions on α we deduce

$$2(\alpha_{\nu+1}-\alpha_\nu)\le 2\alpha_{\nu-1}\Big(\alpha-\frac{1}{\sqrt{\alpha}}\Big)\le\sigma|z|. \tag{3.1}$$

Recall that $A_\nu=\{\zeta:\alpha_{\nu-1}<|\zeta|\le\alpha_\nu\}$, $\nu\in\mathbb{N}$. We choose $\varphi(t)=h'(t)t\psi(\log^+ t)$. Fixing $\nu\in\mathbb{N}$ we apply Lemma 2 to the zero set $\{a_\mu\}$ of f, $R_0=\infty$. There exists a finite collection of disks $D_{\nu j}=D(z_{\nu j},r_{\nu j})$ $(1\le j\le j_\nu)$ such that

$$\begin{aligned}
&\Big|\sum_{|a_\mu|\le\alpha_{\nu+1}}\frac{1}{z-a_\mu}\Big|+\Big|\sum_{|a_\mu|\le\alpha_{\nu+1}}\frac{1}{z-\frac{\alpha_{\nu+1}^2}{\bar a_\mu}}\Big|\le\max_{\nu-1\le s\le\nu+1}\frac{2\pi\alpha_s n(\alpha_{\nu+1})}{\alpha_{s-1}(\alpha_s-\alpha_{s-1})}+\\
&\le\frac{2\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}}n_z\Big(\frac{2(\alpha_{\nu+1}-\alpha_{\nu-2})}{|z|}\Big)\sqrt{\log^+ n_z\Big(\frac{2(\alpha_{\nu+1}-\alpha_{\nu-2})}{|z|}\Big)+1}\le\\
&\le\frac{2\pi\alpha n(\alpha_{\nu+1},0,f)}{r(\sqrt{\alpha}-1)}+\frac{2h'(\alpha_{\nu+1})\psi(\log(\alpha r))}{\sqrt{\alpha}}n_z(\sigma;0,f)\sqrt{\log^+ n_z(\sigma;0,f)+1}\le\\
&\le\frac{2\pi\alpha N(\alpha_{\nu+2},0,f)}{r(\sqrt{\alpha}-1)\log\sqrt{\alpha}}+\frac{Ch'(r)\psi(\log(\alpha r))}{\sqrt{\alpha}}n_z(\sigma;0,f)\sqrt{\log^+ n_z(\sigma;0,f)+1}\\
&\le\frac{4\pi\alpha T(\alpha^{3/2}r,0,f)}{r(\sqrt{\alpha}-1)\log\alpha}+\frac{Ch'(r)\psi(\log(\alpha r))}{\sqrt{\alpha}}n_z(\sigma;0,f)\sqrt{\log^+ n_z(\sigma;0,f)+1}\le\\
&\le C(\beta)\Big(\frac{T(\beta r,f)}{r}+h'(r)\psi(\log(\beta r))n_z(\sigma;0,f)\sqrt{\log^+ n_z(\sigma;0,f)+1}\Big),\quad z\in A_\nu\setminus\bigcup_{j=1}^{j_\nu}D_{\nu j},
\end{aligned} \tag{3.2}$$

where $C(\beta)=O\big((\beta-1)^2\big)$ $(\beta\downarrow 1)$.

Similarly, there exists a finite collection of disks $D^*_{\nu j}=D(z^*_{\nu j},r^*_{\nu j})$ $(1\le j\le j^*_\nu)$ such that

$$\Big|\sum_{|b_\nu|\le\alpha_{\nu+1}}\frac{1}{z-b_\nu}\Big|+\Big|\sum_{|b_\nu|\le\alpha_{\nu+1}}\frac{1}{z-\frac{\alpha^2_{\nu+1}}{\bar b_\nu}}\Big|\le$$
$$\le C(\beta)\Big(\frac{T(\beta r,f)}{r}+h'(r)\psi(\log(\beta r))n_z(\sigma;\infty,f)\sqrt{\log^+ n_z(\sigma;\infty,f)+1}\Big),\quad z\in A_\nu\setminus\bigcup_{j=1}^{j^*_\nu}D^*_{\nu j}.$$

