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O. Shukel'

FUNCTORS OF FINITE DEGREE AND ASYMPTOTIC DIMENSION ZERO

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For any finitary normal functor F in the category of compact Hausdorff spaces one can define its counterpart on the category of proper metric spaces and coarse maps. The aim of this note is to show that the obtained functor preserves the class of proper metric spaces of asymptotic dimension zero in the sense of Gromov.

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Для каждого финитного нормального функтора F в категории компактных хаусдорфовых пространств можно определить его аналог в категории собственных метрических пространств и грубых отображений. Цель этой заметки — показать, что полученный функтор сохраняет класс собственных метрических пространств асимптотической размерности нуль в смысле Громова.

1. Introduction. In [1] E. Shchepin introduced the class of normal functors in the category **Comp** of compact Hausdorff spaces. In this note we define counterparts of this notion in the category of proper metric spaces. A construction of functors in a related category of coarse spaces is given in [2].

Fix a natural number n. Let $\mathcal{K}_n$ denote the category of sets of cardinality $\leq n$ and Set_f the category of finite sets. It is known that every normal functor of degree $\leq n$ in the category **Comp** is uniquely determined by its restriction onto $\mathcal{K}_n$.

The following definition is due to Gromov [3] who defined the spaces of arbitrary asymptotic dimension. We say that a metric space X is of *asymptotic dimension zero* (written $\operatorname{asdim} X = 0$) if, for every $D > 0$, the set X can be covered by a uniformly bounded D-disjoint family. Examples of spaces whose asymptotic dimension equals zero, in particular, generalized sequences, counterparts of the classical Cantor set and Baire space in asymptotic topology, can be found in [4].

In this note we consider the problem of preservation of metric spaces of asymptotic dimension zero by certain functors acting in the asymptotic category [5].

2. Metric on the set $F(X)$. As a motivation of the forthcoming construction, recall that in the category **Comp** every finitary normal functor of degree $\leq n$ can be obtained by the following construction. Let $F : \mathcal{K}_n \to \mathrm{Set}_f$ be a functor that satisfies the following properties:

1. $F(\varnothing) = \varnothing$;
2. $F(\{*\}) = \{*\}$;

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3. if $i\colon X \to Y$ is an embedding, then so is $F(i)\colon F(X) \to F(Y)$. (In the sequel, if $X \subset Y$, then we identify $F(X)$ with the subset $F(i)(F(X))$ of $F(Y)$);
4. $F(A \cap B) = F(A) \cap F(B)$; (In the sequel, if $a \in F(X)$, then the set $\mathrm{supp}(a) = \bigcap\{A \subset X | a \in F(A)\}$ is called the *support* of a).
5. if $f\colon X \to Y$ is a map and $a \in F(X)$, then $\mathrm{supp}(F(f)(a)) = f(\mathrm{supp}(a))$;
6. if $f\colon X \to Y$ is an onto map, then so is $F(f)\colon F(X) \to F(Y)$.

Then, for any compact Hausdorff space X, the space $F(X)$ is defined the a suitably topologized set $\varinjlim\{f(A)|A \subset X \text{ is finite}\}$; see [1] for the details.

Examples. Let us consider few examples of such functors.

1) The n-th hypersymmetric power functor $\exp_n$. For any finite set A, let $\exp_n A = \{B \subseteq A \mid 1 \leq |B| \leq n\}$, for a map $f : A \to A^{'}$, the map $\exp_n f$ acts by the formula $\exp_n f(B) = f(B) \subset A^{'}$.

2) The G-symmetric power functor SP^n_G (see, e.g. [2]).

3) The composition of two functors F_1, F_2 of degrees of n_1 and n_2 respectively is a functor of degree $n = n_1 n_2$. Indeed, if $a \in F_1(F_2(X))$ and $\mathrm{supp}_{F_1}(a) = \{b_1, ..., b_{k_1}\}$, $(k_1 \leq n_1)$ and $\mathrm{supp}_{F_2}(b_i) = \{x_{i1}, ..., x_{il_i}\}$, $(l_i \leq n_2)$, then $\mathrm{supp}_{F_1F_2}(a) = \bigcup\{\mathrm{supp}_{F_2}(b) \mid b \in \mathrm{supp}_{F_1}(a)\}$ (see [1] for the notion of support).

We therefore conclude that $|\mathrm{supp}_{F_1F_2}(a)| \leq n_1 n_2$ (see, e.g. [6]).

