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COARSE TREES

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Using Gromov product, we give a characterization of the metric spaces coarsely equivalent to $\mathbb{R}$-trees.

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Используя произведение Громова, мы приводим характеризацию метрических пространств, грубо эквивалентных $\mathbb{R}$-деревьям.

Let (X, d), (Y, ρ) be metric spaces. A mapping $f\colon X \to Y$ is called a *coarse embedding* of (X, d) into (Y, ρ) if, for every $r > 0$, there exists $s > 0$ such that, for all x_1,$x_2 \in X$,

$$d(x_1, x_2) \leq r \Rightarrow \rho(f(x_1), f(x_2)) \leq s,$$
$$\rho(f(x_2), f(x_2)) \leq r \Rightarrow d(x_1, x_2) \leq s.$$

The metric spaces (X, d), (Y, ρ) are called *coarsely equivalent* if there exists a coarse embedding $f\colon X \to Y$ and $t > 0$ such that, for every $y \in Y$, there exists $z \in f(X)$ such that $\rho(y, z) \leq t$.

A geodesic metric space (T, ρ) is called an $\mathbb{R}$-*tree* if any two points of T are joined by a unique geodesic segment, and if two geodesic segments meet only at a common endpoint, then their union is a geodesic segment.

In this note we characterize the class of metric spaces which are coarsely embeddable and coarsely equivalent to $\mathbb{R}$-trees. To this end, given a metric space (X, d) with a distinguished point w, we use the Gromov product on X, defined as

$$(x|y)_w = \frac{1}{2}(d(x, w) + d(y, w) - d(x, y)).$$

For geometric interpretation of the Gromov product see [1] or [2, Chapter 6].

If a metric space (X, d) is an $\mathbb{R}$-tree, then for any points $x, y, z \in X$ we have

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} \qquad (\star)$$

By [2, Proposition 6.12], a metric space satisfies $(\star)$ if and only if it can be isometrically embedded into an $\mathbb{R}$-tree, and a geodesic metric space satisfying $(\star)$ is an $\mathbb{R}$-tree.

A metric space (X, d) is called δ-hyperbolic, for some $\delta \geq 0$, if

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - \delta$$

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for any $x, y, z \in X$. By above the paragraph, a geodesic metric space is an $\mathbb{R}$-tree if and only if it is 0-hyperbolic.

For a metric space (X, d), $w \in X$ and $\delta \geq 0$, we consider the following new condition

$$(x_1|x_n)_w \geq \min\{(x_1|x_2)_w, ..., (x_{n-1}|x_n)_w\} - \delta \tag{$\star\star$}$$

for any $x_1, ..., x_n \in X$.

Theorem 1. *If a metric space (X, d) satisfies $(\star\star)$ then there exists an $\mathbb{R}$-tree (T, ρ) and a mapping $f\colon X \to T$ such that*

(i) $d(x, w) = \rho(f(x), f(w))$ *for every* $x \in X$*;*

(ii) $\rho(f(x), f(y)) \leq d(x, y)$*for any* $x, y \in X$*;*

(iii) $\operatorname{codiam} f \leq 2\delta$*where* $\operatorname{codiam} f = \sup\limits_{x,y \in X} |d(x, y) - \rho(f(x), f(y))|$.

In particular, if a metric space satisfies $(\star\star)$, then it is coarsely embeddable into an $\mathbb{R}$-tree.

Proof. Following [1, Proposition 6.1A], we introduce a new product $(x|y))^{'}_w$ on X. Let us consider the family S of all finite sequences $x_1, ..., x_n$ in X such that $x = x_1$,$y = x_n$, and put

$$(x|y)^{'}_w = \sup_S \min(x_i|x_{i+1})_w.$$

Then we put

$$d^{'}(x, y) = d(x, w) + d(y, w) - 2(x|y)^{'}_w$$

and notice that $d^{'}$ is a 0-hyperbolic metric on X. Moreover, $d(x, w) = d^{'}(x, w)$, $d(x, y) - 2\delta \leq d^{'}(x, y) \leq d(x, y)$.

