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ON TRANSITION FUNCTIONS OF MEALY AUTOMATA OF FINITE GROWTH

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The transition functions of Mealy automata that generate finite automaton transformation semigroups are considered in the paper. We formulate conditions on transition functions such that any Mealy automaton with this transition generates a finite semigroup. Also we prove that these requirements are "maximal", because for any transition function that does not satisfies them there exists an output function such that the Mealy automaton defines an infinite semigroup.

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В статье рассматриваются функции переходов автоматов Мили, которые порождают конечные полугруппы. Получены условия на функции переходов, при которых все автоматы Мили порождают конечные полугруппы. Доказано, что эти условия являются "максимальными", то есть для функции переходов, которая не удовлетворяет указанным выше условиям, существует функция выходов такая, что соответствующий автомат Мили порождает бесконечную полугруппу.

1. Introduction. The groups and semigroups of automaton transformations are actively studied since 60ies of the last century [1, 2, 3]. In these papers close relation between Mealy automata properties and properties of semigroups and groups of automaton transformations that are generated by these automata is shown. The review of recent results in the area of automata and groups of automaton transformations is represented in the paper [4]. For a recent review of properties of automaton transformation semigroups see [5]. However, there are a great number of open questions characterization of different properties of groups and semigroups generated by automata in general case (see [4]).

One of the most interesting questions concerning semigroups defined by Mealy automata is the question on finiteness or infinity of a semigroup. This property has a natural interpretation. Mealy automaton determines a particular transformation of words at each state, and finiteness of semigroup means that successive application of these transformations produce only a finite set of automaton transformations. In addition, any automaton generates finite semigroup if and only if it has finite growth order. The problem of the determining the finiteness of the (semi)group generated by a Mealy automaton is mentioned in the list of open problems [4]. All two-state Mealy automata over the two-symbol alphabet which defines finite semigroups are investigated in [6] (see also [7]). However the problem has not been solved in the general case.

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For arbitrary transition function there exists an output function such that the automaton generates a finite semigroup. Different classes of Mealy automata that generate finite (semi)groups for all output functions described by limitations on transition functions were investigated in [8, 9] and in the verbal message of Reznykov and Sushchansky). The class of invertible automata is considered in the paper [9], and finiteness of groups of automata transformations generated by such automata is proved.

The class of transition functions that define automata of finite growth for any output function is considered in the paper. This class of transition functions includes all classes from the papers mentioned above. Moreover, we show "maximality" of this class of transition functions. That is for any transition function that does not belong to the class there exists an output function such that the corresponding automaton generates an infinite semigroup. This result allows to simplify the research of Mealy automata growth, because it covers a wide class of automata of finite growth order.

In section "Preliminaries" we give preliminaries of Mealy automata and semigroups defined by them and set up notations. In section "Automata without branches" the automata without branches [8] and the class of their transition functions are considered. Such automata define only finite semigroups. We extend this class of transition functions to the class of transition functions with limitary cycle in Section "Mealy automata of finite growth". Any a automaton with the latter functions also generates only finite semigroups. In Section "Transition functions without limitary cycle" it is shown that this class is "maximal".

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2. Preliminaries.

Definition 1 ([10, 11])**.** A *finite Mealy automaton* is an ordered quintuple $A = (X, Y, Q, \pi, \lambda)$, where X is the input alphabet, Y is the output alphabet, Q is the finite nonempty set of states, $\pi\colon X \times Q \to Q$ is the transition functions and $\lambda\colon X \times Q \to X$ is the output function. X and Y are finite nonempty sets.

We will consider only finite automata whose input and output alphabet coincide ($X = Y$). We denote such automata by quadruples $A = (X, Q, \pi, \lambda)$. For convenience, we fix a finite alphabet $X = \{0, 1, \ldots, m-1\}$, where $m \geq 2$ is some natural number.

Let $T_X = \{f \mid f\colon X \to X\}$ be the semigroup of all transformation of the alphabet X (the full symmetric semigroup), X^* be the set of all finite words over X, X^ω be the set of all ω-words (infinite words) over X.

It is convenient to describe finite automata by the Moore diagrams. We will use the following modification of them. The Moore diagram of an automaton A is an edge-labeled and vertex-labeled directed multigraph D_A with the set of vertices Q. Vertices q_i and q_j of the graph D_A are connected by the oriented edge in direction from q_i to q_j marked by the label x, if $\pi(x, q_i) = q_j$. Here $x \in X, q_i, q_j \in Q$. Every vertex q is labeled by the transformation $\lambda_q \in T_X$ of the alphabet X that corresponds to the output function at the state q, i.e. $\lambda_q(x) = \lambda(x, q)$, where $x \in X, q \in Q$.

