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PRONORMALITY AND PERMUTABILITY IN PERIODIC FC–GROUPS

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In this paper we study periodic locally soluble FC–groups in which permutability is a transitive relation and the class of groups in which all finite normal subgroups of Sylow subgroups are pronormal. The choice of class of periodic FC-groups is made, because it has sufficiently developed structure and is similar to the class of finite groups.

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Рассматриваются периодические локально разрешимые FC–группы, в которых свойство переставляемости является транзитивным, и класс групп, в которых все конечные нормальные подгруппы силовских подгрупп — пронормальны. Выбор класса периодических FC–групп обусловлен тем обстоятельством, что он имеет достаточно развитую силовскую структуру и по многим аспектам близок к классу конечных групп.

A subgroup H of a group G is said to be *permutable* (or *quasinormal*) in G, in symbols $H \operatorname{per} G$, if $HK = KH$ for all subgroups K of G.

The study of the some basic properties of the permutable subgroups started rather long time ago (see, for example [4]). In particular, there were described the groups (finite and infinite) in which every subgroup is permutable [4, Theorem 2.4]. Like normality, the relation "to be permutable subgroup" is not transitive. A finite group in which permutability is a transitive relation is said to be a PT-group. Finite soluble PT-groups have been described by G. Zacher [6]. The structure of arbitrary finite PT-groups has been considered by D.J.S. Robinson [3].

The study of infinite PT-groups is not started yet. In paper [7], [8] we have been started to consider periodic locally soluble FC-groups in which permutability is a transitive relation. The choice of the class of periodic FC-groups is made, because it has sufficiently developed Sylow structure, and many another its properties are similar to those of finite groups. Therefore it is natural to begin consideration of infinite PT-groups and their characterization in this class.

In this paper we consider the relationship between properties of permutability and pronormality of infinite periodic FC-groups. In particular, we extend the main result of the paper [1, Theorem A and Theorem D] to the class of infinite periodic locally soluble FC-groups.

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Definition 1. A group G is said to be *modular*, if the lattice of all subgroups of G is modular, that is for all subgroups X, Y, Z of G such that $X \leq Z$ the equality $\langle X, Y\rangle \cap Z = \langle X, Y \cap Z\rangle$ holds.

By Iwasava's theorem (see, for example, [4, Theorem 2.4.14]) a locally finite p-group, p is a prime, is modular if and only if each subgroup of G is permuted in G. The structure of locally finite modular p-groups G is described in the following way (see, for example, [4, Theorem 2.4.14]).

Proposition. *Let G be a locally finite modular p-group.*

(i) $p \neq 2$, *then* $G = B\langle a\rangle$ *where* B *is a normal abelian subgroup of exponent* p^k, *and there is a positive integer* t *such that* $t = 1 + p^m, m \leq k \leq m + d$ *where* $p^d = |G/B|$ *and* $a^{-1}ba = b^t$ *for all* $b \in B$;

(ii) $p = 2$, *then either* G *is a Dedekind group or* $G = B\langle a\rangle$ *where* B *is a normal abelian subgroup of exponent* p^k, *and there is a positive integer* t *such that* $t = 1 + p^m$, $2 \leq m \leq k \leq m + d$ *where* $p^d = |G/B|$ *and* $a^{-1}ba = b^t$ *for all* $b \in B$.

In both cases G *is bounded and nilpotent.*

Definition 2. Let p be a prime. We say that a group G *belongs to the class* $\mathfrak{P}_p$ if G satisfies the following two conditions:

(i) each Sylow p-subgroup of G is modular;

(ii) if P is the Sylow p-subgroup of G, then each normal subgroup of P is pronormal in G.

Recall that a subgroup H of a group G is said to be *pronormal* in G if for each element $g \in G$ the subgroups H, H^g are conjugate in $\langle H, H^g\rangle$.

In the paper [1, Theorem A and Theorem D] it is proved that a finite soluble group G is a PT-group if and only if $G \in \wp_p$ for all $p \in \Pi(G)$. In this connection it is interesting to consider the infinite groups from the class $\mathfrak{P}_p$.

Here we will consider not only the class $\mathfrak{P}_p$ but also the following its extension.

Definition 3. Let p be a prime. We say that a group G *belong to the class* $\mathfrak{P}^*_p$ if G satisfies the following two conditions:

(i) each Sylow p-subgroup of G is modular;

(ii) if P is the Sylow p-subgroup of G, then each finite normal subgroup of P is pronormal in G.