By (2.2)

$$|I|\le\frac{4\alpha_{\nu+1}}{(\alpha_{\nu+1}-r)^2}(T(\alpha_{\nu+1},f)+O(1))\le\frac{4\alpha}{(\sqrt{\alpha}-1)^2r}(T(\alpha r,f)+O(1)),\quad r\to+\infty.$$

Substituting the latter estimates in (2.1) we obtain ($|z|=r$)

$$\Big|\frac{f'(z)}{f(z)}\Big|\le C(\beta)\Big(\frac{T(\beta r,f)}{r}+h'(r)\psi(\log(\beta r))n_z(\sigma;0,\infty,f)r\sqrt{\log^+ n_z(\sigma;0,\infty,f)+1}\Big),\tag{3.3}$$

$z\in\mathbb{C}\setminus\Omega$, where $\Omega=\bigcup_{\nu=1}^\infty\Big(\bigcup_{j=1}^{j_\nu}D_{\nu j}\cup\bigcup_{j=1}^{j^*_\nu}D^*_{\nu j}\Big)$. It remains to estimates the exceptional set Ω. By Lemma 2 we have

$$\alpha_{\nu+1}\Big(\frac{1}{\sqrt{\alpha^3}}-\frac{c}{h'(\alpha_{\nu+1})\alpha_{\nu+1}\psi(\log\alpha_{\nu+1})}\Big)\le|z_{\nu j}|\le\alpha_{\nu+1}\Big(\sqrt{\alpha^3}+\frac{c}{h'(\alpha_{\nu+1})\alpha_{\nu+1}\psi(\log\alpha_{\nu+1})}\Big).$$

Using the properties of the function h, we deduce $\alpha^{-2}\le|z_{\nu j}|/\alpha_{\nu+1}\le\alpha^2$ for all ν greater than some ν_1. Hence,

$$\sum_{\nu=\nu_1}^\infty\sum_{j=1}^{j_\nu}h'(|z_{\nu j}|)r_{\nu j}\le\sum_{\nu=\nu_1}^\infty Ch'(\alpha_{\nu+1})\frac{\alpha_\nu}{\varphi(\alpha_{\nu+1})}\le\sum_{\nu=1}^\infty\frac{C(\beta)}{\psi(\log\alpha_{\nu+1})}=\sum_{\nu=1}^\infty\frac{C(\beta)}{\psi(\frac{\nu}{2}\log\alpha)}<\infty.$$

Similarly, $\sum_{\nu=1}^\infty\sum_{j=1}^{j^*_\nu}r^*_{\nu j}h'(|z^*_{\nu j}|)<\infty$. Theorem 1 is proved. □

Proof of Theorem 3. Let $\varepsilon(t)=\psi(t)/t$. We shall write $T(r)$ instead of $T(r,f)$.

We define a sequence (α_ν) by the induction. Let α_0 be such that $T(\alpha_0)>0$,

$$\alpha_{\nu+1}=\alpha_\nu+\frac{1}{(1+B)^2h'(\alpha_\nu)\varepsilon\Big(T(\frac{\alpha^2_{\nu+1}}{\alpha_\nu})\Big)},\quad \nu\in\mathbb{Z}_+.\tag{3.4}$$