The functions π and λ can be extended naturally to mappings of the set $X^* \times Q$ into the sets Q and X^* by the following equalities [11]:

$$\begin{aligned} &\pi(\Lambda, q) = q, \quad \pi(wx, q) = \pi\left(x, \pi(w, q)\right), \\ &\lambda(\Lambda, q) = \Lambda, \quad \lambda(wx, q) = \lambda(w, q)\lambda\left(x, \pi(w, q)\right), \end{aligned}$$

where $\Lambda \in X^*$ is the empty word, $q \in Q, w \in X^*, x \in X$. The function λ can also be extended in a natural way to a mapping $\lambda\colon X^\omega \times Q \to X^\omega$ (see for example, [11]).

Let $\pi(X_1, Q_1) = \{\pi(x,q) | x \in X_1, q \in Q_1\}$ and $\lambda(X_1, Q_1) = \{\lambda(x,q) | x \in X_1, q \in Q_1\}$ for an arbitrary $X_1 \subseteq X$, $Q_1 \subseteq Q$.

Definition 2 ([11])**.** The transformation $f_q\colon X^* \to X^*$ ($f_q\colon X^\omega \to X^\omega$), defined by the equality $f_q(u) = \lambda(u,q)$, where $u \in X^*$ ($u \in X^\omega$), is called an *automaton transformation defined by the automaton $A = (X, Q, \pi, \lambda)$ at the state q.*

Mealy automaton $A = (X, Q, \pi, \lambda)$, where $Q = \{q_0, q_1, \ldots, q_{n-1}\}$, determines the set $F_A = \left\{f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}}\right\}$ of automaton transformations over X^*. The Mealy automaton A is called *invertible* if all transformations from the set F_A are bijections. It is easy to show (see for example [4]) that A is invertible if and only if the transformation λ_q is a permutation of X for each state $q \in Q$.

Definition 3 ([11])**.** Mealy automata $A_i = (X, Q_i, \pi_i, \lambda_i)$, for $i = 1, 2$, are called *isomorphic* if there exist permutations $\xi, \psi \in \mathrm{Sym}(X)$ and $\theta\colon Q_1 \to Q_2$ such that

$$\theta\pi_1(x,q) = \pi_2(\xi x, \theta q), \qquad \psi\lambda_1(x,q) = \lambda_2(\xi x, \theta q)$$

for all $x \in X$ and $q \in Q_1$.

Definition 4 ([11])**.** Mealy automata A_i for $i = 1, 2$, are called *equivalent* if $F_{A_1} = F_{A_2}$.

Proposition 1 ([11])**.** *Each class of equivalent Mealy automata over an alphabet X contains, up to isomorphism, a unique automaton that is minimal with respect to the number of states (such an automaton is called reduced).*

The minimal automaton can be found using the standard algorithm of minimization.

Definition 5 ([12])**.** For $i = 1, 2$ let $A_i = (X, Q_i, \pi_i, \lambda_i)$ be arbitrary Mealy automata. The automaton $A = (X, Q_1 \times Q_2, \pi, \lambda)$ such that its transition and output functions are defined in the following way:

$$\pi(x, (q_1, q_2)) = (\pi_1(\lambda_2(x, q_2), q_1), \pi_2(x, q_2)), \quad \lambda(x, (q_1, q_2)) = \lambda_1(\lambda_2(x, q_2), q_1),$$

where $x \in X$ and $(q_1, q_2) \in Q_1 \times Q_2$, is called the *product* of the automata A_1 and A_2.

Proposition 2 ([12])**.** *For any states $q_1 \in Q_1$ and $q_2 \in Q_2$ and an arbitrary word $u \in X^*$ the following equality holds:*

$$f_{(q_1,q_2),A}(u) = f_{q_1,A_1}(f_{q_2,A_2}(u)).$$

The power A^n is defined for any automaton A and any positive integer n. Let us denote by $A^{(n)}$ the minimal Mealy automaton equivalent to A^n. It follows from definition of the product that $|Q_{A^{(n)}}| \le |Q_A|^n$.

Definition 6 ([13])**.** The function γ_A of a natural argument defined by $\gamma_A(n) = |Q_{A^{(n)}}|$, $n \in \mathbb{N}$, is called the *growth function* of a Mealy automaton A.

Definition 7. A semigroup generated by the set $F_A = \left\{f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}}\right\}$ of transformations defined by a Mealy automaton in all its states A is called *the semigroup generated by automaton A*. In the case of invertible automaton A the group generated by F_A is called the *group generated by the automaton A*.

