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GROWTH OF ANALYTIC FUNCTIONS IN THE UNIT DISC AND COMPLETE MEASURE IN THE SENSE OF GRISHIN

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Let $\rho_T[f]$ and $\rho_M[f]$ be the orders of an analytic function f in the unit disc defined by the Nevanlinna characteristic and the maximum modulus function, respectively. Given $0 \le \sigma \le \rho \le \sigma+1$, $\rho \ge 1$, we describe the class A_σ^ρ of analytic function in $\mathbb{D}$ such that $\rho_T[f]=\sigma$, $\rho_M[f]=\rho$ in terms of the so called complete measure (in the sense of Grishin) of an analytic function. The proofs are based on results of C.N.Linden and recent results of the author.

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Пусть $\rho_T[f]$ и $\rho_M[f]$ — порядки аналитической в единичном круге функции f, определяемые по характеристике Неванлинны и максимуму модуля соотвественно. По заданным $0 \le \sigma \le \rho \le \sigma+1$, $\rho \ge 1$, мы описываем класс A_σ^ρ аналитических функций в $\mathbb{D}$ таких, что $\rho_T[f]=\sigma$, $\rho_M[f]=\rho$ в терминах так называемой полной меры (в смысле Гришина) аналитической функции. Доказательства основаны на результатах Линдена и недавних результатах автора.

1. Introduction.

1.1. Analytic functions in the unit disc. Let $D(z,t)=\{\zeta\in\mathbb{C}\colon |\zeta-z|<t\}$, $z\in\mathbb{C}$, $t>0$, $\mathbb{D}=D(0,1)$. Denote by $A(\mathbb{D})$ the class of analytic functions in $\mathbb{D}$. For a meromorphic function f, $f(0)\notin\{0,\infty\}$, we define the maximum modulus $M(r,f)=\max\{|f(z)|\colon |z|=r\}$, and the Nevanlinna characteristic $T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta+N(r,\infty,f)$, $0<r<1$, where $x^+=\max\{x,0\}$, $N(r,\infty,f)=\int_0^r\frac{n(t,\infty,f)}{t}dt$, $n(r,\infty,f)$ is the number of poles of f in the closed disc $\overline{D(0,r)}$. In the sequel, the symbol C with indices stands for some positive constants. We write $a\asymp b$ if $C_1a\le b\le C_2a$ for some positive constants C_1 and C_2, and $a(r)\sim b(r)$ if $\lim_{r\to1}a(r)/b(r)=1$.

Usually, the orders of the growth of analytic functions in $\mathbb{D}$ are defined as

$$\rho_M[f]=\limsup_{r\uparrow1}\frac{\log^+\log^+M(r,f)}{-\log(1-r)},\ \rho_T[f]=\limsup_{r\uparrow1}\frac{\log^+T(r,f)}{-\log(1-r)}.$$

It is well known that

$$\rho_T[f]\le\rho_M[f]\le\rho_T[f]+1, \tag{1}$$

and all cases are possible. Given $\rho>1$, σ satisfying $\sigma\le\rho\le\sigma+1$, C. N. Linden ([4]) also constructed a canonical product with the property $\rho_T[g]=\sigma$, $\rho_M[g]=\rho$.

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Professor O. Skaskiv has drown our attention to the following natural problem.

Problem 1. *Given* $0 \le \sigma \le \rho \le \sigma + 1$, *describe the class* A_σ^ρ *of analytic function in* $\mathbb{D}$ *such that* $\rho_T[f] = \sigma$, $\rho_M[f] = \rho$.

In order to solve Problem 1 one needs a representation of functions analytic in $\mathbb{D}$ of finite order of the growth.

In 1952 A.G. Naftalevich [1] and in 1955 M.Tsuji [2] independently proved that if $w(z)$ is meromorphic and of order $\rho_T[w] = \rho$ in $\mathbb{D}$ then

$$w(z) = \frac{P(z)}{Q(z)} e^{g(z)},$$

where $g(z)$ is an analytic function in $\mathbb{D}$, $P(z)$ and $Q(z)$ are canonical products constructed by the zeros and poles of $w(z)$, respectively. Each of the functions $P(z)$, $Q(z)$, $e^{g(z)}$ is analytic and of order at most ρ. The canonical product $P(z)$ has the form

$$P(z, (a_n), q) = \prod_{n=1}^{\infty} E\Big(\frac{1-|a_n|^2}{1-\bar{a}_n z}, q\Big), \quad \sum_n (1-|a_n|)^{q+1} < \infty, \tag{2}$$

where $E(w, q) = (1-w)\exp\{w + w^2/2 + \cdots + w^q/q\}$, $q \in \mathbb{Z}_+$, is the Weierstrass primary factor, (a_n) are the sequence of zeros of $w(z)$, q is the smallest integer such that $\sum_n (1-|a_n|)^{q+1} < \infty$.

Asymptotic properties of canonical products (2) were systematically studied by C. N. Linden ([3]–[7]). In particular, he established a connection between $\rho_M[P]$ and the zero distribution of P.