4) Let $1 \leq m \leq n$. Given a functor F of degree n, we define a subfunctor $F^{'}$ of F by $F^{'}X = \{a \in FX \mid |\mathrm{supp}(a)| \leq m\}$. Then $F^{'}$ is a functor of degree m.

Now, given a functor $F : \mathcal{K}_n \to \mathrm{Set}_f$, we proceed in a similar manner in order to define a functor in asymptotic categories.

Let (X, d) be a metric space. The family $\exp_f X$ of nonempty finite subsets in X is partially ordered by inclusion. We define the set $F(X)$ to be the direct limit of the direct system $\{F(A),\ F(\iota_{AB});\ \exp_f X\}$ (here, for $A, B \in \exp_f X$ with $A \subset B$, we denote by $\iota_{AB}\colon A \to B$ the inclusion map). For every $A \in \exp_f X$, we identify $F(A)$ with the corresponding subset of $F(X)$ along the map $F(\iota_A)$, where $\iota_A\colon A \to X$ is the limit inclusion map. For any $a \in F(X)$, there exists a unique minimal $A \in \exp_f X$ such that $a \in F(A)$. Then we say that A is the *support* of a and write $\mathrm{supp}(a) = A$. Note that this notion of support agrees with that defined above. Note that the cardinality of the supports does not exceed n. We express this by saying that the degree of F is at most n.

Given two maps $f, g : X \to Y$ where (X, d) is a metric space, we define the distance between f and g as $\sup_{x \in X} d(f(x), g(x))$. We keep the notation d for this distance.

Given a normal functor $F : \mathcal{K}_n \to \mathrm{Set}_f$ and a metric space (X, d), define a metric space $(F(X), \widehat{d})$ as follows.

We are going to define a metric $\widehat{d}$ on $F(X)$. Given $a, b \in F(X)$, we let

$$\widehat{d}(a,b) = \inf\Bigg\{\sum_{i=1}^{m} d(f_{2i-1}, f_{2i}) \mid f_{2i-1}, f_{2i}\colon A_i \to X \text{ are such that there exist } c_i \in F(A_i),$$
$$\mathrm{supp}(c_i) = A_i,\ i \in \{1, \ldots, m\},\ \text{with } a = F(f_1)(c_1),\ F(f_2)(c_1) = F(f_3)(c_2), \ldots,$$
$$F(f_{2m-1})(c_m) = F(f_{2m-2})(c_{m-1}),\ F(f_{2m})(c_m) = b\Bigg\}.$$

Hereafter, we say that $f_1, \ldots, f_{2m}$ and $c_1, \ldots, c_m$ form a *chain* connecting a and b. The number m is then called the *lenght* of this chain.

Theorem 1. *The function* $\widehat{d}\colon F(X)\times F(X)\to\mathbb{R}$ *is a metric on* $F(X)$.

Proof. Note first that the function $\widehat{d}$ is well-defined, i.e. we always have the infimum of a nonempty set when we evaluate $\widehat{d}(a,b)$. Indeed, let $c_i\in F(A_i)$, $i=1,2$, $\mathrm{supp}(c_i)=A_i$, $f_1\colon A_1\to X$, $f_2\colon A_2\to X$, be such that $F(f_1)(c_1)=a$, $F(f_4)(c_2)=b$. Let $x_0\in X$ and $f_2\colon A_1\to X, f_3\colon A_2\to X$ be the constant map with value x_0. Then $F(f_2)(c_1)=F(f_3)(c_2)$ and therefore $\widehat{d}(a,b)\le d(f_1,f_2)+d(f_3,f_4)$.

It is clear that $\widehat{d}(a,b)\ge 0$, for every $a,b\in F(X)$.

1) $\widehat{d}(a,b)=0\Leftrightarrow a=b$.

If $a=b$, then evidently $\widehat{d}(a,b)=0$. We have to show that, if $\widehat{d}(a,b)=0$, then $a=b$. Suppose the contrary and consider two cases.