Definition 8. An automaton A is called an *automaton of finite growth (order)* if its growth function is bounded.

It is easy to show that an automaton A is an automaton of finite growth if and only if the semigroup generated by it is finite.

Definition 9 ([14])**.** The *wreath product* by infinite sequence $(H_1, M_1), (H_2, M_2), \ldots$ (by finite sequence $(H_1, M_1), (H_2, M_2), \ldots, (H_k, M_k)$) of transformation semigroups is the semigroup of all transformations h of the set $M = \prod_{i=1}^{\infty} M_i$ ($M = \prod_{i=1}^{k} M_i$) satisfying the following conditions:

1. if $(y_1, y_2, \ldots) = (x_1, x_2, \ldots)^h$ then y_i depends only on i first coordinates $x_1, x_2, \ldots$ for $i = 1, 2, \ldots$;
2. if $x_1^0, \ldots, x_{i-1}^0$ is fixed then the transformation $h_i\left(x_1^0, \ldots, x_{i-1}^0\right) = g_i, h_i \colon x_i \to y_i$ induced by h is a transformation from the semigroup H_i.

We denote the defined wreath product of transformation semigroups by $H = \wr_{i=1}^{\infty}(H_i, M_i) = \wr_{i=1}^{\infty} H_i$ (in the finite sequence case $H = \wr_{i=1}^{k}(H_i, M_i) = \wr_{i=1}^{k} H_i = H_1 \wr H_2 \wr \cdots \wr H_k$).

3. Automata without branches. In this section we consider a special set of Mealy automata that define finite (semi)groups. We will use the following classification of automaton states [8].

Let Q be a nonempty finite set of states and $\pi \colon X \times Q \to Q$ be a fixed transition function. Let $q \in Q$ be an arbitrary state.

Definition 10 ([8])**.** We say that q is a *rest state* if the automaton stays in this state under the action of any input symbol, i.e.

$$\forall x \in X \colon \pi\,(x, q) = q.$$

Definition 11 ([8])**.** We say that q is an *unconditional jump state* if there exists the state $q' \in Q$, $q' \neq q$, such that the automaton moves from the state q to the state q' under the action of any input symbol, i.e.

$$\exists q' \in Q,\, q' \neq q \colon \forall x \in X, \pi\,(x, q) = q'.$$

Definition 12. We say that q is a *state without branches* if it is either a rest state or an unconditional jump one, otherwise we say that q is a *branch state.*

A state $q \in Q$ is a state without branches if and only if for all $x_1, x_2 \in X$ the equality $\pi\,(x_1, q) = \pi\,(x_2, q)$ holds.

Examples of a rest state, an unconditional jump state and a branch state for the case $X = \{0, 1\}$ are shown, respectively, in Figures 1 a, b and c.

Definition 13. We say that a Mealy automaton $A = (X, Q, \pi, \lambda)$ has no branches (and, correspondingly, its transition function π has no branches) if any state $q \in Q$ is a state without branches (i.e. either a rest state or an unconditional jump state).

Note that an automaton would be an automaton without branches if and only if its transition function depends only on the state and does not depend on the input symbol. Hence, we will denote the transition function of an automaton without branches by the symbol $s(q) = \pi\,(x, q)$, for all $q \in Q, x \in X$.

Let $g = g_1 g_2 \ldots \in (T_X)^{\omega}$ be an arbitrary infinite word. Let $F_g \colon X^{\omega} \to X^{\omega}$ be the transformation over the set of infinite words defined by the equality

$$F_g(x_1 x_2 \ldots) = g_1(x_1) g_2(x_2) \ldots.$$

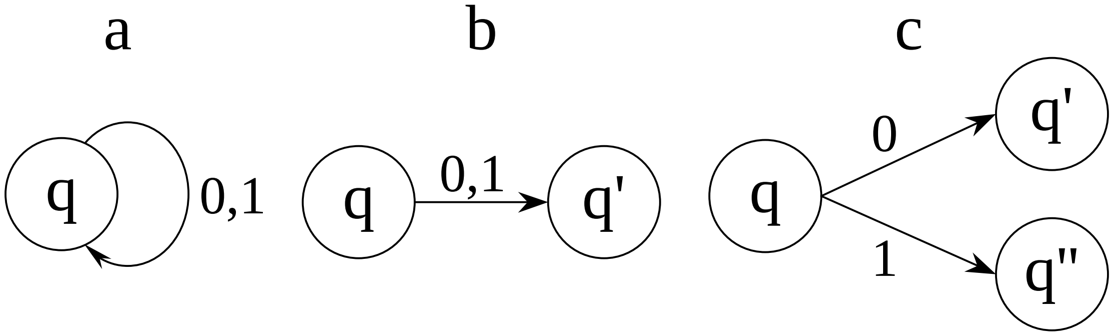

Fig. 1: A rest state, an unconditional jump state and a branch state

In the sequel, we will call such transformations as "symbol-by-symbol" transformations. It is clear that $F_u \circ F_v = F_{u \circ v}$, where $u = u_1 u_2 \ldots \in (T_X)^\omega$, $v = v_1 v_2 \cdots \in (T_X)^\omega$, $u \circ v = (u_1 \circ v_1)(u_2 \circ v_2)\ldots$, is an elementwise composition.