It is easy to see that $\alpha_\nu\uparrow+\infty$ and $\alpha_{\nu+1}\sim\alpha_\nu$ as $\nu\to+\infty$, by (1.6). By our assumption (1.5) we have $1/Bh'(\alpha_{\nu-1})\le h'(r)\le Bh'(\alpha_{\nu-1})$, for all $r\in[\alpha_{\nu-1},\alpha_\nu]$, $\nu\in\mathbb{N}$. Moreover, denoting $\frac{\alpha^2_{\nu+1}}{\alpha_\nu}=\alpha^*_\nu$ we deduce

$$\alpha_{\nu+1}=\alpha_{\nu-1}+\frac{1}{(1+B)^2h'(\alpha_\nu)\varepsilon(T(\alpha^*_\nu))}+\frac{1}{(1+B)^2h'(\alpha_{\nu-1})\varepsilon(T(\alpha^*_{\nu-1}))}\le$$
$$\le\alpha_{\nu-1}+\frac{1}{(1+B)h'(\alpha_{\nu-1})\varepsilon(T(\alpha^*_{\nu-1}))}<\alpha_{\nu-1}+\frac{1}{h'(\alpha_{\nu-1})\varepsilon(T(\alpha_{\nu-1}))}.$$

Therefore, by (1.5), $1/Bh'(\alpha_{\nu-1}) \le h'(r) \le Bh'(\alpha_{\nu-1})$, for all $r \in [\alpha_{\nu-1}, \alpha_{\nu+1}]$

Without loss of generality we may assume that $f(0) = 1$. Then, it is well-known that for $\Delta > 0$, $\rho > 0$, $\rho' = \rho e^{\Delta}$, $n(\rho, a) \le \frac{T(\rho')}{\Delta}$ for $a \in \{0, \infty\}$. We choose $\rho = \alpha_{\nu+1}$, $\rho' = \alpha_\nu^*$. Then

$$n(\alpha_{\nu+1}, a) \le \frac{T(\alpha_\nu^*)}{\log \frac{\alpha_{\nu+1}}{\alpha_\nu}} = (1 + o(1))(1 + B)^2 \alpha_\nu h'(\alpha_\nu) T\big(\alpha_\nu^*\big)\varepsilon\big(T(\alpha_\nu^*)\big), \quad a \in \{0, \infty\}. \tag{3.5}$$

Now we apply Lemma 2 with $\varphi(\alpha_\nu) = h'(\alpha_\nu)\alpha_\nu\psi(\nu)$. By (3.5) we have

$$\begin{aligned}\Big|\sum_{|a_\mu|\le\alpha_{\nu+1}} \frac{1}{z - a_\mu}\Big| + \Big|\sum_{|a_\mu|\le\alpha_{\nu+1}} \frac{1}{z - \frac{\alpha_{\nu+1}^2}{\bar a_\mu}}\Big| &\le \max_{\nu-1\le s\le\nu+1} \frac{2\pi\alpha_s}{\alpha_{s-1}(\alpha_s - \alpha_{s-1})} n(\alpha_{\nu+1}, 0, f) + \\ &+ \frac{2\varphi(\alpha_{\nu+1})}{\alpha_{\nu+1}} n_z\Big(\frac{2(\alpha_{\nu+1} - \alpha_{\nu-2})}{|z|}\Big)\sqrt{\log^+ n_z\Big(\frac{2(\alpha_{\nu+1} - \alpha_{\nu-2})}{|z|}\Big) + 1} \le \\ &\le (2\pi + o(1))B\alpha_\nu\Big((1+B)^2 h'(\alpha_\nu)\varepsilon(T(\alpha_\nu^*))\Big)^2 T(\alpha_\nu^*) + \\ +2h'(\alpha_{\nu+1})\psi(\nu+1) n_z\Big(\frac{2}{r(1+B)^2 h'(\alpha_\nu)\varepsilon(T(\alpha_\nu^*))}\Big)&\sqrt{\log^+ n_z\Big(\frac{2}{r(1+B)^2 h'(\alpha_\nu)\varepsilon(T(\alpha_\nu^*))}\Big) + 1} \le \\ &\le C_5 r(h'(r)\varepsilon(T(\alpha_\nu^*)))^2 T(\alpha_\nu^*) + \\ &+ C_6 h'(r)\psi(\nu+1) n_z\Big(\frac{1}{rh'(r)\varepsilon(T(r))}\Big)\sqrt{\log^+ n_z\Big(\frac{1}{rh'(r)\varepsilon(T(r))}\Big) + 1},\end{aligned} \tag{3.6}$$

where $C_5 = 7(1 + B)^4 B^3$, $C_6 = 2B$,

$$z \in A_\nu \setminus \bigcup_{j=1}^{j_\nu} D_{\nu j}, \quad \sum_j r_{\nu j} \le \frac{\alpha_{\nu+1}}{\varphi(\alpha_{\nu+1})}.$$