If G is a group, then we denote by $\mathrm{Norm}(G)$ the subgroup generated by all cyclic subgroups which are normal in G.

As we have already observed above if G is a modular locally finite p-group, p is a prime, then either P is abelian or $P = A\langle b\rangle$ for some element b where A is normal in P abelian subgroup and every cyclic subgroup of A is P-invariant. In particular, for the second case $A = \mathrm{Norm}(P)$.

Lemma 1. *Let G be a locally finite group and P be the Sylow p-subgroup of G, p is a prime. If $G \in \mathfrak{P}^*_p$ and D is a subgroup of P, then every subgroup of* $\mathrm{Norm}(P) \cap D$ *is normal in $N_G(D)$. In particular, if P is abelian, then every subgroup D is normal in $N_G(D)$.*

Proof. Let $H \leq \mathrm{Norm}(P) \cap D$ and $y \in H$. Then $Y = \langle y \rangle$ is normal in P and therefore Y is pronormal in G. It follows that the subgroups Y, Y^x are conjugate in $\langle Y, Y^x \rangle$ for each $x \in N_G(D)$. In other words, there is an element $u \in \langle Y, Y^x \rangle$ such that $Y^u = Y^x$. Y is normal in P, so Y is normal in D and $\langle Y, Y^x \rangle \leq D$. Hence $u \in D$ that implies $Y^u = Y$ and $Y = Y^x$. In other words, every cyclic subgroup of H is normal in $N_G(D)$. Then H is also normal in $N_G(D)$. □

Corollary 1. *Let G be a locally finite group and P be the Sylow p-subgroup of G, p is a prime. If $G \in \mathfrak{P}_p^*$ and D is a normal p-subgroup of G, then every subgroup of* $\mathrm{Norm}(P) \cap D$ *is normal in G.*

Corollary 2. *Let G be a locally finite group and P be the Sylow p-subgroup of G, p is a prime. Suppose that $G \in \mathfrak{P}_p^*$. Then every normal subgroup of P is normal in $N_G(P)$. In particular, every subgroup of P, including $[P, P]$, is normal in $N_G(P)$.*

Lemma 2. *Let G be a periodic FC-group and P be the Sylow p-subgroup of G, p is a prime, $p \neq 2$. Suppose that $G \in \mathfrak{P}_p^*$ and P is non–abelian. Then $N_G(P) = PC_G(P)$.*

Proof. Suppose the contrary. Then there is a p'-element g such that $g \notin C_G(P)$. As we observed above $P = A\langle b \rangle$ for some element b where A is normal in P abelian bounded subgroup and $b^{-1}vb = v^d$ for all $v \in A$ where $d = 1 + p^s$. By Corollary 2 of Lemma 1 A is normal in $N_G(P)$. Suppose that $g \notin C_G(A)$. Then there is an element $a \in A$ such that $ag \neq a$. Choose a finite normal subgroup K such that $a, g, b \in K$. Since $|g|$ is a p'-number, $B = A \cap K = C_B(g) \times [B, g]$ (see, for example, [2, Theorem 5.2.3]). Every subgroup of A is normal in P and therefore it is $\langle g \rangle$-invariant by Corollary 2 of Lemma 1. By the choice of the element g we have $B \neq C_B(g)$, thus $[B, g] \neq \langle 1 \rangle$. Choose an element c of order p in $C_B(g)$. Let now u be an element of $[B, g]$ having order p. Since $u \notin C_B(g)$, $u^g = u^d$ where d is a p'-number, moreover $d \not\equiv 1 \pmod p$. We have $(uc)^g = u^g c^g = u^d c$. On the other hand, since $uc \notin C_P(g), (uc)^g = (uc)^t$ where t is also a p'-number such that $t \not\equiv 1 \pmod p$. Hence $u^d c = (uc)^t = u^t c^t$, therefore $d \equiv t \pmod p$ and $t \equiv 1 \pmod p$. This contradiction proves that $C_B(g) = \langle 1 \rangle$ and hence $[B, g] = B$. Since every subgroup of A is normal in $H = (P \cap K)\langle g \rangle, H/C_H(A)$ is abelian (see, for example, [4, Theorem 1.5.1]). Put $L = (P \cap K)$. Then L is a Sylow p-subgroup of K (see, for example, [5, Theorem 5.4]). Since L/B is cyclic, $C_L(B)$ is abelian, so that $C_L(B) \neq P$. It follows that $[L, g] \neq L$ and $[L, L] \leq [L, g]$. Using again Theorem 5.2.3 of a [2], we obtain

$$L/[L, L] = C_{L/[L,L]}(g) \times [L/[L, L], g] = C_{L/[L,L]}(g) \times [L, g]/[L, L].$$