Lemma 1 ([8])**.** *Let $q \in Q$ be an arbitrary state of an automaton without branches $A = (X, Q, \pi, \lambda)$. Then the transformation defined by the automaton A at the state q is a symbol-by-symbol one.*

Proof. Let f_q be an automaton transformation generated by an automaton without branches at the state q and let v be an input word. The automaton goes to the fixed (independent of the first symbol) state $s(q)$ under the action of the first symbol. Similarly, it goes to the state $s(s(q)) = s^2(q)$ after reading two first symbols, and so on. Therefore the equality $f_q = F_u$ holds, where $u = \lambda_q \lambda_{s(q)} \lambda_{s^2(q)} \cdots \lambda_{s^k(q)} \cdots$. □

Let $P_{u,v} = F_{uv^*}$, where $u, v \in T_X^*, v^* = vv\cdots$ For any $u_1, u_2, v_1, v_2 \in T_X^*$ such that $|u_1| = |u_2|$, $|v_1| = |v_2|$ the equality $P_{u_1,v_1} \circ P_{u_2,v_2} = P_{u_1 \circ u_2, v_1 \circ v_2}$ holds. Here $|u|$ denotes the length of a word u.

Lemma 2. *Let $A = (X, Q, \pi, \lambda)$ be an automaton without branches, $|Q| = n$. Then there exist integers $l, k \in \mathbb{N}$, $l < n$, such that the transformation generated by the automaton A at any state $q \in Q$ has the form of $P_{u,v}$, where $|u| = l, |v| = k$.*

Proof. The proof is similar to the proof of the "pumping" lemma [15]. Let $q \in Q$. Since $|Q| = n$, there are two equal states among states $q, s(q), s^2(q), \ldots, s^n(q)$. Let such first states be $s^a(q) = s^b(q)$, $a < b$ and let $l_q = a - 1$, $k_q = b - a$. It is easily seen that the sequence $\{s^i(q)\}_{i=1}^\infty$ is an ultimately periodic sequence with the least period k_q. Indeed for all natural $p > l_q$ we have $s^{p+k_q}(q) = s^{p-l_q-1}\left(s^{l_q+1+k_q}(q)\right) = s^{p-l_q-1}\left(s^{l_q+1}(q)\right) = s^p(q)$.

Let $l = \max_{q \in Q}(l_q)$, and let k be the least common multiple of all k_q where $q \in Q$. It is clear that $l < n$ and k is not more than the least common multiple of all integers between 2 and n. Then all sequences $\{s^i(q)\}_{i=1}^\infty$, where $q \in Q$, are ultimately periodic with period k and we can regard that the periodic parts start at the position l. By Lemma 1 the automaton A at the state q defines the transformation $F_{\lambda_q \lambda_{s(q)} \lambda_{s^2(q)} \cdots \lambda_{s^k(q)} \cdots} = P_{u_q, v_q}$, where $u_q = \lambda_q \lambda_{s(q)} \lambda_{s^2(q)} \cdots \lambda_{s^{l-1}(q)}$,$v_q = \lambda_{s^l(q)} \lambda_{s^{l+1}(q)} \cdots \lambda_{s^{l+k-1}(q)}$, $|u_q| = l$,$|v_q| = k$. □

Theorem 1. *A semigroup S is isomorphic to some semigroup generated by a finite automaton without branches over an alphabet X ($|X| = m$), if and only if S is isomorphic to a* *semigroup* $S' \leq (T_X)^k$ *for some natural* k *such that there exists a natural number* l *such that* $1 \leq l \leq k$ *and*

$$v_1 v_2 \cdots v_k \in S' \Rightarrow v_2 \cdots v_k v_l \in S' \tag{1}$$

where $v_i \in T_X$ *for* $1 \leq i \leq k$.

Proof. The proof falls naturally into two parts. First we will construct a semigroup $S' \leq (T_X)^k$ with property (1) such that it is isomorphic to a semigroup S generated by an automaton without branches. Then for some semigroup $S' \leq (T_X)^k$ with property (1) we will construct an automaton without branches such that semigroup generated by it is isomorphic to S'.