I. Let $\mathrm{supp}(a)\neq\mathrm{supp}(b)$. Without loss of generality, one may assume that there exists $x\in\mathrm{supp}(b)\setminus\mathrm{supp}(a)$. Let $d(x,\mathrm{supp}(a))=\alpha>0$. Then there exist elements $c_i\in F(A_i)$, $\mathrm{supp}(c_i)=A_i$, and maps $f_{2i-1},f_{2i}:A_i\to X$, $i\in\{1,\dots,k\}$, such that
$a=F(f_1)(c_1),\ F(f_2)(c_1)=F(f_3)(c_2),\dots,\ F(f_{2k-1})(c_k)=F(f_{2k-2})(c_{k-1}),\ F(f_{2k})(c_k)=b$ and $\sum_{i=1}^k d(f_{2i-1},f_{2i})<\alpha/2$. Then $\mathrm{supp}(a_i)\subset O_{d(f_{2i-1},f_{2i})}(\mathrm{supp}(a_{i-1}))$ and $\mathrm{supp}(b)\subset$ $\subset O_{\sum_{i=1}^k d(f_{2i-1},f_{2i})}(\mathrm{supp}(a))\subset O_{\frac{\alpha}{2}}(\mathrm{supp}(a))$. We obtain a contradiction.

II. Let $\mathrm{supp}(a)=\mathrm{supp}(b)$. Then $\alpha=\min\{d(x,y)\mid x,y\in\mathrm{supp}(a),x\neq y\}>0$. Like in case I (and we keep the notations) one may assume that $\sum_{i=1}^k d(f_{2i-1},f_{2i})<\alpha/2$. Let U be the $\frac{\alpha}{2}$-neigborhood of $\mathrm{supp}(a)$ and $r:U\to\mathrm{supp}(a)$ be the retraction such that $r(x')=x$, whenever $x'\in O_{\frac{\alpha}{2}}(x)$. Replace the map f_i by $f_i'=rf_i$, $i\in\{1,\dots,2k\}$. Then for all $i\in\{1,\dots,k\}$ we have $d(f_{2i-1}',f_{2i}')=0$ because the maps f_{2i-1}' and f_{2i}' are equal for the points of the same $\frac{\alpha}{2}$-neighborhood. Then $a=b$. Once again we obtain a contradiction.

2) It is obvious that $\widehat{d}(a,b)=\widehat{d}(b,a)$.

3) We are going to prove the triangle inequality for $\widehat{d}$. Let $a,b,c\in F(X)$. Given $\varepsilon>0$, find $f_{2i-1},f_{2i}\colon A_i\to X$ and $c_i\in F(A_i),\mathrm{supp}(c_i)=A_i$, $i\in\{1,\dots,m\}$, such that

$$\begin{aligned} a=&F(f_1)(c_1),\ F(f_2)(c_1)=F(f_3)(c_2),\dots,\\ &F(f_{2m-1})(c_m)=F(f_{2m-2})(c_{m-1}),\ F(f_{2m})(c_m)=b \end{aligned}\tag{1}$$

and $\sum_{i=1}^{m} d(f_{2i-1},f_{2i})\le\widehat{d}(a,b)+\varepsilon$ and $g_{2i-1},g_{2i}\colon B_i\to X$ and $d_i\in F(B_i)$, $\mathrm{supp}(d_i)=B_i$, $i\in\{1,\dots,l\}$, such that
$b=F(g_1)(d_1),\ F(g_2)(d_1)=F(g_3)(d_2),\dots,\ F(g_{2l-1})(d_l)=F(g_{2l-2})(d_{l-1}),\ F(g_{2l})(d_l)=c$
and $\sum_{i=1}^{l} d(g_{2i-1},g_{2i})\le\widehat{d}(b,c)+\varepsilon$. It easily follows from the definition that then

$$\widehat{d}(a,c)\le\sum_{i=1}^{m} d(f_{2i-1},f_{2i})+\sum_{i=1}^{l} d(g_{2i-1},g_{2i})\le\widehat{d}(a,b)+\widehat{d}(b,c)+2\varepsilon.$$

Letting $\varepsilon\to0$, we are done. □

In the conditions as above, given a map $f:X\to Y$, we define $F(f):F(X)\to F(Y)$ as follows. If $a\in F(X)$, then $a\in F(\mathrm{supp}(a))$ and $F(f)(a)\in F(f(\mathrm{supp}(a)))\subset F(Y)$ is well-defined. We obtain a functor in the asymptotic category; see [2] for details.

Definition 1. If $F:\mathcal{K}_n\to\mathrm{Set}_f$ is a functor as above, we say that the obtained functor in the asymptotic category (for which we preserve the notation F) is a *normal functor* of degree $\le n$.

Let us consider a special case $F = \exp_n$. Given a metric space (X, d), one can also equip the set $\exp_n$ with the Hausdorff metric d_H:

$$d_H(A, B) = \inf\{\varepsilon > 0|\ A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}.$$

Proposition 1. $d_H \le \widehat{d} \le 3d_H$.