It follows that $C_{L/[L,L]}(g) \neq \langle 1 \rangle$. Every subgroup of $L/[L, L]$ is $\langle g \rangle$-invariant by Corollary 2 of Lemma 1. Using the same arguments, we can obtain that $C_{L/[L,L]}(g) \neq \langle 1 \rangle$ implies an equality $C_{L/[L,L]}(g) = L/[L, L]$. In other words, $H/[L, L]$ is abelian. In this case H is nilpotent by the result due to P. Hall (see, for example, [3, Theorem 5.2.10]). However, then $g \in C_G(P)$, and we obtain a contradiction.

Suppose now that $g \in C_G(A)$. Let $m = p^s$. Since $b^{-1}vb = v^{1+m} = vv^m$ for all $v \in A$, P/A^m is abelian. A subgroup A is bounded, therefore $A \neq A^m$. It follows that $A \neq [P, P]$. In other words, $C_{P/[P,P]}(g) \neq \langle 1 \rangle$ because $A/[P, P] \leq C_{P/[P,P]}(g)$. As above using a local technique, we obtain that $C_{P/[P,P]}(g) = P/[P, P]$ and we arrive to contradiction. This contradiction proves that $N_G(P) = PC_G(P)$. □

Corollary 3. *Let G be a periodic FC-group and P be the Sylow p-subgroup of G, p is a prime, $p \neq 2$. Suppose that $G \in \mathfrak{P}_p^*$ and P is non–abelian. Then $G = O_{p'}(G) \leftthreetimes P$.*

Proof. It follows from Lemma 2 and Proposition 1 of [7]. □

Lemma 3. *Let G be a periodic FC-group and H be the normal subgroup of G having finite index. If $G \in \mathfrak{P}_p^*$ for some prime p, then $G/H \in \mathfrak{P}_p$.*

Proof. Let P/H be the Sylow p-subgroup of G/H. There is a finite subgroup F such that $P = FH$. Choose a (finite) Sylow p-subgroup Q in F. Since $F/(F \cap H) \cong FH/H = P/H$ is a p-group, $F = Q/(F \cap H)$ and hence $P = QH$. There is a Sylow p-subgroup S of G such that $Q \leq S$. Then SH/H is a p-subgroup of G/H and $P/H = QH/H \leq SH/H$. On the other hand, P/H is the Sylow p-subgroup of G/H, therefore $P/H = SH/H$. Since $G \in \mathfrak{P}_p^*$, a subgroup S is modular, hence $P/H = SH/H$ is modular too. Let L/H be a normal subgroup of P/H. The equality $P/H = SH/H$ shows that S includes a normal subgroup T such that $L/H = TH/H$. Since L/H is finite, there is a finite p-subgroup R of T such that $LH = RH$. Put $X = R^P$, then $X \leq T$. Since G is an FC-group, a subgroup X is finite. We have

$$LH = RH \leq XH \leq TH = LH.$$

That implies $LH = XH$ and $LH/H = XH/H$. The relation $G \in \mathfrak{P}_p^*$ shows that X is pronormal in G. Then its homomorphic image $XH/H = L/H$ is pronormal in G/H. This shows that $G/H \in \mathfrak{P}_p$. □

Corollary 4. *Let G be a periodic locally soluble FC-group and R be the locally nilpotent radical of G. If $G \in \mathfrak{P}_p^*$ for all $p \in \Pi(G)$, then G/R is abelian.*

Proof. Let H is a normal subgroup having finite index in G. Lemma 3 proves that $G/H \in \mathfrak{P}_p$ for all $p \in \Pi(G)$. Theorems D, A of [1] show that G/H is a finite soluble PT-group. Let R_H/H be the nilpotent radical of G/H. By the description of finite soluble PT-groups given by Zacher [6] we obtain that $(G/H)/(R_H/H) \cong G/R_H$ is abelian. Let $\aleph$ be the family of all normal subgroups of G having finite indexes in G. Put $L = \bigcap_{H \in \aleph} R_H, Z = \bigcap \aleph$, then Z is a subgroup of the center of G (see, for example, [5, Theorem 1.9]). Let F be an arbitrary finite subgroup of L. Since L/Z is residually finite, there is a subgroup $U \in \aleph$ such that $FZ/Z \cap U/Z = \langle 1 \rangle$. It follows that