Let $\square(re^{i\varphi}) = \Big\{\zeta\colon r \le |\zeta| \le \dfrac{1+r}{2}, |\arg\zeta - \varphi| \le \pi(1-r)\Big\}$, $\nu(re^{i\varphi})$ be the number of zeros of P in $\square(re^{i\varphi})$. We define

$$\nu_1(\varphi) = \varlimsup_{r\uparrow 1} \frac{\log^+ \nu(re^{i\varphi}, P)}{-\ln(1-r)}, \quad \nu[P] = \sup_{\varphi} \nu_1(\varphi), \quad \rho_n[P] = \varlimsup_{r\uparrow 1} \frac{\log^+ n(r, P)}{-\log(1-r)},$$

where $n(r, P)$ is the number of zeros in $\overline{D(0, r)}$. Then

Theorem A ([3, Theorem V]). *With the notation above we have*

$$\rho_T[P] = (\rho_n[P] - 1)^+, \tag{3}$$

$$\rho_M[P] \begin{cases} = \nu[P], & \rho_M[P] \ge 1, \\ \le \nu[P] \le 1, & \rho_M[P] < 1. \end{cases} \tag{4}$$

Remark 1. We note that relation (3) essentially follows from [1, 2]. Moreover, $(\rho_n[P]-1)^+$ is equal to the convergence exponent of the zero sequence of P defined by

$$\mu[P] = \inf\Big\{\mu \ge 0\colon \sum_n (1-|a_n|)^{\mu+1} < \infty\Big\}.$$

A parametric representation of functions analytic in $\mathbb{D}$ was obtained M. M. Djrbashian using the Riemann-Liouville fractional integral ([9]).

Following Djrbashian, consider two subclasses of $A(\mathbb{D})$, $\alpha > 0$,

$$A_\alpha\colon \sup_{0<r<1} \int_0^{2\pi} \Big(\int_0^r (r-t)^{\alpha-1} \log|f(te^{i\varphi})|\, dt\Big)^+ d\varphi < +\infty,$$

$$A^*_\alpha\colon \sup_{0<r<1}\int_0^{2\pi}\left(\int_0^r (r-t)^{\alpha-1}\log^+|f(te^{i\varphi})|\,dt\right)d\varphi<+\infty.$$

Obviously, $A^*_\gamma\subset A^*_\alpha\subset A_\alpha\subset A_\beta$, $\gamma<\alpha<\beta$. Note that $f\in A^*_\alpha$ means $\int_0^1 T(t,f)(1-t)^{\alpha-1}dt<+\infty$, i.e. f belongs to the convergence class of order α.

Throughout this paper by $(1-w)^\alpha$, $w\in\mathbb{D}$, $\alpha\in\mathbb{R}$, we mean the branch of the power function such that $(1-w)^\alpha\Big|_{w=0}=1$.

Theorem B. *The class A_α, $\alpha>-1$, coincides with the class of functions represented in the form*

$$f(z)=C_\lambda z^\lambda B_\alpha(z)\exp\left\{\int\limits_0^{2\pi}\frac{d\psi(\theta)}{(1-e^{-i\theta}z)^{\alpha+1}}\right\}\equiv C_\lambda z^\lambda B_\alpha(z)\exp\{g_\alpha(z)\},\tag{5}$$

where $\psi\in BV[0,2\pi]$, (z_k) is the zero sequence of $f(z)$ such that $\sum_k(1-|z_k|)^{\alpha+1}<+\infty$,$B_\alpha(z)=\prod_k\left(1-\frac{z}{z_k}\right)\exp\{-W_\alpha(z,z_k)\}$ is Djrbashian's product,

$$W_\alpha(z,\zeta)=\int_{|\zeta|}^1\frac{(1-x)^\alpha}{x}dx+\sum_k\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)\Gamma(1+k)}\times$$

$$\times\left((\overline{\zeta}z)^k\int_{|\zeta|}^1\frac{(1-x)^\alpha}{x^{k+1}}dx-\left(\frac{z}{\zeta}\right)^k\int_0^{|\zeta|}(1-x)^\alpha x^{k-1}\,dx\right).$$

In view of a complicated structure of $B_\alpha(z)$ it is easier to work with product (2) of Naftalevich-Tsuji. It is clear that if $\rho_T[f]$ is finite or, which is equivalent, $\rho_M[f]$ is finite, f can be represented in the form $f(z)=cz^\lambda P(z,(a_n),q))g_q(z)$ for an appropriate choice of q (cf. proof of Theorem 4).

Recently the author solved [8] Problem 1 in the class of non-zero analytic functions in $\mathbb{D}$, using the technic of fractional integration. The solution is based on works of M.Djrbashian [9], G.Hardy and J.Littlewood [11], F.Shamoyan [13] and the author [14]. In order to formulate results we need preliminaries concerning fractional integration.

1.2. Fractional integration. The Riemann-Liouville fractional integral of order $\alpha>0$ for $h\in L(0,1)$ is defined by the formulas [9, Chap. IX,§1], [10, V.2,Chap.XII,§8]

$$D^{-\alpha}h(r)=\frac{1}{\Gamma(\alpha)}\int_0^r(r-x)^{\alpha-1}h(x)\,dx,$$

$$D^0h(r)\equiv h(r),\quad D^\alpha h(r)=\frac{d^p}{dr^p}\{D^{-(p-\alpha)}h(r)\},\ \alpha\in(p-1;p].$$

If we deal with periodic functions, in particular with trigonometric series, the definition of Riemann-Liouville is not suitable. Then we consider a definition due to H.Weyl [10, Chap.XII,§8]. Let $f\in L(0,2\pi)$. Suppose that

$$\int_0^{2\pi}f(x)\,dx=0.\tag{6}$$

If $\sum_{n\in\mathbb{Z}} c_n e^{inx}$, $c_0 = 0$ is the Fourier series of f, then the fractional integral (derivative) $I_\alpha[f]$ of order α is defined by the Fourier series