Let S be a semigroup generated by the automaton without branches $A = (X, Q, \pi, \lambda)$. By Lemma 2 there exist integers $l', k' \in \mathbb{N}$ such that the transformation generated by the automaton A at any state $q \in Q$ has the form of $P_{u,v}$, where $|u| = l'$, $|v| = k'$, $u \in T_X^{l'}, v \in T_X^{k'}$. Let $k = l' + k'$. The desired isomorphism is $F_1 \colon S \to S' \leq (T_X)^k$, where $F_1(z) = uv$, $z = P_{u,v} \in S$. The isomorphism F_1 is well defined because $uv \in (T_X)^k$ and the words u and v are uniquely defined by $z \in S$. It is clear that $F_1(P_{u_1,v_1} \circ P_{u_2,v_2}) = F_1(P_{u_1 \circ u_2, v_1 \circ v_2}) = (u_1 \circ u_2)(v_1 \circ v_2) = u_1 v_1 \circ u_2 v_2 = F_1(P_{u_1,v_1}) \circ F_1(P_{u_2,v_2})$. Let us prove property (1). Let $l = l' + 1$. Suppose $v = v_1 v_2 \ldots v_k \in S'$. Then $v' = F_1^{-1}(v) = F_1^{-1}(v_1 v_2 \ldots v_k) = P_{v_1 v_2 \ldots v_{l'}, v_{l'+1} \cdots v_k} \in S$ and let $v' = f_{q_1} \circ f_{q_2} \circ \cdots \circ f_{q_r}$, where $q_1, \ldots, q_r \in Q$ are not necessarily different states, $r \in \mathbb{N}$. Let $v'' = f_{s(q_1)} \circ f_{s(q_2)} \circ \cdots \circ f_{s(q_r)}$. It is clear that $v'' \in S$ and $v'' = P_{v_2 \ldots v_{l'} v_{l'+1}, v_{l'+2} \cdots v_k v_{l'+1}}$. So $F_1(v'') = v_2 \ldots v_{l'} v_{l'+1} v_{l'+2} \ldots v_k v_{l'+1} = v_2 \ldots v_k v_l \in S'$.

Let $S' \leq (T_X)^k$ be a semigroup such that property (1) holds. Let us construct the automaton $A = (X, S', \pi, \lambda)$, where $\pi(x, v_1 v_2 \ldots v_k) = v_2 \ldots v_k v_l$ and $\lambda(x, v_1 v_2 \ldots v_k) = v_1(x)$, where $v_1 v_2 \ldots v_k \in S'$. The automaton is well defined because $v_2 \ldots v_k v_l \in S'$ by property (1) and $v_1(x) \in X$, because $v_1 \in T_X$. It is clear that A is an automaton without branches. Moreover, the automaton transformation generated by state $w = v_1 v_2 \ldots v_k$ has the form $F_{w w_2 w_2 \cdots w_2 \cdots} = P_{w_1, w_2}$. Here $w_1 = v_1 v_2 \cdots v_{l-1}$, $w_2 = v_l v_{l+1} v_{l+2} \cdots v_k$. So we construct the isomorphism $F_1 \colon S \to S' \leq (T_X)^k$ as above, where A generates the semigroup S. □

Thus, it follows from Theorem 1 that automata without branches always generate finite semigroups independently of their output function.

4. Mealy automata of finite growth. Let Q be a nonempty finite set of states. We say that a transition function $\pi \colon X \times Q \to Q$ generates an infinite semigroup (group) of finite automaton transformation if there exists an output function $\lambda \colon X \times Q \to Q$ such that the noninitial automaton $A = (X, Q, \pi, \lambda)$ generates infinite semigroup (group) of automaton transformations. Otherwise we say that transition function generates only finite semigroups (groups) of finite automaton transformations.

Transition functions without branches generate only finite semigroups and groups. However, there exists an other transition function and, therefore, other automata classes which generates only finite semigroups. A more general (but not "maximal") class was given in Reznykov's verbal message by the following theorem.

Proposition 3. *Let* $A = (X_m, Q_n, \pi, \lambda)$ *be a Mealy automaton such that the transition function satisfies the following condition. Let there exists the number* $k \geq 1$ *such that for any* $q \in Q_n$ *and for arbitrary words* $u_1, u_2 \in X_m^*$ *of length* k *the equality* $\pi(u_1, q) = \pi(u_2, q)$ *holds. Thus, the automaton* A *generates the finite semigroup and the growth function* γ_A *is almost constant.*

We will formulate a "maximal" class of transition functions which generates finite semigroups. Similar class for groups is considered in [9].