For $\alpha_{\nu-1} < r \le \alpha_\nu$ we have

$$\begin{aligned}T(\alpha_\nu^*) = T\Big(\frac{\alpha_{\nu+1}^2}{\alpha_\nu}\Big) &= T\Big(\alpha_\nu + \frac{2 + o(1)}{(1+B)^2 h'(\alpha_\nu)\varepsilon(T(\alpha_\nu^*))}\Big) \le \\ \le T\Big(\alpha_{\nu-1} + \frac{1}{(1+B)^2 h'(\alpha_{\nu-1})\varepsilon(T(\alpha_{\nu-1}^*))} &+ \frac{2 + o(1)}{(1+B)^2 h'(\alpha_\nu)\varepsilon(T(\alpha_\nu^*))}\Big) \le \\ \le T\Big(r + \frac{4B}{(1+B)^2 h'(r)\varepsilon(T(r))}\Big) &\le T\Big(r + \frac{4}{h'(r)\varepsilon(T(r))}\Big).\end{aligned} \tag{3.7}$$

Applying Lemma 3 with $u(x) = T(x)$ and $\varepsilon(x) = 4B\varepsilon(x)$, we obtain that

$$T\Big(r + \frac{4B}{h'(r)\varepsilon(T(r))}\Big) \le eT(r), \quad r \to +\infty,\ r \notin E_1, \tag{3.8}$$

where $E_1 \subset [1, \infty)$ is of finite h-measure.

Lemma 4. *For the sequence (α_ν) defined by (3.4) we have*

$$\varepsilon(T(\alpha_{\nu-1}^*))h'(\alpha_\nu) \le \nu \le (B + 1)^2\varepsilon(T(\alpha_{\nu-1}^*))h(\alpha_\nu), \quad \nu \to +\infty, \tag{3.9}$$

Proof of the lemma. By the definition of α_ν we have

$$\nu=\sum_{k=0}^{\nu-1}\frac{k-(k-1)}{\alpha_{k+1}-\alpha_k}(\alpha_{k+1}-\alpha_k)=\sum_{k=0}^{\nu-1}C\varepsilon(T(\alpha_k^*))h'(\alpha_k)\le$$

$$\le(1+B)^2\varepsilon(T(\alpha_{\nu-1}^*))\sum_{k=0}^{\nu-1}h'(\alpha_k)(\alpha_{k+1}-\alpha_k)\le B(1+B)^2\varepsilon(T(\alpha_{\nu-1}^*))\int_{\alpha_0}^{\alpha_\nu}dh(t)\le$$

$$\le C_7\varepsilon(T(\alpha_{\nu-1}^*))h(\alpha_\nu).$$

Since $T(r,f)$ is non-decreasing and $r\le\alpha_\nu\le\alpha_{\nu-1}^*$, similarly we obtain

$$\nu=\sum_{k=1}^{\nu-1}(B+1)\varepsilon(T(\alpha_k^*))h'(\alpha_k)\ge(1+B)\varepsilon(T(\alpha_{\nu-1}^*))h'(\alpha_\nu)\ge\varepsilon(T(r))h'(r).$$