$$FZ/Z \cong (FZ/Z)/(FZ/Z \cap U/Z) \cong ((FZ/Z)(U/Z))/(U/Z) \cong FU/U.$$

An inclusion $F \leq L$ implies $FU/U \leq LU/U \leq R_U/U$, that implies the nilpotency of FU/U and hence FZ/Z. Since $Z \leq \zeta(G)$, a subgroup FZ is nilpotent, so and F is nilpotent. Hence L is a locally nilpotent normal subgroup of G and therefore $L \leq R$. The equality $L = \bigcap_{H \in \aleph} R_H$ together with Remak's theorem give an embedding

$$G/L \hookrightarrow \prod_{H \in \aleph} G/R_H.$$

Since G/R_H is abelian for each $H \in \aleph, G/L$ is likewise abelian. The inclusion $L \leq R$ proves that G/R is abelian. □

Corollary 5. *Let G be a periodic locally soluble FC-group. If $G \in \mathfrak{P}_p^*$ for all $p \in \Pi(G)$, then G is hypercyclic.*

Proof. We wil use here the notations of the proof of Corollary 1. Since $Z \leq \zeta(G)$, it suffices to prove that G/Z is hypercyclic. In turn for this it suffice to prove that G/Z contains a normal cyclic subgroup using transfinite induction. Let L be the arbitrary finite normal subgroup of G. There is a normal subgroup $H \in \aleph$ such that $LZ/Z \cap H/Z = \langle 1 \rangle$. By Lemma 3 $G/H \in \mathfrak{P}_p$. Theorems D, A of [1] show that G/H is a finite soluble PT-group. The description of finite soluble PT-groups in [6] shows that G/H is hypercyclic. In particular, its normal subgroup LH/H contains a G-invariant cyclic subgroup C/H. Let c be the element of L satisfying $C/H = \langle c \rangle H/H$. Then

$$[\langle c \rangle H/H, G/H] \leq \langle c \rangle H/H$$

that implies

$$[\langle c \rangle, G] \leq \langle c \rangle H.$$

On the other hand, $c \in L$ and $[\langle c \rangle, G] \leq L$ because L is normal in G. Hence

$$[\langle c \rangle, G] \leq \langle c \rangle H \cap L \leq Z \text{ } or \text{ } [\langle c \rangle Z/Z, G/Z] \leq \langle c \rangle Z/Z.$$

So $\langle c \rangle Z/Z$ is a normal cyclic subgroup of G/Z. □

Lemma 4. *Let G be a periodic locally soluble FC-group and L be the locally nilpotent residual of G. If $G \in \mathfrak{P}_p^*$ for all $p \in \Pi(G)$, then G/L is modular.*

Proof. Since a periodic FC-group is locally finite, repeating the arguments of the proof of Corollary 1 of Lemma 3 we can prove that G/L is locally nilpotent. Let S/L be the Sylow p-subgroup of G/L. Then there is a Sylow p-subgroup P of G such that $S/L = PL/L$ (see, for example, [5, Theorem 5.4]). It follows that S/L is modular and the fact that G/L is locally nilpotent implies that the whole factor–group G/L is modular. □

Lemma 5. *Let G be a periodic locally soluble FC-group and suppose that $G \in \mathfrak{P}_p^*$ for all primes p. Let*

$$\eta = \{p | p \text{ is a prime such that p} \neq 2, \text{ the Sylow p} - \text{subgroups of G are abelian}$$
$$\text{and not lie in the center of its normalizer}\}.$$

Then the Sylow p-subgroup S_p of G is normal in G for each $p \in \eta$ and $\times_{p \in \eta} S_p$ is the locally nilpotent residual of G. In particular, the locally nilpotent residual of G is abelian.