Recall that a state $q \in Q$ is a state without branches if for all $x_1, x_2 \in X$ the equality $\pi(q, x_1) = \pi(q, x_2)$ holds, otherwise the state $q \in Q$ is called a branch state. A state $q \in Q$ is called cyclic if there exists a nonempty word $w \in X^*$ such that $\pi(w, q) = q$.

Let $Q_1 \subseteq Q$, write $\Pi(Q_1) = \pi(X, Q_1) = \{\pi(x, q) \,|\, q \in Q_1, x \in X\}$. Set $\Pi^0(Q_1) = Q_1$. Let us prove by induction that $\Pi^k(Q_1) = \Pi(\cdots \Pi(Q_1) \cdots) = \{\pi(v, q) \,|\, q \in Q_1, v \in X^*, |v| = k\}$. For $k = 0$ we have $\Pi^0(Q_1) = Q_1 = \{\pi(\Lambda, q) \,|\, q \in Q_1\}$, where Λ is the empty word. Assuming $\Pi^l(Q_1) = \{\pi(v, q) \,|\, q \in Q_1, v \in X^*, |v| = l\}$ for $k = l$, we have for $k = l + 1$

$$\begin{aligned}\Pi^{l+1}(Q_1) =& \Pi\left(\Pi^l(Q_1)\right) = \Pi\left(\{\pi(v, q) \,|\, q \in Q_1, v \in X^*, |v| = l\}\right)\\ =& \{\pi(x, \pi(v, q)) \,|\, q \in Q_1, v \in X^*, |v| = l, x \in X\}\\ =& \{\pi(vx, q) \,|\, q \in Q_1, v \in X^*, |v| = l, x \in X\}\\ =& \{\pi(v', q) \,|\, q \in Q_1, v' \in X^*, |v'| = l + 1\}\end{aligned}$$

Let Q_1 be a nonempty set such that $\Pi(Q_1) \subseteq Q_1$ (in particular, one can set $Q_1 = Q$ since $\Pi(Q) \subseteq Q$). Let $k \in \mathbb{N}$. If $q \in \Pi^k(Q_1)$ then there exists a word $v = v_1 v_2 \dots v_k \in X^*$ and a state $q' \in Q_1$ such that $\pi(v, q') = q$ and, consequently, $q = \pi(v_2 \dots v_k, \pi(v_1, q'))$, $\pi(v_1, q') \in \Pi(Q_1)$. It follows that $\pi(v_1, q') \in Q_1, |v_2 \dots v_k| = k - 1$, whence $q \in \Pi^{k-1}(Q_1)$. Therefore if $\Pi(Q_1) \subseteq Q_1$ then the equality $\Pi^k(Q_1) \subseteq \Pi^{k-1}(Q_1)$ holds for any integer $k \geq 1$.

Furthermore, for any natural k we have $\Pi^k(Q_1) \neq \varnothing$. Let $\Pi^\infty(Q_1) = \bigcap_{k=1}^\infty \Pi^k(Q_1)$.

If $\Pi^k(Q_1) = \Pi^{k+1}(Q_1)$ for some $k \in \mathbb{N}$, then $\Pi^k(Q_1) = \Pi^{k+1}(Q_1) = \Pi^{k+2}(Q_1) = \cdots = \Pi^\infty(Q_1)$. Since the set Q_1 is finite, there always exists $k \in \mathbb{N}, k \leq |Q|$ such that $\Pi^k(Q_1) = \Pi^{k+1}(Q_1) = \Pi^\infty(Q_1)$. Moreover $\Pi^\infty(Q) = \Pi^k(Q) \neq \varnothing$. Thus, we have

$$Q_1 \supseteq \Pi(Q_1) \supseteq \Pi^2(Q_1) \supseteq \cdots \supseteq \Pi^k(Q_1) = \Pi^{k+1}(Q_1) = \cdots = \Pi^\infty(Q_1). \quad (2)$$

Summarizing the properties of Π, we have the following lemma.

Lemma 3. *Let Q_1 be a nonempty set of states such that $\Pi(Q_1) \subseteq Q_1$. The sets $\Pi^i(Q_1)$, $i \geq 0$, satisfy the following equalities:*

1. *$\Pi(\Pi^\infty(Q_1)) = \Pi^\infty(Q_1)$;*
2. *there exists $k \leq |Q_1|$ such that the equality $\Pi^\infty(Q_1) = \Pi^k(Q_1)$ holds;*
3. *for all $l \geq k$ the equality $\Pi^l(Q_1) = \Pi^k(Q_1)$ holds.*

Recall that the above statements are valid in the case $Q_1 = Q$, too. In this case we have the following characterization of the set $\Pi^\infty(Q)$.