□

Let $F_\nu=\bigcup_j[|z_{\nu j}|-r_{\nu j},|z_{\nu j}|+r_{\nu j}]$. By Lemma 2 $\sum_j r_{\nu j}\le\frac{1}{h'(\alpha_{\nu+1})\psi(\nu+1)}$. Using the lower estimate from Lemma 4 we deduce for $F=\bigcup_\nu F_\nu$ $(\nu\ge\nu_1\ge 2)$

$$\int_F dh(r)\le\sum_{\nu=\nu_1}^{\infty}\sum_j\int_{|z_{\nu j}|-r_{\nu j}}^{|z_{\nu j}|+r_{\nu j}}dh(r)\le\sum_{\nu=\nu_1}^{\infty}2Bh'(\alpha_{\nu+1})\sum_j r_{\nu j}\le\sum_\nu\frac{1}{\psi(\nu)}<+\infty.$$

It means that the h-measure of F is finite. Therefore (3.6)–(3.8) yield $(r\to+\infty, r\notin E_1\cup F)$

$$\left|\sum_{|a_\mu|\le\alpha_{\nu+1}}\frac{1}{z-a_\mu}\right|+\left|\sum_{|a_\mu|\le\alpha_{\nu+1}}\frac{1}{z-\frac{\alpha_{\nu+1}^2}{\bar a_\mu}}\right|\le C_1r(h'(r)\varepsilon(T(r)))^2T(r)+$$

$$+C_2h'(r)\psi(h(r)\varepsilon(T(r)))n_z\Big(\frac{1}{rh'(r)\varepsilon(T(r))}\Big)\sqrt{\log^+n_z\Big(\frac{1}{rh'(r)\varepsilon(T(r))}\Big)+1}. \tag{3.10}$$

Similarly,

$$\left|\sum_{|b_\mu|\le\alpha_{\nu+1}}\frac{1}{z-b_\mu}\right|+\left|\sum_{|b_\mu|\le\alpha_{\nu+1}}\frac{1}{z-\frac{\alpha_{\nu+1}^2}{\bar b_\mu}}\right|\le C_1r(h'(r)\varepsilon(T(r)))^2T(r)+$$

$$+C_2h'(r)\psi\Big(h(r)\varepsilon(T(r))\Big)n_z\Big(\frac{1}{rh'(r)\varepsilon(T(r))}\Big)\sqrt{\log^+n_z\Big(\frac{1}{rh'(r)\varepsilon(T(r))}\Big)+1}, \tag{3.11}$$

as $r\to+\infty$, $r\notin E_2$, where $E_2\subset[1,+\infty)$ is of finite h-measure.

Using (2.1), (3.7) and (3.8) we have

$$I\le\frac{4\alpha_{\nu+1}(T(\alpha_{\nu+1})+O(1))}{(\alpha_{\nu+1}-r)^2}\le C(1+B)^4rT(r)(h'(\alpha_\nu)\varepsilon(T(\alpha_\nu^*))^2\le$$

$$\le 5B^2e(1+B)^4rT(r)(h'(r)\varepsilon(T(r))^2,\quad r\to+\infty, r\notin E_3. \tag{3.12}$$

Hence, (3.10)–(3.12) yield (1.7).

Theorem 3 is proved. □

4. An Example.

Proof of Theorem 2. We modify an example constructed in [13]. Given $\rho \in (0, +\infty)$ we set $q = [\rho]$, $r_n = 4^{\frac{n}{\rho}}$, $n \in \mathbb{Z}_+$ and define a Weierstrass product of the form

$$B(z) = \prod_{n=1}^{\infty} \prod_{k=1}^{4^n} E\Big(\frac{z}{a_{nk}}, q\Big),$$

where $E(w, q)$ is a Weierstrass primary factor of genus q ([1], [2]), zeros a_{nk} satisfy $r_{n-1} < |a_{nk}| \le r_n$, $1 \le k \le 4^n$, and will be specified later. Now we just note that since $n(r_n, 0, B) = 4(4^n - 1)/3$, we have $n(t, 0, B) \asymp t^\rho$ as $t \to +\infty$. It is well-known that the product is absolutely convergent in $\mathbb{C}$. Moreover,