Proof. Let $A, B \in \exp_n X$ and $d_H(A, B) < \varepsilon$, for some $\varepsilon > 0$. For every $a \in A$, choose arbitrary $f(a) \in B$ such that $d(a, f(a)) < \varepsilon$. Also, for every $b \in B$, choose arbitrary $g(b) \in A$ such that $d(b, g(b)) < \varepsilon$.

Let $A_1 = f(A) \subset B$. Define a retraction $r : B \to A_1$ as follows: $r|_{A_1} = 1_A$, $r|_{B\setminus A_1} = fg|_{B\setminus A_1}$. Then $d(r, 1_B) \le d(fg, 1_B) < 2\varepsilon$ and we conclude that $\widehat{d}(A, B) \le d(1_A, f) + d(r, 1_B) < \varepsilon + 2\varepsilon = 3\varepsilon$.

On the other hand, suppose that $\widehat{d}(A, B) < \varepsilon$. Then there exists a diagram

$$A \xleftarrow{f_1} C_1 \xrightarrow{f_2} f_2(C_1) = f_3(C_2) \xleftarrow{f_3} C_3 \xrightarrow{f_4} \dots \xleftarrow{f_{2m-1}} C_m \xrightarrow{f_{2m}} B,$$

where $C_1, ..., C_m$ are nonempty sets of cardinality $\le n$, such that $\sum_{i=1}^{m} d(f_{2i-1}, f_{2i}) < \varepsilon$.
Given $a \in A$, one can find $c_i \in C_i$, $i \in \{1, ..., m\}$, such that $a = f_1(c_1)$, $f_{2i}(c_{2i-1}) = f_{2i+1}(c_{2i})$, $i \in \{1, ..., m-1\}$.

Then $f_{2m}(c_m) \in B$ and $d(a, f_{2m}(c_m)) < \varepsilon$. Therefore $A \subset O_\varepsilon(B)$. Similarly, one can prove that $B \subset O_\varepsilon(A)$. We conclude that $d_H(A, B) < \varepsilon$. □

Now, consider the case $F = SP^n_G$. If d is a metric on X, then in the set $SP^n_G X$ we consider the metric $\widetilde{d}([x], [y]) = \min_\sigma \max_{1\le i\le n} d(x_i, y_{\sigma(i)})$.

Proposition 2. $\widetilde{d} = \widehat{d}$.

Proof. Let $x, y \in SP^n_G X$, $x = [x_1, ..., x_n]$, $y = [y_1, ..., y_n]$, and $\widetilde{d}(x, y) < \varepsilon$. Then $\min_\sigma \max_{1\le i\le n} d(x_i, y_{\sigma(i)}) < \varepsilon$. There exists $\sigma \in S_n$ such that $\max d(x_i, y_{\sigma(i)}) < \varepsilon$. Consider a discrete space $Z = \{z_1, ..., z_n\}$ with $|Z| = n$. Let $z = [z_1, ..., z_n]$. Define maps $f_1, f_2 : Z \to X$ by the formula $f_1(z_i) = x_i$, $f_2(z_i) = y_{\sigma(i)}$. Then $SP^n_G f_1(z) = [f_1(z_1), ..., f_1(z_n)] = [x_1, ..., x_n] = x$; $SP^n_G f_2(z) = [f_2(z_1), ..., f_2(z_n)] = [y_{\sigma(1)}, ..., y_{\sigma(n)}] = y$.

Therefore $\widehat{d}(x, y) \le d(f_1, f_2) = \max d(x_i, y_{\sigma(i)}) < \varepsilon$. We conclude that $\widehat{d} < \widetilde{d}$.

Suppose now that $x, y \in SP^n_G X$ and $\widehat{d}(x, y) < \varepsilon$. Then there exists a diagram

$$X \xleftarrow{f_1} Z_1 \xrightarrow{f_2} X \xleftarrow{f_3} Z_2 \xrightarrow{f_4} \dots \xleftarrow{f_{2m-1}} Z_m \xrightarrow{f_{2m}} X,$$

where $Z_1, ..., Z_m$ are finite sets, elements $z_i = [z_{i1}, ..., z_{in}] \in SP^n_G X$ satisfying the properties:

$$1)\ SP^n_G(f)(z_1) = x;\ 2)\ SP^n_G(f)(z_m) = y;$$
$$3)\ SP^n_G(f_{2i})(z_i) = SP^n_G(f_{2i+1})(z_{i+1}),\quad i \in \{1, ..., m-1\}.$$

Let $x = [x_1, ..., x_n]$, $y = [y_1, ..., y_n]$. Without loss of generality, we may assume that $f(z_{1i}) = x_i$, for every $i \in \{1, ..., n\}$. For every $i \in \{1, ..., n\}$, there exists a permutation $\sigma_i \in G$ such that $f_{2i}(z_{ij}) = f_{2i-1}(z_{i+1,\sigma_i(j)})$, $i \in \{1, ..., m\}$; $j \in \{1, ..., n\}$.