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{c_n e^{inx}}{(in)^\alpha} \overset{\alpha\ge 0}{=} \frac{1}{2\pi}\int_0^{2\pi} f(t)\Psi_\alpha(x-t)\,dt. \tag{7}$$

where $\Psi_\alpha(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{int}(in)^{-\alpha}$ is convergent almost everywhere on $[0, 2\pi]$ as $\alpha > 0$. In what follows $i^\alpha = e^{i\pi\alpha/2}$. Let $f_\alpha = i^\alpha I_\alpha[f]$. The integral in (7) exists almost everywhere, its value is integrable, and the series (7) is convergent almost everywhere and is the Fourier series of f_α. Moreover f_α satisfies (6). The operator I_α has the same properties as D^α. If $\alpha \in (0, 1)$ the derivative $f_{-\alpha}$ of order α is defined by $f_{-\alpha}(x) = \frac{d}{dx} f_{1-\alpha}(x)$. If $\alpha > 0$, $n-1 \le \alpha < n$, $n \in \mathbb{N}$, then $f_{-\alpha}(x) = \frac{d^n}{dx^n} f_{n-\alpha}(x)$. There is the following connection between the definitions of Riemann-Liouville and Weyl

$$f_\alpha(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)\Psi_\alpha(t)\,dt = \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}\,dt + \frac{1}{2\pi}\int_0^{2\pi} f(t) r_\alpha(x-t)\,dt,$$

where $r_\alpha(x)$ is an analytic function of x, $\alpha > 0$.

Let $\psi\colon [0, 2\pi] \to \mathbb{R}$. We denote by $\mathrm{BV}[0, 2\pi]$ and $\mathrm{AC}[0, 2\pi]$ the classes of functions of bounded variations and absolutely continuous functions on $[0, 2\pi]$, respectively. If $\psi_{-\beta} \in \mathrm{AC}[0, 2\pi]$, $\beta > 0$, we shall write $\psi \in \mathrm{AC}^\beta[0, 2\pi]$. Similarly $\psi \in \mathrm{BV}^\beta[0, 2\pi]$, if $\psi_{-\beta} \in \mathrm{BV}[0, 2\pi]$. Let

$$\omega(\delta, \psi) = \sup\{|\psi(x) - \psi(y)| \colon x, y \in [0, 2\pi], |x-y| < \delta\}$$

be the modulus of continuity of $\psi \in \mathrm{BV}[0, 2\pi]$.

Following [10] we say that $\psi \in \Lambda_\gamma$ if $\omega(\delta; \psi) = O(\delta^\gamma)$ $(\delta \downarrow 0)$. We define $\tau[\psi] = \sup\{\gamma \ge 0 \colon \psi \in \Lambda_\gamma\}$, $\gamma[\psi] = \sup\{\tau \ge 0 \colon \psi \in \mathrm{AC}^\tau[0, 2\pi]\} = \sup\{\tau \ge 0 \colon \psi \in \mathrm{BV}^\tau[0, 2\pi]\}$.

We note that the operator I_α maps Λ_β onto $\Lambda_{\beta+\alpha}$ provided $\alpha, \beta, \alpha+\beta \in (0, 1)$ (see Theorems (8.13), (8.14) [10, V.2,Ch.XII]). Therefore the map $\tau_\psi(\eta) = \tau[\psi_\eta]$ defined for $\eta \ge 0$ is continuous. This fact allows us to define $\tau[\psi] = \lim_{\eta\downarrow 0} \tau[\psi_\eta]$ for $\psi \in \bigcap_{\beta<0} \mathrm{BV}^\beta[0, 2\pi]$.

Let

$$S_\alpha(z) = \Gamma(1+\alpha)\Big(\frac{2}{(1-z)^{\alpha+1}} - 1\Big), \quad P_\alpha(r, t) = \operatorname{Re} S_\alpha(re^{it}).$$

Note that $S_0(z)$ is the Schwartz kernel, $P_0(r, t)$ is the Poisson kernel; $P_\alpha(r, t) = D^\alpha(r^\alpha P_0(r, t))$.

The following theorems solve Problem 1 in the class of non-vanishing analytic functions in $\mathbb{D}$.

Theorem C ([8, Theorem 3]). *Let $g(z) = \exp\{h(z)\}$, where*

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} S_\alpha(ze^{-it})\,d\psi(t) + i\operatorname{Im} h(0), \tag{8}$$

$\alpha \ge 0$, $\psi \in \mathrm{BV}[0, 2\pi]$. Then $\rho_T[g] = (\alpha - \gamma[\psi])^+$.

Theorem D ([8, Corollary 2]). *Let $g(z) = \exp\{h(z)\}$, where h has form (8), $\alpha \ge 0$, $\psi \in \mathrm{BV}[0, 2\pi]$. Then $\rho_T[g] = (\alpha - \gamma[\psi])^+$, $\rho_M[g] = \rho_T[g] + 1 - \tau[\psi_{\rho_T[g]-\alpha}]$.*

Theorem E ([8, Theorem 4]). *Let $0 \le \sigma \le \rho \le \sigma + 1 < +\infty$, g be analytic and $g(z) \ne 0$ in $\mathbb{D}$. Then $g \in A^\rho_\sigma$ iff for any $\alpha > \sigma$ we have $g(z) = \exp\{h(z)\}$, where $h(z)$ has form (8) with ψ which is determined by g and satisfies $\sigma = (\alpha - \gamma[\psi])^+$ and $\tau[\psi_{\sigma-\alpha}] = 1 - \rho + \sigma$.*

Moreover, $\psi \in \bigcap_{\beta<\alpha-\sigma} \mathrm{BV}^\beta[0, 2\pi]$.

2. Complete measure and main results. In view of the results cited in the introduction we have all components to solve Problem 1 under restriction $\rho_M[f] \ge 1$. The problem is that different approaches are used for canonical products and non-vanishing analytic functions. In the present paper we propose an integrated approach based on the concept of so called *complete measure of an analytic function in the sense of Grishin.* This concept was originally introduced by A.Grishin in [15, 16] for a subclass of subharmonic functions in the half-plane with non-positive boundary values. The advantage of the complete measure is that it takes into account not only Riesz' measure (zero distribution) of subharmonic function (analytic function) but the boundary measure as well. It is also possible to describe the local growth of an argument of a bounded analytic function in terms of the complete measure ([17]).