To construct the tree T we consider the family $\{I_x\colon x \in X\}$ of copies of the segment $\{[0, d^{'}(w, x)]\colon x \in X\}$ and denote by B the disjoint union of the family $\{I_x\colon x \in X\}$. Then we define a relation $\sim$ on B by the rule

$$(t, x) \sim (t', x) <=> t = t' \leq (x|x')_w.$$

Due to the inequality $(x|y)^{'}_w \geq \min\{(x|z)^{'}_w, (y|z)^{'}_w\}$, this is an equivalence. We put $T = B/\sim$. By [2, Proposition 6.11], T is an $\mathbb{R}$-tree and the mapping $f\colon X \to T$, where $f(x)$ is the equivalence class containing $[0, d^{'}(w, x)]$, is an isometric embedding. By the construction of $d^{'}$, f satisfies *(i), (ii), (iii).* □

Given a metric spaces (X, d), (Y, ρ) and $\lambda > 0$, $c \geq 0$, a mapping $f\colon X \to Y$ is called a (λ, c)-*isometry* if

$$\lambda^{-1} d(x, y) - c \leq \rho(f(x), f(y)) \leq \lambda d(x, y) + c$$

for any $x, y \in X$. In this case we say that (X, d) is *quasi-isometrically* embeddable into (Y, ρ). In the proof of the next theorem we use the following observation: if (X, d), (Y, ρ) are geodesic metric spaces, then $f\colon X \to Y$ is a coarse embedding if and only if f is a (λ, c)-isometry.

Theorem 2. *If a geodesic metric (X, d) is coarsely embeddable into an $\mathbb{R}$-tree (T, ρ), then (X, d) satisfies $(\star\star)$.*

Proof. Let $x_1, x_2, ..., x_n \in X$, $f\colon X \to Y$ be a (λ, c)-isometry. Since (T, ρ) is an $\mathbb{R}$-tree, $[f(x_1), f(x_n)]$ is contained in the union of geodesic segments

$$[f(x_1), f(x_2)], [f(x_2), f(x_3)], ..., [f(x_{n-1}), f(x_n)].$$

By [2, Theorem 6.17], there exists $\delta_1 > 0$ depending only on λ and c such that $f([x_1, x_n])$, $f([x_1, x_2])$, $f([x_2, x_3]), ..., f([x_{n-1}, x_n])$ are of Hausdorff distance at most δ_1 from the geodesic segments $[f(x_1), f(x_n)], [f(x_1), f(x_2)], ..., [f(x_{n-1}), f(x_n)]$. Hence, $f([x_1, x_n])$ is in the δ_1-neighbourhood of

$$f([x_1, x_2]) \cup f([x_2, x_3]) \cup ... \cup f([x_{n-1}, x_n]).$$

Since f is a (λ, c)-isometry, there is $\delta_2 > 0$ depending only on δ_1, λ, c such that $[x_1, x_2]$ is contained in the δ_2-neighbourhood of $[x_1, x_2] \cup [x_2, x_3] \cup ... \cup [x_{n-1}, x_n]$. By [Corollaries 6.20 and 6.9], there exists $\delta_3 > 0$ such that, for every geodesic segment $[y, z]$ in X, we have

$$d(w, [y, z]) - \delta_3 \le (y|z)_w \le d(w, [y, z]).$$

Thus, to satisfy $(\star\star)$ we can choose $\delta > 0$ depending only on δ_2 and δ_3. □

Theorem 3. *For every geodesic metric space (X, d), the following statements are equivalent*

(i) (X, d) is coarsely embeddable into an $\mathbb{R}$-tree;

(ii) (X, d) is coarsely equivalent to an $\mathbb{R}$-tree;

(iii) (X, d) satisfies $(\star\star)$.

Proof. The equivalence $(i) <=> (iii)$ follows from Theorems 1 and 2, so we should prove only $(i) \Rightarrow (ii)$. Let (T, ρ) be an $\mathbb{R}$-tree, $f\colon X \to T$ be a coarse embedding. We put $T' = \bigcup\{[f(x), f(y)]\colon x, y \in X\}$ and show that (X, d) is coarsely equivalent to (T', ρ). Let z be an arbitrary point from T'. Choose $x, y \in X$ such that $z \in [f(x), f(y)]$. By [2,Theorem 6.17], there exists $\delta > 0$ depending only on f such that $[f(x), f(y)]$ is in the δ-neighbourhood of $f([x, y])$. Hence, there exists $z' \in f(X)$ such that $d(z, z') \le \delta$. □

The following example shows that the geodesity assumption could not be dropped in Theorem 2.

Example. Let $X = \{(x, y) \in \mathbb{R}^2\colon y = x^2\}$, d be the euclidian metric on X. We put $T = X$ and for any $a, b \in X$, denote by $\rho(a, b)$ the length of the arc of T with endpoints a, b. Clearly, (T, ρ) is an $\mathbb{R}$-tree and the identity mapping $(X, d) \to (T, \rho)$ is a coarse equivalence, but (X, d) is not even hyperbolic, so (X, d) does not satisfy $(\star\star)$.

Question. *Let a metric space (X, d) be quasi-isometrically embeddable into an $\mathbb{R}$-tree. Does (X, d) satisfy $(\star\star)$?*

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