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \sum_{k=1}^{4^n} \frac{E'(\frac{z}{a_{nk}}, q)}{E(\frac{z}{a_{nk}}, q)} = \sum_{n=1}^{\infty} \sum_{k=1}^{4^n} \frac{z^q}{a_{nk}^q (z - a_{nk})}. \tag{4.1}$$

We use a construction due to V.Eiderman and J.Anderson ([15], [16]). Set $E^{(0)} = [-\frac{1}{2}, \frac{1}{2}]$ and at the ends of $E^{(0)}$ take subintervals $E_j^{(1)}$ of length $\frac{1}{4}$, $j \in \{1, 2\}$. Let

$$E^{(1)} = \bigcup_{j=1}^{2} E_j^{(1)} = \Big[-\frac{1}{2}, -\frac{1}{4}\Big] \cup \Big[\frac{1}{4}, \frac{1}{2}\Big].$$

We then construct, in a similar manner, two sub-intervals $E_{j,i}^{(2)}$ of length 4^{-2} in each $E_j^{(1)}$ and denote by $E^{(2)}$ the union of the four intervals $E_{j,i}^{(2)}$. Continuing this process we obtain a sequence of sets $E^{(n)}$ consisting of 2^n intervals of length 4^{-n}. We define $E_n = E^{(n)} \times E^{(n)}$ the Cartesian product, and note that E_n consists of 4^n squares $E_{n,k}$, $k \in \{1, \ldots, 4^n\}$ with sides parallel to the coordinate axes.

Theorem C. *Let $P > 0$ be given and set $\mathcal{E} = (100P)^{-1}\sqrt{n}4^n E_n$ is the set defined above. Let ν be the measure formed by 4^{n+1} Dirac masses located at the corners z_k of the squares which form E_n. Then for any covering $\{D(w_j, \rho_j)\}$ of the set*

$$\mathcal{Z} = \Big\{z \in \mathbb{C} : \Big|\sum_{k=1}^{4^{n+1}} \frac{1}{z - z_k}\Big| > P\Big\}$$

we have $\sum_j \rho_j \ge c_2 \frac{4^{n+1}}{P} \sqrt{\log 4^{n+1}}$ where $c_2 > 0$ is an absolute constants.

Moreover, the projection of $\bigcup_j D(w_j, \rho_j)$ onto the straight line $y = x/2$ has measure at least $c_3 \frac{4^{n+1}}{P} \sqrt{\log 4^{n+1}}$ where $c_3 > 0$ is an absolute constant.

Remark 6 [15, 16]. Let $z'_{n,k}$ be the centers of $E_{n,k}$ which form $\mathcal{E}$ in Theorem C. Then $\mathcal{Z} \supset \bigcup_{k \in \mathcal{K}_n} D(z'_{n,k}, 0.001\sqrt{n}/P)$, and $\operatorname{card} \mathcal{K}_n \ge c_4 4^n$ where $c_4 > 0$ is an absolute constant.

Let h be a positive increasing unbounded differentiable function on $(0, \infty)$ with $h'(2r) \asymp h(r)$ and $h'(r) r \psi(\ln r) \ge 1$ as $r \to +\infty$.

We choose $P_n = 4^{n+1} h'(r_{n+1}) \sqrt{\log 4^{n+1}} \psi(\log r_{n+1})$,

$$\widetilde{E}_n = \frac{\sqrt{n}4^n}{100 P_n} \mathcal{E} = \frac{1}{400\sqrt{(1 + 1/n)\log 4}\, h'(r_{n+1}) \psi(\log r_{n+1})} \mathcal{E}.$$

Hence the side length d_n of $\widetilde{E}_n$ satisfies

$$\frac{1}{800h'(r_{n+1})\psi(\log r_{n+1})} \le d_n \le \frac{1}{400h'(r_{n+1})\psi(\log r_{n+1})}. \tag{4.2}$$

We rotate $\widetilde{E}_n$ on the angle $\pi/6$ clockwise and move along Ox such that the right vertex of the square coincides with r_{n+1}. We denote by $a_{n+1,k}$, $1 \le k \le 4^{n+1}$ the vertices of the squares that form the obtained square E_n^*.