Let f be of the form

$$f(z) = C_q z^\lambda P(z, (a_k), q) \exp\Big\{\frac{1}{2\pi}\int\limits_0^{2\pi} S_q(ze^{-i\theta}) d\psi^*(\theta)\Big\}, \tag{9}$$

where $P(z,(a_n),q)$ has form (2), $\psi^* \in BV[0,2\pi]$, (a_k) is the zero sequence of $f(z)$ such that $\sum_k (1-|a_k|)^{q+1} < +\infty$, $\lambda \in \mathbb{Z}_+$, $C_q \in \mathbb{C}$. Let M be Borel's subset of $\overline{\mathbb{D}}$ such that $M \cap \partial\mathbb{D}$ is measurable with respect to the Lebesgue measure on $\partial\mathbb{D}$. A *complete measure* λ_f *of genus* q *in the sense of Grishin* is defined by

$$\lambda_f(M) = \int_{\mathbb{D}\cap M} (1-|\zeta|)^{q+1}\, d\mu_f(\zeta) + \psi(M \cap \partial\mathbb{D}), \tag{10}$$

where μ_f is the Riesz measure of $\log|f|$, i.e. $\mu_f(\zeta) = \sum_n \delta(\zeta - a_n)$, $\delta(\zeta)$ the unit mass supported at ζ, ψ is the Stieltjes measure generated by ψ^*.

Remark 3. There is the following interplay between $f(z)$ and $\psi^*(\varphi)$ (see [9, Chap.IX,§4]):

$$\psi^*(\varphi) = \lim_{n\to\infty} \int_0^\varphi D^{-q} \log|f(r_n e^{i\theta})| d\theta - \lim_{n\to\infty} \int_0^\varphi D^{-q} \log|P(r_n e^{i\theta})| d\theta - \varphi^q \frac{\log|C_q|}{q!},$$

where (r_n) is a sequence on $(0,1)$, $r_n \uparrow 1$ $(n \to +\infty)$. If $q=0$, then [18] there exists a finite limit

$$\lim_{n\to\infty} \int_0^\varphi \log|P(r_n e^{i\theta})| = \lim_{n\to\infty} \int_0^\varphi \ln \prod_{k=1}^\infty \Big|\frac{\overline{a}_k(a_k - r_n e^{i\theta})}{1 - r_n e^{i\theta}\overline{a}_k}\Big| d\theta = \varphi \log \prod_{k=1}^\infty |a_k|.$$

Thus, the contribution of the canonical product to the boundary measure ψ is $C\varphi$, which is absolutely continuous with all its derivatives, so has no influence on the growth of f.

The measure $\lambda = \lambda_f$ has the following properties:

(1) λ is finite on $\overline{\mathbb{D}}$;

(2) λ is positive on $\mathbb{D}$, i.e. $\lambda(B) \ge 0$ for any Borel set $B \subset \mathbb{D}$;

(3) λ is a zero measure outside $\overline{\mathbb{D}}$;

(4) $d\lambda\Big|_{\partial\mathbb{D}}(\zeta) = d\psi(\zeta)$;

(5) $d\lambda\Big|_{\mathbb{D}}(\zeta) = (1-|\zeta|)^{q+1}\, d\mu_f(\zeta) \equiv d\mu_*(\zeta)$.

For a complete measure λ and $\alpha \ge 0$, we define λ^α by $d\lambda^\alpha(\zeta) = (1-|\zeta|)^{q+1-\alpha} d\mu(\zeta) + d\psi^\alpha(\zeta)$, where $\psi^\alpha = i^{-\alpha} I_{-\alpha}[\psi]$. We say that $\lambda \in \mathrm{BV}^\alpha(\overline{\mathbb{D}})$ if

$$\int_{\overline{\mathbb{D}}} |d\lambda^\alpha(\zeta)| = \int_{\mathbb{D}} (1-|\zeta|)^{q+1-\alpha} d\mu(\zeta) + \int_{\partial\mathbb{D}} |d\psi^\alpha(\zeta)| < \infty. \tag{11}$$

Let $\gamma[\lambda] = \sup\{\tau \ge 0 \colon \lambda \in \mathrm{BV}^\tau[0, 2\pi]\}$.

Theorem 1. *Let $q \in \mathbb{Z}_+$. Let $f(z)$ be an analytic function in $\mathbb{D}$ of form (9). Let $\lambda = \lambda_f$ be the complete measure of f of genus q, and $\lambda \in \mathrm{BV}(\mathbb{D})$. Then $\rho_T[f] = (q - \gamma[\lambda])^+$.*

Proof of Theorem 1. Let $\gamma[\psi] = \sup\{\tau \colon \psi \in \mathrm{BV}^\tau[0, 2\pi]\}$,

$$\gamma[\mu_*] = \sup\{\tau \colon \mu^* \in \mathrm{BV}^\tau(\mathbb{D})\} = \sup\Big\{\tau \colon \int_{\mathbb{D}} (1-|\zeta|)^{q+1-\tau} |d\mu(\zeta)| < +\infty\Big\}.$$

We note that $q - \gamma[\mu_*]$ is the exponent of convergence of the zero sequence of the function f. According to the theorem of Tsuji ([2, Theorem 2])

$$\rho_T[P] = (q - \gamma[\mu_*])^+ \le \rho_T[f]. \tag{12}$$

By the definitions, we have $\gamma[\lambda] = \min\{\gamma[\psi], \gamma[\mu_*]\}$.