Let $\sigma = \sigma_m^{-1} ... \sigma_1^{-1}$. We see that $d(x_i, y_{\sigma(i)}) < \varepsilon$, for every $i \in \{1, ..., n\}$, whence $\widetilde{d}(x, y) < \varepsilon$.

The equality $\widehat{d} = \widetilde{d}$ then follows from the above inequalities. □

Note that, of $G = \{e\}$, we obtain the sup-($= l_\infty$-)metric on the product X^n.

3. Main result. The main result of this note is the following one.

Theorem 2. *If* $\operatorname{asdim} X = 0$, *then* $\operatorname{asdim} F(X) = 0$.

Proof. Consider a space X with $\operatorname{asdim} X = 0$. Let $D > 0$ and $\mathcal{U}$ be a D-discrete uniformly bounded open cover of X.

Given a set Y and two maps $f, g\colon Y \to X$, we say that f and g are $\mathcal{U}$-close, if, for every $y \in Y$, there exists an element $U \in \mathcal{U}$ such that $\{f(y), g(y)\} \subset U$.

We say that $a, b \in F(X)$ are $\mathcal{U}$-chained, if there exist elements $c_i \in F(A_i)$, $\operatorname{supp}(c_i) = A_i$, $i \in \{1, \dots, k\}$, and maps $f_{2i-1}, f_{2i}, \ : A_i \to X$, $i \in \{1, \dots, k\}$ such that $a = F(f_1)(c_1)$, $F(f_{2i})(c_i) = F(f_{2i+1})(c_{i+1})$ for every $i \in \{1, \dots, k-1\}$, $F(f_{2k})(c_k) = b$, and the maps f_{2i-1}, f_{2i} are $\mathcal{U}$-close ($i \in \{1, \dots, k\}$).

For any $a \in F(X)$ we call the set $\{b \in F(X) \mid a \text{ and } b \text{ are } \mathcal{U}\text{-chained}\}$ the $\mathcal{U}$-component of a. Obviously, every two $\mathcal{U}$-components are either disjoint or coincide. We denote by $\widehat{\mathcal{U}}$ the family of all $\mathcal{U}$-components of the points in $F(X)$.

We have to show that the family $\widehat{\mathcal{U}}$ is D-disjoint and uniformly bounded.

1) Suppose that U, V are two distinct elements of $\widehat{\mathcal{U}}$, $a \in U$ and $b \in V$.

If $c_i \in F(A_i)$, $\operatorname{supp}(c_i) = A_i$, and f_{2i-1}, $f_{2i}\colon A_i \to X$, $i \in \{1, \dots, k\}$ are such that

$$a = F(f_1)(c_1),\ F(f_{2i})(c_i) = F(f_{2i+1})(c_{i+1}),\ i \in \{1, \dots, k-1\}, \text{ and } F(f_{2k})(c_k) = b,$$

then there exists $j \in \{1, \dots, k\}$ such that f_{2j-1}, f_{2j} are not $\mathcal{U}$-close. Thus, $d(f_{2j-1}, f_{2j}) > D$, whence $\widehat{d}(a, b) \geq D$, and we are done.

2) We are going to show that the family $\widehat{\mathcal{U}}$ is uniformly bounded.

Consider the triples of the form (c, f, g), where $c \in F(A)$, $\operatorname{supp}(c) = A$, where A is a finite set, and $f\colon A \to B$, $g\colon A \to C$ are onto maps. Let us say that triples (c', f', g') and (c'', f'', g'') are adjoint if $c' \in F(A')$, $\operatorname{supp}(c') = A'$, $c'' \in F(A'')$, $\operatorname{supp}(c'') = A''$ and there exists bijective map $t\colon A' \to A''$, $r\colon f'(A') \to f''(A'')$, $s\colon g'(A') \to g''(A'')$ such that the diagram

$$\begin{array}{ccccc}
f'(A') & \xleftarrow{f'} & A' & \xrightarrow{g'} & g'(A') \\
\downarrow r & & \downarrow t & & \downarrow s \\
f''(A'') & \xleftarrow{f''} & A'' & \xrightarrow{g''} & g''(A'')
\end{array}$$

is commutative and $F(t)(c') = c''$.