Then by Theorem C for any covering $\{D(w_{n,j}, \rho_{n,j})\}$ of the set

$$M_n \stackrel{\text{def}}{=} \Big\{z : \Big|\sum_{k=1}^{4^{n+1}} \frac{1}{z - a_{n+1,k}}\Big| \ge 4^{n+1}\sqrt{\log 4^{n+1}}h'(r_{n+1})\psi(\log r_{n+1})\Big\}$$

we have $\sum_j \rho_{n,j} \ge c_2/(h'(r_{n+1})\psi(\log r_{n+1}))$.

Let $z \in M_n$. According to Remark 6 we can assume that

$$z \in \bigcup_{k\in\mathcal{K}_n} D(z'_{n,k}, 10^{-3}\sqrt{n+1}/P_n) \subset M_n.$$

Then, by (4.2)

$$\begin{aligned} |z| &\le r_{n+1} + 0.001\frac{\sqrt{n+1}}{P_n} \le r_{n+1} + \frac{1}{100h'(r_{n+1})\psi(\log r_{n+1})} \le 1.01 r_{n+1}, \\ |z| &\ge r_{n+1} - d_n\frac{2}{\sqrt{3}} - 0.001\frac{\sqrt{n+1}}{P_n} \ge r_{n+1} - \frac{1}{100h'(r_{n+1})\psi(\log r_{n+1})} \ge 0.99 r_{n+1}. \end{aligned} \tag{4.3}$$

We split the sum from (4.1):

$$\sum_{m=1}^{\infty}\sum_{k=1}^{4^m} \frac{z^q}{a_{mk}^q(z - a_{mk})} = \sum_{m=1}^{n}\sum_{k=1}^{4^m} \frac{z^q}{a_{mk}^q(z - a_{mk})} + \\ + \sum_{k=1}^{4^{n+1}} \frac{z^q}{a_{n+1,k}^q(z - a_{n+1,k})} + \sum_{m=n+2}^{\infty}\sum_{k=1}^{4^m} \frac{z^q}{a_{mk}^q(z - a_{mk})} \equiv \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{4.4}$$

Estimates of Σ_1 and Σ_3 for $z \in M_n$, $|z| = r$ are similar to that in [13, (4.5),(4.6)]. We have

$$|\Sigma_1| + |\Sigma_3| \le C_6(\rho) \cdot \begin{cases} r^{\rho-1}, & \rho \notin \mathbb{N} \\ r^{\rho-1}\log r, & \rho \in \mathbb{N} \end{cases} \le C_6\frac{n(r)\log r}{r}, \quad r \to +\infty. \tag{4.5}$$

Finally, we estimate the difference

$$\Big|\sum_{k=1}^{4^{n+1}} \frac{z^q}{a_{n+1,k}^q(z - a_{n+1,k})} - \sum_{k=1}^{4^{n+1}} \frac{1}{z - a_{n+1,k}}\Big| \le \\ \le \sum_{k=1}^{4^{n+1}} \frac{|z^{q-1} + z^{q-2}a_{n+1,k} + \cdots + a_{n+1,k}^{q-1}|}{|a_{n+1,k}|^q} \le C_6(\rho)\frac{q4^{n+1}}{r_{n+1}} \le C_7(\rho)\frac{n(r)}{r}.$$