We consider two cases. First, let $\gamma[\mu_*] > \gamma[\lambda]$. Then $\gamma[\psi] = \gamma[\lambda]$. In this case by Theorem D we have $\rho_T[g] = (q - \gamma[\psi])^+ = (q - \gamma[\lambda])^+$. By (12), we have $\rho_T[P] = (q - \gamma[\mu_*])^+ < (q - \gamma[\lambda])^+ = \rho_T[g]$ unless the case $q < \gamma[\lambda]$. Therefore $\rho_T[f] = \rho_T[Pg] = (q - \gamma[\lambda])^+$. If $q < \gamma[\lambda]$, then $\rho_T[P] = \rho_T[f] = 0 = (q - \gamma[\lambda])^+$.

Now, let $\gamma[\psi] \ge \gamma[\lambda] = \gamma[\mu_*]$. By Theorem D $\rho_T[g] = (q - \gamma[\psi])^+ \le (q - \gamma[\lambda])^+$. Suppose that $\rho_T[f] < (q - \gamma[\lambda])^+$. Then $\sum_n (1 - |a_n|)^{q+1-\gamma[\lambda]-\varepsilon} < \infty$ for some positive ε. Hence, the exponent of convergence of the zeros of f is smaller than $q - \gamma[\lambda] = q - \gamma[\mu_*]$. It is a contradiction. □

For a measure μ supported by $\overline{\mathbb{D}}$ and $\zeta \in \partial\mathbb{D}$ we denote by $\widehat{\mu}$ the restriction of μ on the $S(\zeta) \cup \partial\mathbb{D}$ where

$$S(\zeta) = \{z \in \mathbb{D} \colon |1 - z\bar{\zeta}| \le \sqrt{1 + 4\pi^2}(1 - |z|)\}$$

is a Stolz angle with the vertex ζ. It is easy to check that $\square(re^{i\varphi}) \subset S(e^{i\varphi})$. We define an asymptotic modulus of continuity $\omega(\delta, \widehat{\mu}) = \sup_{\zeta \in \partial\mathbb{D}} |\widehat{\mu}|(\overline{D(\zeta, \delta)} \cap \overline{\mathbb{D}})$. We write $\widehat{\mu} \in \Lambda_\gamma$ if $\omega(\delta, \widehat{\mu}) = O(\delta^\gamma)$ $(\delta \downarrow 0)$. Let also $\tau[\widehat{\mu}] = \sup\{\tau \ge 0 \colon \omega(\delta, \widehat{\mu}) \in \Lambda_\tau\}$.

Now we formulate a theorem equivalent to Theorem A in terms of the complete measure.

Theorem 2. *Let $q \in \mathbb{Z}_+$. Let $P(z)$ be a product of form (2). Then*

1) $\rho_T[P] = (q - \gamma[\mu_*])^+$;

2a) $\rho_M[P] = \rho_T[P] + 1 - \tau[\widehat{\mu}_*^{\,q-\rho_T[P]}] = q + 1 - \gamma[\mu_*] - \tau[\widehat{\mu}_*^{\,\gamma[\mu_*]}]$ *if* $\rho_M[P] \ge 1$;

2b) $\rho_M[P] \le \rho_T[P] + 1 - \tau[\widehat{\mu}_*^{\,q-\rho_T[P]}] \le 1$ *if* $\rho_M[P] < 1$.

The proof follows immediately from Theorem A and the following lemma.

Lemma 1. *Let $q \in \mathbb{Z}_+$, $0 \le |a_n| < 1$, $\sum_n (1 - |a_n|)^{q+1} < \infty$. If $P(z) = P(z; (a_n), q)$ is defined by (2), then $\nu[P] = \rho_T[P] + 1 - \tau[\widehat{\mu}]$, where*

$$\mu(E) \stackrel{\text{def}}{=} \mu_*^{q-\rho_T[P]}(E) = \int_E (1 - |\zeta|)^{\rho_T[P]+1} \, d\mu_P(\zeta) = \sum_{a_n \in E} (1 - |a_n|)^{\rho_T[P]+1}, E \subset \overline{\mathbb{D}}. \tag{13}$$

Proof of the lemma. Let $\sigma = \rho_T[f]$. Since $\square(re^{i\varphi}) \subset S(e^{i\varphi})$, we have for any fixed $\varepsilon > 0$

$$\nu(re^{i\varphi}) \le \frac{\mu(\square(re^{i\varphi}))}{(\frac{1-r}{2})^{\sigma+1}} \le 2^{\sigma+1}\frac{\omega(2\sqrt{\pi^2+1}(1-r),\widehat{\mu})}{(1-r)^{\sigma+1}} \le C(\sigma,P)(1-r)^{\tau[\widehat{\mu}]-\sigma-1-\varepsilon}, \quad r \uparrow 1.$$

Hence,

$$\nu[P] \le 1 - \tau[\widehat{\mu}] + \sigma. \tag{14}$$

On the other hand, by additivity of the measure μ^ε, $\varepsilon > 0$,

$$\omega(2^{-j},\widehat{\mu^\varepsilon}) = \sup_{\zeta\in\partial\mathbb{D}} \mu^\varepsilon(\overline{D(\zeta,2^{-j})}\cap S(\zeta)) \le \sup_{\zeta\in\partial\mathbb{D}} \sum_{k=j}^{\infty}\sum_{m=-1}^{1} \mu^\varepsilon(\square((1-2^{-k})\zeta e^{im\pi 2^{-k}})) \le$$

$$\le 3\sum_{k=j}^{\infty} \sup_{\zeta\in\partial\mathbb{D}} \nu((1-2^{-k})\zeta)2^{-k(\sigma+1+\varepsilon)} \le 3\sum_{k=j}^{\infty} 2^{-k(\sigma+1+\varepsilon-\nu[P]-\varepsilon/2)} \le C(\varepsilon,P)2^{-j(\sigma+1+\varepsilon-\nu[P]-\varepsilon/2)}$$

as $j \to \infty$. Since $\omega(\delta)$ is nondecreasing, we deduce $\omega(\delta,\widehat{\mu^\varepsilon}) = O(\delta^{\sigma+1-\nu[P]+\varepsilon/2})$, $\delta \downarrow 0$, and consequently $\tau[\widehat{\mu^\varepsilon}] \ge \sigma + 1 - \nu[P] + \varepsilon/2$. Tending $\varepsilon \to 0$ we arrive to the inequality $\tau[\widehat{\mu}] \ge \sigma + 1 - \nu[P]$, which together with (14) completes the proof of item 2a) of the lemma. Item 2b) follows from Theorem A. □