Proof. Let $p \in \Pi(G)$ and let S_p be the Sylow p-subgroup of G. Denote by L the locally nilpotent residual of G. First consider $N_G(S_2)$. Being hypercyclic by Corollary 2 of Lemma 3, G is locally supersoluble. Let L be an arbitrary finite normal subgroup of G. Since L is supersoluble, its Sylow $2'$-subgroup is normal. It follows that the Sylow $2'$-subgroup of G is normal. By a result due to S.N. Chernikov (see, for example, [5, Theorem 5.25]), $G = O_{2'}(G) \leftthreetimes S_2$, in particular, $2 \notin \Pi(L)$. Let now $p \neq 2$. If S_p is not abelian, then by Corollary of Lemma 2, $G = O_{p'}(G) \leftthreetimes S_p$, and we obtain that $p \notin \Pi(L)$. This means that $\Pi(L) \subseteq \eta$. Conversely, suppose that $p \in \eta$. Assume to the contrary that $p \notin \Pi(L)$. Then $S_p \bigcap L = \langle 1 \rangle$. Since S_pL/L is the Sylow p-subgroup of G/L (see, for example, [5, Theorem 5.4]) and G/L is locally nilpotent, G has a normal Sylow p'-subgroup Q. Using again a theorem of S.N. Chernikov (see, for example, [5, Theorem 5.25]) we obtain a decomposition $G = O_{p'}(G) \leftthreetimes S_p$. It follows that $N_G(S_p) = H \times P$, where $H = O_{p'}(G) \cap N_G(S_p)$. Since P is abelian, $P \leq \zeta(N_G(S_p))$, that contradicts to the choice of p. This contradiction proves the equality $\Pi(L) = \eta$. By Corollary 1 of Lemma 3, L is locally nilpotent, thus every Sylow p-subgroup with $p \in \eta$ is normal in G and $L = \times_{p \in \eta} S_p$. □

Corollary 6. *Let G be a periodic locally soluble FC-group and let L be the locally nilpotent residual of G. If $G \in \mathfrak{P}_p^*$ for all primes p, then there exists a locally nilpotent modular subgroup T such that $G = L \rtimes T$.*

Proof. Indeed, Lemma 5 approves that L is normal Sylow p-subgroup of G. By a result due to S.N. Chernikov (see, for example, [5, Theorem 5.25]) we obtain $G = L \rtimes T$ for some subgroup T. The isomorphism $T \cong G/L$ together Lemma 4 complete the proof. □

Theorem. *Let G be a periodic locally soluble FC-group. Then $G \in \mathfrak{P}_p^*$ for all primes p if and only if G is a PT-group.*

Proof. Suppose first that $G \in \mathfrak{P}_p^*$ for all primes p. Let L be the locally nilpotent residual of G. By Lemma 5, L is abelian and $\Pi(L) \cap \Pi(G/L) = \varnothing$. By Lemma 4, every subgroup of G/L is permutable. Corollary 1 of Lemma 1 shows that every subgroup of L is normal in G and Corollary 2 of Lemma 3 proves that G is hypercyclic. An application of Theorem of [8] proves that G is a PT-group.

Conversely, let G be a locally soluble FC-group and suppose that G is a PT-group. Choose an arbitrary prime $p \in \Pi(G)$. Again we denote by L the locally nilpotent residual of G. Let P be the Sylow p-subgroup of G and choose in P a normal finite subgroup D. If $p \in \Pi(L)$, then by Theorem of [8], $P \leq L$. By the same result each subgroup of P is normal in G, in particular, D is normal in G and hence pronormal. Assume now that $p \in \Pi(G/L)$. Let $g \in G$. Since G/L is locally nilpotent and DL/L is normal in PL/L, $DL/L = (DL/L)^{gL} = D^gL/L$. In other words, $DL = D^gL$. It follows that D and D^g are the Sylow p-subgroups of DL. Then D and D^g are the Sylow p-subgroups of $\langle D, D^g\rangle$. Being finite Sylow p-subgroups of a periodic FC-group $\langle D, D^g\rangle$, D and D^g are conjugate, which implies that D is pronormal in G. Consequently, $G \in \mathfrak{P}_p^*$ for all primes p. □

Corollary 7. *Let G be a periodic locally soluble FC-group. If $G \in \mathfrak{P}_p$ for all primes p, then G is a PT-group.*

In fact, if $G \in \mathfrak{P}_p$ for all primes p, then $G \in \mathfrak{P}_p^*$ for all primes p and we may obtain Theorem.

The following class of groups has been introduced in [1].

Definition 4. *Let p be a prime. We say that a group G belong to the class $\mathfrak{X}_p$ if G satisfies the following condition:*

if P is the Sylow p-subgroup of G, then each subgroup of P is permutable in $N_G(P)$.

Periodic locally soluble FC-groups from the class $\mathfrak{X}_p$ have been studied in [7]. From the main result of this paper we obtain

Corollary 8. *Let G be a periodic locally soluble FC-group. Then the following statements are equivalent:*

(i) *G is a PT-group;*

(ii) *if p is an arbitrary prime and P is a Sylow p-subgroup of G, then each subgroup of P is permutable in $N_G(P)$;*

(iii) *if p is an arbitrary prime and P is a Sylow p-subgroup of G, then P is modular and each normal subgroup of P is pronormal in G for all primes p.*

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