Hence, it follows from (4.1)–(4.5), the latter estimate and the definition of M_n and the properties of h that for $z \in M_n$, $|z| = r$

$$\left|\frac{B'(z)}{B(z)}\right| \geq \left|\sum_{k=1}^{4^{n+1}} \frac{1}{z - a_{n+1,k}}\right| - \left|\sum_{k=1}^{4^{n+1}} \frac{z^q}{a_{n+1,k}^q (z - a_{n+1,k})} - \sum_{k=1}^{4^{n+1}} \frac{1}{z - a_{n+1,k}}\right| -$$

$$-\left|\sum_{m \neq n+1} \sum_{k=1}^{4^m} \frac{z^q}{a_{mk}^q (z - a_{mk})}\right| \geq C_8(\rho) h'(r) n(r) \sqrt{\log n(r)} \psi(\log r) - O\Big(\frac{n(r) \log r}{r}\Big) \geq$$

$$\geq C_9(\rho) h'(r) n_z\Big(0.01\Big) \sqrt{\log n_z\Big(0.01\Big)} \psi(\log r) r, \quad r \to +\infty.$$

On the other hand,

$$\sum_n \sum_j h'(|z_n|) \rho_{n,j} \geq c_2 \sum_n \sum_j \frac{h'(|z_n|)}{h'(r_{n+1}) \psi(\log r_{n+1})} \geq c_3 \sum_n \frac{1}{\psi(n \log 4)} = +\infty.$$

Theorem 2 is proved. □

REFERENCES

1. Гольдберг А.А., Островский И.В. Распределение значений мероморфных функций, М.Наука, 1970, 592 с.
2. Hayman W. K. Meromorphic functions, Clarendon Press, Oxford, 1964.
3. Стрелиц Ш. Асимптотические свойства аналитических решений дифференциальных уравнений. – Вильнюс, Минтис, 1972, 468 с.
4. Gundersen G. *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates,* J. London Math. Soc. (2) **37** (1988) 88-104.
5. Chyzhykov I., Gundersen G.G., Heittokangas J. *Linear differential equations and logarithmic derivative estimates,* Proc. London Math. Soc. (3) **86** (2003), 735–754.
6. Heittokangas J. *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss. **122** (2000) 1-54.
7. Chiang Y.-M., Feng S.-J. *On the growth of logarithmic diffrences, difference quotients and logarithmic derivatives of meromorphic functions,* Trans. Amer. Math. Soc. (to appear)
8. Macintyre A.J. *Wiman's method and the 'flat regions' of integral functions,* Quart. J. Math. **9** (1938), 81–88.
9. Гольдберг А.А., Гринштейн В.А. *О логарифмической производной мероморфной функции,* Мат.заметки **19** (1976), no.4, 525–530. Math.Notes **19** (1976), 320–323.
10. Heittokangas J., Korhonen R., Rättyä J., *Generalized logarithmic derivative estimates of Gol'dberg-Grinshtein type,* Bull. London Math. Soc. **36** (2004), 105–114.
11. Miles J. *A sharp form of the lemma of the logarithmic derivative,* J. London Math. Soc. **45** (1992), 243–254.
12. Jankowski M. *An estimate for the logarithmic derivative of meromorphic functions,* Analysis **14** (1994), 185–194.
13. Chyzhykov I.E. *Sharp logarithmic derivative estimates for meromorphic functions,* Matem. Studii **27** (2007), no.2, 120–138.

14. Strelitz Sh. *On upper bounds for the logarithmic derivative of a meromorphic function,* Complex variables, **23** (1993), 131-143.

15. Anderson J.M., Eiderman V.Ya. *Estimates for the Cauchy transforms of point masses (the logarithmic derivative of a polynomial),* Doklady Ross Akad. Nauk **401** (2005), no.5, 583–586.

16. Anderson J.M., Eiderman V.Ya. *Cauchy transforms of point masses (the logarithmic derivative of polynomials),* Annals of Math. **163** (2006), no.3, 1057–1076.

17. Levin B. Ja. *Distribution of zeros of entire functions*, revised edition, Transl. Math. Monographs, Volume 5, translated by R. P. Boas *et al* (Amer. Math. Soc., Providence, 1980).

18. Salo T.M., Skaskiv O.B., Trakalo O.M. On the best possible description of exeptional set in Wiman-Valiron theory for entire function, Matematychni Studii **16** (2001), № 2, 131–140.

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