Lemma 4. *The set $\Pi^\infty(Q)$ consists of all states that are accessible from the cyclic ones.*

Proof. Let us denote by S the set of all states that are accessible from the cyclic ones. Let q be a cyclic state, i.e. for some nonempty word $w \in X^*$ we have $\pi(w, q) = q$. Let $l = |w|$, then $l > 0$. We have $q = \pi(w, q) \in \Pi^l(Q)$, $q = \pi(ww, q) \in \Pi^{2l}(Q)$, ..., $q \in \Pi^{il}(Q)$, ..., $i \geq 1$. Applying (2), for each natural j we obtain $q \in \Pi^j(Q)$, i.e. $q \in \Pi^\infty(Q)$. Let q' be accessible from q, i.e. for some word $v \in X^*$ ($|v| = t$) we have $q' = \pi(v, q)$. Since $q \in \Pi^\infty(Q)$, we have $q' \in \Pi^t(\Pi^\infty(Q))$. By Lemma 3, $\Pi^t(\Pi^\infty(Q)) = \Pi^\infty(Q)$. Therefore, $q' \in \Pi^\infty(Q)$. Hence, all states accessible from the cyclic ones belong to $\Pi^\infty(Q)$, i.e. $S \subseteq \Pi^\infty(Q)$.

Let us prove that $\Pi^\infty(Q) \subseteq S$. Let $q \in \Pi^\infty(Q)$ and let $|Q| = n$, then $q \in \Pi^n(Q)$ by Lemma 3, i.e. there exists q_0 such that $q = \pi(w, q_0)$, where $w = w_1 w_2 \cdots w_{n-1} w_n \in X^*, q_0 \in Q$. Set $q_i = \pi(w_1 w_2 \cdots w_{i-1} w_i, q_0) = \pi(w_i, q_{i-1})$, where $0 < i \leqslant n$. Since the number of states $|Q| = n$, there are two equal states among the $(n+1)$ states $q_0, q_1, \ldots, q_{n-1}, q_n$. Let $q_j = q_{j+r}, r > 0, 0 \leqslant j < j + r \leqslant n$. Since $q_j = q_{j+r} = \pi(w_{j+1} w_{j+2} \ldots w_{j+r}, q_j)$, q_j is a cyclic state. Moreover, $q = q_{n+1} = \pi(w_{j+1} w_{j+2} \cdots w_n w_{n+1}, q_j)$ is accessible from the cyclic state q_j, i.e. $q \in S$. We obtain $\Pi^\infty(Q) \subseteq S$, thus $\Pi^\infty(Q) = S$, which proves the lemma. □

Similarly to the proof of Lemma 4, if $\Pi(Q_1) \subseteq Q_1 \subset Q$, then the set $\Pi^\infty(Q_1)$ consists of all states that are accessible from the cyclic states that belong to Q_1.

Definition 14. Let Q be a finite set of states. A transition function $\pi\colon X \times Q \to Q$ is called *a transition function with limitary cycle* if the set $\Pi^\infty(Q)$ consists of states without branches, otherwise it is called *a transition function without limitary cycle.*

Theorem 2. *Let $\pi\colon X \times Q \to Q$ be a transition function with limitary cycle and $A = (X, Q, \pi, \lambda)$ be an arbitrary automaton with this transition function. Then A generates a finite automaton transformation semigroup.*

Proof. Let S_A denote the semigroup generated by the automaton A. Let $\Pi^\infty(Q)$ contains only states without branches, and let $\Pi^\infty(Q) = \Pi^k(Q)$ (such a number k always exists by Lemma 3). Let us denote by A_k the automaton with the set of states $\Pi^k(Q)$, the same alphabet X and the corresponding restrictions of the transition function π and the output function λ. This automaton is well defined, because $\pi\left(X, \Pi^k(Q)\right) = \Pi^{k+1}(Q) = \Pi^k(Q)$ by the definition of k. Note that the automaton A_k is an automaton without branches. Therefore it defines a finite semigroup of automaton transformations S_{A_k}.