Since the cardinality of the sets $F(A)$, where $|A| \leq n$, is bounded, there are at most finitely many mutually non-adjoint triples. Let us denote by N the maximal cardinality of a family of mutually non-adjoint triples. Now, let $a, b \in F(X)$ be $\mathcal{U}$-chained elements. Find $f_{2i-1}, f_{2i}\colon A_i \to X$ and $c_i \in F(A_i)$, $\operatorname{supp}(c_i) = A_i$, $i \in \{1, \dots, m\}$, such that (1) holds and all the maps $f_1, \dots, f_{2m}$ are $\mathcal{U}$-close. If $m > N$, one can find $k, l \in \{1, \dots, m\}$, $k < l$, for which the triples (c_k, f_{2k-1}, f_{2k}) and (c_k, f_{2l-1}, f_{2l}) are adjoint.

We then see that $f_1, f_2, \dots, f_{2k-1}, f_{2l}, \dots, f_{2m}$ and $c_1, c_2, \dots, c_{k-1}, c_k = c_l, c_{l+1}, \dots, c_m$ form a chain connecting a and b and of length $m-1$. Therefore, in the definition of the distance $\widehat{d}(a, b)$, we can restrict ourselves with the chains of length not exceeding N. Note also that if we start with a chain in which all the subsequent maps are $\mathcal{U}$-close, then the resulting chain has the same property. This allows us to estimate the distance between a and b as follows. Let $\operatorname{mesh}\mathcal{U} = \sup\{\operatorname{diam} U \mid U \in \mathcal{U}\}$, where, as usual, $\operatorname{diam} U = \sup\{d(x, y) \mid x, y \in U\}$. Since the family $\mathcal{U}$ is uniformly bounded, we see that $\operatorname{mesh}\mathcal{U} < \infty$ and $\widehat{d}(a, b) \leq N \cdot \operatorname{mesh}\mathcal{U}$. This proves that the family $\widehat{\mathcal{U}}$ is uniformly bounded.

We therefore conclude that asdim $F(X) = 0$. □

Recall that a metric space is of the Assouad-Nagata dimension 0 (written $\dim_{AN} X = 0$) if there exists $c > 0$ such that, for any $D > 0$, there exists a uniformly bounded D-disjoint cover $\mathcal{U}$ of X such that mesh $\mathcal{U} \le c \cdot D$ (see [7]).

By using a similar technique, one can prove the following result.

Theorem 3. *Let X be a proper metric space of the Assouad-Nagata dimension 0. Then $\dim_{AN} F(X) = 0$, where F is a functor as above.*

Proof. We see that one can take $c = N \cdot c'$, where N is taken from the proof of Theorem 2 and c' is the constant from the definition of $\dim_{AN} F(X)$ (i. e. mesh$\mathcal{U} \le c' \cdot D$). □

If d is an ultrametric on X, then define a function $\widehat{d} : F(X) \times F(X) \to \mathbb{R}$ as follows:

$$\widehat{d}(x, y) = \max \{d(f_{2i-1}, f_{2i})\ i \in \{1, ..., k\}\}.$$

The proof of the following is obvions.

Proposition 3. *The function $\widehat{d}$ is an ultrametric on $F(X)$.*

4. Remarks and open questions. In the theory of functors in the category of compact Hausdorff spaces, one has the following result by Basmanov [8, 9]: the functors of finite degree preserve the class of finite-dimensional spaces. Moreover, if the dimension of X does not exceed m and the degree of F does not exceed n, then the dimension of $F(X)$ does not exceed nm. The counterpart of this result in the asymptotic category (see [2] for some properties of functors in this category) requires a different technique than that used in this note.

In [2], a related question of definition of a coarse structure on the sets of the form $F(X)$ is considered. It looks plausible that, if a coarse structure is generated by a metric, then the corresponding coarse structure on $F(X)$ defined in [2] coincides with that defined above.

A metric space is said to be a space of *bounded geometry* if, for every $r > 0$ the ball $B_r(x)$ contains at most $c(r)$ points that are 1 apart, where $c(r)$ is a constant independent of x. For spaces of bounded geometry, one can provide a more direct proof of the main result.

A counterpart of the obtained result can be also proved for some functors of infinite degree, e.g., for the hyperspace functor. We leave as an open problem that of extension of the results of this note onto functors of infinite degree.

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Faculty of Mechanics and Mathematics
Ivan Franko National University of L'viv

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