Remark 4. Example 1 at the end of the paper shows that Theorem 2 and Lemma 1 are not valid with $\widehat{\mu}_*$ replaced by μ_*.

For the Stieltjes measure ψ and the measure μ_* associated with an analytic function of form (9) we define

$$\Delta(\psi) = \gamma[\psi] + \tau[\psi^{\gamma[\psi]}],\ \Delta(\mu_*) = \gamma[\mu_*] + \tau[\widehat{\mu}_*^{\gamma[\mu_*]}],\ \Delta(\lambda_f) = \min\{\Delta(\psi),\Delta(\mu_*)\}.$$

Theorem 3. *Let $q \in \mathbb{Z}_+$. Let $f(z)$ be an analytic function in $\mathbb{D}$ of form (9) where ψ is of bounded variation on $[0,2\pi]$. If $\rho_M[f] \ge 1$, then*

$$\rho_M[f] = q + 1 - \Delta(\lambda_f). \tag{15}$$

Proof. Since $\rho_M[f] \ge 1$, we have $\max\{\rho_M[P],\rho_M[g]\} \ge 1$.

We prove that if $\rho_M[g] \ge 1$ then $q \ge \gamma[\psi]$. In fact, suppose the contrary, $q < \gamma[\psi]$. Then, by Theorem D $\rho_T[g] = 0$, $\rho_M[g] = 1$, consequently, $\tau[\psi_{-q}] = 0$. On the other hand, $\widetilde{\psi} = \psi_{-(q+\gamma[\psi])/2} \in \mathrm{AC}$, by the definition of $\gamma[\psi]$. Hence, $\psi_{-q} = I_\eta[\widetilde{\psi}]$ where $\eta = (\gamma[\psi]-q)/2 > 0$. By Theorem (8.3) [10, Chap.XII] $\psi_{-q} \in \Lambda_\eta$ (assume that $\eta < 1$), i.e. $\tau[\psi_{-q}] \ge \eta > 0$. This is a contradiction. Therefore, $q \ge \gamma[\psi]$ provided $\rho_M[g] \ge 1$. Thus, $\rho_T[g] = q - \gamma[\psi]$, $\rho_M[g] = q + 1 - \Delta(\psi)$.

Consider three cases. First, let $\rho_M[g] \ge 1$, and $\rho_M[P] \ge 1$. Then, by Theorem 2 we have $\rho_M[P] = q+1-\Delta(\mu_*)$. If $\Delta(\mu_*) > \Delta(\psi) = \Delta(\lambda)$ then $\rho_M[P] = q+1-\Delta(\mu_*) < q+1-\Delta(\psi) = \rho_M[g]$. Thus, $\rho_M[f] = \rho_M[g] = q + 1 - \Delta(\lambda)$.

Suppose then that $\Delta(\lambda) = \Delta(\mu_*) \le \Delta(\psi)$. Then $\rho_M[f] \le \max\{\rho_M[P],\rho_M[g]\} = q + 1 - \Delta(\mu_*)$. Suppose that, $\rho_M[P] < q + 1 - \Delta(\mu_*)$. By a theorem of Linden ([5]), $\nu[f] \le \max\{\rho_M[f],1\} < q+1-\Delta(\mu_*) = \nu[f]$. This is a contradiction. Hence, $\rho_M[f] = q+1-\Delta(\mu_*) = q + 1 - \Delta(\lambda)$, as required.

Then, let $\rho_M[g] \ge 1$, and $\rho_M[P] < 1$. If $\gamma[P] \le q$, then $\rho_T[P] = q-\gamma[\mu_*]$, and by Theorem 2 $\rho_M[P] \le \nu[P] = q + 1 - \Delta(\mu^*) \le 1 \le q + 1 - \Delta(\psi)$. Hence, $\Delta(\mu_*) \ge \Delta(\psi) = \Delta(\lambda)$, and we have (15). If $\gamma[p] > q$, taking into account that $\rho_M[g] \ge 1$ implies $\Delta(\psi) \le q$, we obtain $\Delta(\mu_*) > \Delta(\psi) = \Delta(\lambda)$, and (15).

Finally, let $\rho_M[g] < 1$, $\rho_M[P] \ge 1$. By Theorem 2 we have $\rho_M[P] = q + 1 - \Delta(\mu_*)$. If $q \ge \gamma[\psi]$, then $\rho_M[g] = q + 1 - \Delta(\psi)$, and we easily get the required conclusion. Otherwise, $\Delta(\psi) \ge \gamma[\psi] > q$ and $\Delta(\mu) \le q$, because $\rho_M[P] \ge 1$. Hence, $\Delta(\lambda) = \Delta(\mu_*) < \Delta(\psi)$, and $\rho_M[f] = \rho_M[P] = q + 1 - \Delta(\lambda)$. □

Now, we are ready to give a solution of Problem 1 under the restriction $\rho \ge 1$.

Theorem 4. *Let $0 \le \sigma \le \rho \le \sigma + 1 < +\infty$, and $\rho \ge 1$. Then, $f \in A_\sigma^\rho$ iff f can be represented in form (9) with $q = [\sigma] + 1$, and the complete measure λ_f of genus q has properties $\sigma = q - \gamma[\lambda_f]$, $\rho = q + 1 - \Delta(\lambda_f)$.*

Proof of Theorem 4. The *sufficiency* follows from Theorems 1 and 3.

Necessity. Since $\rho_T[f] = \sigma$, from the results of M.Djrbashian [9, Ch.9] (cf. [8, Proposition 1]) it follows that $f \in A_\alpha$ for any $\alpha > \sigma$. In particular, $f \in A_q$, $q \ge [\sigma] + 1$, and can be represented in the form (9). We also have, $\sum_n (1 - |a_n|)^{q+1} < +\infty$, where (a_n) is the zero sequence of f, because σ is not lesser than the exponent of convergence. Thus, $P(z, (a_n), q) \in A_q$. Hence, the non-vanishing analytic function $g(z) = f(z)/(P(z)z^\lambda)$ belongs to A_q as well ($\rho_T[g] \le \sigma$). Therefore, $g(z) = \exp\left\{\frac{1}{2\pi}\int_{|\zeta|=1} S_q(z\bar\zeta)d\psi(\zeta)\right\}$, where ([9, Chap. IX, §4])

$$\psi^*(\theta) = \lim_{n\to\infty} \int_0^\theta D^{-q} \log|g(r_n e^{i\varphi})|\, d\varphi,$$

for some sequence (r_n), $r_n \uparrow 1$ as $n \to +\infty$. By Theorem 1 we have $\sigma = \rho_T[f] = q - \gamma[\lambda]$. The case $q < \gamma[\lambda]$ is impossible, because in this case, by the definitions of $\Delta(\lambda)$ we would have $\rho < 1$.

Then, by Theorem 3 we obtain $\rho = \rho_M[f] = q + 1 - \Delta[\lambda]$. The necessity is proved. □

Let us discuss the condition $\rho \ge 1$. This growth bound frequently appears when we describe asymptotic properties of analytic functions in the unit disc. For example, if $\rho_M > 1$ we have ([6]) that $\rho_\infty[f] := \lim_{p\to+\infty} \rho_p[f] = \rho_M[f]$, where $\rho_p[f]$ is the order of L_p-norm of $\log|f(re^{i\varphi})|$ over the circle $|z| = r$. This is not the case when $\rho_M[f] < 1$, we have only $\rho_M[f] \le \rho_\infty[f]$. Theorem A gives another example of such type. It turns out that $\rho_M[f]$ does not characterize asymptotic properties of the minimum modulus of f when $\rho_M[f] < 1$. My opinion is that one should use $\rho_\infty[f]$ instead of $\rho_M[f]$ for describing asymptotic properties of analytic functions in $\mathbb{D}$ with $\rho_M < 1$.

Conjecture. *Let $0 \le \sigma \le \rho \le \sigma + 1 < +\infty$. Then, $\rho_T[f] = \sigma$, $\rho_\infty[f] = \rho$ if and only if f can be represented in form (9) with $q = [\sigma] + 1$, and the complete measure λ_f of genus q has the properties $\sigma = (q - \gamma[\lambda_f])^+$, $\rho = q + 1 - \Delta(\lambda_f)$.*

3. Examples. Example 1. Given $0 \le \sigma_1 \le \rho_1 \le \sigma_1 + 1 < +\infty$, $\rho_1 \ge 1$, we choose an integer $q \ge [\sigma_1] + 1$, and define $\delta = (\sigma_1 + 1)/\rho_1 - 1$. Following Linden [4] we consider the sequence of complex numbers

$$a_{k,m} = (1 - 2^{-\frac{k}{\rho_1}})e^{im2^{-k/\rho_1}}, \quad 1 \le m \le [2^{k\delta}] \tag{16}$$

where each of numbers (16) is counted 2^k times respectively for each $k \in \mathbb{N}$. Then (see the proof of Theorem 1 [4]) for $P(z) = P(z, (a_{k,m}), q)$ we have

$$n(r, P) \asymp (1 - r)^{-\sigma_1 - 1}, \quad \nu(r, P) \asymp (1 - r)^{-\rho_1}, \quad r \uparrow 1.$$

Therefore, by Theorem A $\rho_T[P]=\sigma_1$, $\rho_M[P]=\rho_1$.

Let us calculate the complete measure λ_P of genus q in the sense of Grishin. Taking into account that $\lambda_P=\mu_*$, we obtain $\lambda_P(\zeta)=\sum_{k=1}^{\infty}\sum_{m=1}^{[2^{k\delta}]}2^{k(1-\frac{q+1}{\rho_1})}\delta(\zeta-a_{k,m})$. Hence,

$$\lambda_P^{\tau}(\zeta)=\sum_{k=1}^{\infty}\sum_{m=1}^{[2^{k\delta}]}2^{k(1-\frac{q+1-\tau}{\rho_1})}\delta(\zeta-a_{k,m}),$$

$$\lambda_P^{\tau}(\overline{D(0,1-2^{-j})})=\sum_{k=1}^{j}2^{k(1-\frac{q+1-\tau}{\rho_1})}[2^{k(\frac{\sigma_1+1}{\rho_1}-1)}]\asymp\sum_{k=1}^{j}2^{k\frac{\tau-q-\sigma_1}{\rho_1}}.$$

Thus, $\gamma[\lambda_P]=q-\sigma_1$.

Let $\square_N=\{\zeta\in\mathbb{D}\colon |\zeta|\ge 1-2^{-N/\rho_1}, |\arg\zeta|\le 2^{-N/\rho_1}\}$, $N\in\mathbb{N}$. For simplicity, suppose that $\sigma_1<\rho_1$. Given $\varepsilon>0$, let us estimate $\mu_*^{\varepsilon}(\square_N)$ from below. In order that $a_{mk}\in\square_N$, when $k\ge N$ it is necessary that $2^{-k/\rho_1}m\le 2^{-N/\rho_1}$, i.e. $m\le 2^{\frac{k-N}{\rho_1}}$. Hence, if $k>\frac{N}{\rho_1-\sigma_1}$, then $2^{\frac{k-N}{\rho_1}}>2^{k\delta}$. i.e. $a_{k,m}\in\square_N$ for all admissible m. Therefore,

$$\mu_*^{\varepsilon}(\square_N)\ge\sum_{k=[\frac{N}{\rho_1-\sigma_1}]+1}\sum_{m=1}^{[2^{k\delta}]}2^k(1-|a_{k,m}|)^{\sigma_1+1+\varepsilon}=$$

$$=\sum_{k=[\frac{N}{\rho_1-\sigma_1}]+1}2^k[2^{k(\frac{\sigma_1+1}{\rho_1}-1)}](2^{-\frac{k}{\rho_1}})^{\sigma_1+1+\varepsilon}\ge\frac{1}{2}\sum_{k=[\frac{N}{\rho_1-\sigma_1}]+1}2^{-\frac{k\varepsilon}{\rho_1}}\ge C(\sigma_1,\rho_1)2^{-\frac{N\varepsilon}{\rho_1(\rho_1-\sigma_1)}}.$$

Therefore, $\tau[\mu_*^{\varepsilon}]\le\frac{\varepsilon}{\rho_1(\rho_1-\sigma_1)}$. Consequently, $\tau[\mu_*]=0$. On the other hand, $\tau[\widehat{\mu}_*]=1+\sigma_1-\rho_1\ne\tau[\mu_*]$ unless $\rho_1=\sigma_1+1$.

Example 2. Let us construct an analytic function $g(z)$ without zeros in $\mathbb{D}$ and with given values of the orders.

Given $0\le\sigma_2\le\rho_2\le\sigma_2+1<+\infty$, we define an integer $q\ge[\sigma_2]+1$. Let $\tau_2=\sigma_2+1-\rho_2\in[0;1]$. We put

$$\omega(\delta)=\begin{cases}1/\ln(1/\delta), & \tau_2=0,\\ \delta^{\tau_2}, & \tau_2\in(0,1),\\ \delta/\ln(1/\delta), & \tau_2=1.\end{cases}$$

Let χ be a nondecreasing singular function of Cantor type on $[0,2\pi]$, $\chi(0)=0$, $\chi(2\pi)=1$ with $\omega(\delta,\chi)\asymp\omega(\delta)$ (see [19] for details). We define $\psi=\chi_{q-\sigma_2}i^{q-\sigma_2}$. Then ψ is real-valued, and $\gamma[\psi]=q-\sigma_2$. For the function $g(z)$ defined by (8) using Theorem D we have, $\rho_T[g]=\sigma_2$, and $\rho_M[g]=\sigma_2+1-\tau_2=\rho_2$ as required.

Example 3. Let $f(z)=P(z)g(z)$, where $P(z)$ and $g(z)$ are defined in the previous examples with $q=\max\{[\sigma_1],[\sigma_2]\}+1$. Then 6 cases are possible.

1	$\rho_1\ge\rho_2\ge\sigma_1\ge\sigma_2$	$\Delta(\lambda)=\Delta(\mu_*)\le\Delta(\psi)$, $\gamma[\lambda]=\gamma[\mu_*]\le\gamma[\psi]$, $\gamma[\mu_*]\ge\Delta(\psi)-1$	$\rho_M[f]=\rho_1$, $\rho_T[f]=\sigma_1$
2	$\rho_1\ge\sigma_1>\rho_2\ge\sigma_2$	$\gamma[\mu_*]<\Delta(\psi)-1$	$\rho_M[f]=\rho_1$, $\rho_T[f]=\sigma_1$
3	$\rho_1\ge\rho_2\ge\sigma_2>\sigma_1$	$\Delta(\lambda)=\Delta(\mu_*)\le\Delta(\psi)$; $\gamma[\lambda]=\gamma[\psi]<\gamma[\mu_*]$	$\rho_M[f]=\rho_1$, $\rho_T[f]=\sigma_2$
4	$\rho_2>\rho_1\ge\sigma_1\ge\sigma_2$	$\Delta(\lambda)=\Delta(\psi)<\Delta(\mu_*)$, $\gamma[\lambda]=\gamma[\mu_*]\le\gamma[\psi]$	$\rho_M[f]=\rho_2$, $\rho_T[f]=\sigma_1$
5	$\rho_2>\rho_1\ge\sigma_2>\sigma_1$	$\Delta(\lambda)=\Delta(\psi)<\Delta(\mu_*)$, $\gamma[\lambda]=\gamma[\psi]<\gamma[\mu_*]$, $\gamma[\psi]\ge\Delta(\mu_*)-1$	$\rho_M[f]=\rho_2$, $\rho_T[f]=\sigma_2$
6	$\rho_2\ge\sigma_2>\rho_1\ge\sigma_1$	$\gamma[\psi]<\Delta(\mu_*)-1$	$\rho_M[f]=\rho_2$, $\rho_T[f]=\sigma_2$

For example, the inequality $\gamma[\mu_*] < \Delta(\psi) - 1$ in case 2 is equivalent to $q - \gamma[\psi] > q + 1 - \Delta[\mu_*]$, which means, in view of Theorem 4, that $\sigma_1 = \rho_T[P] > \rho_M[g] = \rho_2$.

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