An automaton transformation defined by the automaton A at the state q has the form $f_q(w_k w) = f_q(w_k) f_{\pi(w_k, q)}(w)$, where $w_k \in X^*, |w_k| = k, w \in X^\omega$, $\pi(w_k, q) \in \Pi^k(Q)$, $f_{\pi(w_k,q)} \in S_{A_k}$. Thus every $f \in S_A$ has the form $f(w_k w) = f'(w_k) f''_{w_k}(w)$, where $f'\colon X^k \to X^k$, $w_k \in X^k$, $f''_{w_k} \in S_{A_k}$. The transformation $f \in S_A$ is uniquely determined by the function $f'\colon X^k \to X^k$ and the function $f'''\colon X^k \to S_{A_k}$, where $f'''(w_k) = f''_{w_k}$, $w_k \in X^k$. Since there are only a finite number of functions $f'\colon X^k \to X^k$ and $f'''\colon X^k \to S_{A_k}$, the number of elements of S_A is finite. Moreover, $S_A \cong S \leq \underbrace{T_X \wr \cdots \wr T_X}_{k} \wr S_{A_k}$. □

This theorem gives a sufficient condition for transition functions that determines only finite semigroups. In the following section we will show that this condition is necessary.

5. Transition functions without limitary cycle. Now we can formulate our main result. A transition function of Mealy automaton determines only finite semigroups if and only if it is a transition functions with limitary cycle. In other words, a transition function determines only finite semigroups (i.e. any automaton with such transition function determines a finite semigroup independently of an output function) if and only if there exists an integer k such that the automaton goes to a state without branches from any initial state under action of any word of length greater than or equal to k. Formally, we have the following theorem.

Theorem 3. *Let Q be a finite set of states. Let $\pi\colon X \times Q \to Q$ be a transition function. Then the following three conditions are equivalent:*

1. *the transition function π defines an infinite semigroup;*

2. *there exists a cyclic branch state for* π*;*
3. *the transition function* π *is transition function without limitary cycle (in other words,* $\Pi^{\infty}(Q)$ *contains a branch state).*

In order to prove this theorem we will analyze transition functions without limitary cycle. We will consider the three types of such transition functions. For an arbitrary transition function π of each type we will construct a output function $\lambda\colon X \times Q \to Q$ such that the noninitial automata $A = (X, Q, \pi, \lambda)$ generates an infinite automaton transformation semigroup S_A. To this end we will select some transformation of the alphabet $\lambda_q(x) = \lambda(x, q) \in T_X$ for each state. Let us fix several elements of the set T_X: $\alpha(x) \equiv 0, \beta(x) \equiv 1, \varepsilon(x) \equiv x$, $\alpha, \beta, \varepsilon \in T_X$, $x \in X$. We will denote by f_q the automaton transformation determined by the automaton A at the state $q \in Q$. In order to prove that the semigroup S_A is infinite we will show infinite order of some element $\overline{f} \in S_A$, namely we will select a word $w \in X^{\omega}$ such that the words $\overline{f}(w), \overline{f}^2(w) = \overline{f}\left(\overline{f}(w)\right), \ldots, \overline{f}^i(w), \ldots$ are pairwise different. This condition provides pairwise difference of $\overline{f}, \overline{f}^2, \ldots, \overline{f}^i, \ldots$ as a transformation over X^{ω}, whence the semigroup S_A is infinite.

Let π be a transition function such that $\Pi^{\infty}(Q)$ contains a branch state. Let us consider all possible sequences of states of the form $q_0, q_1 = \pi(0, q_0), \ldots, q_{i+1} = \pi(0, q_i), \ldots$. Each of them is ultimately periodic. So we can select from them their periodical parts (simple cycles, marked by 0 in the Moore diagram) of the form $C = \{q_z, q_{z+1}, \ldots, q_{z+k} = q_z\}$, where $z \geqslant 0, k \geqslant 1$. Only one of the following three cases is possible.

a) All these cycles consist of states without branches.

b) There is a cycle C, marked by 0, such that there exists a state $q \in C$ and a symbol $x \in X$ such that $\pi(x, q) \notin C$ (i.e. there exists a cycle which we can leave).

c) There exists a cycle with a branch state but there is no cycle described in case b).

We will review all this cases sequentially and construct automata which generate infinite semigroups for each case.

Lemma 5. *Let* π *be a transition function such that all cycles, marked by 0, consists of states without branches but the set* $\Pi^{\infty}(Q)$ *contains a branch state. Then there exists the output function* λ *such that* $A = (X, Q, \pi, \lambda)$ *generates an infinite automaton transformation semigroup.*

Proof. Let us consider the following sets.

- the set Q_1 of all states $q \in Q$ such that there exists a nonnegative integer number $k = k(q)$ such that for each word $w \in X^*$ of length greater than or equal to k ($|w| \geqslant k$) the state $\pi(w, q)$ will be a state without branches,
- the set $Q_2 = \{q \in Q \backslash Q_1 | \exists x \in X : \pi(x, q) \in Q_1\}$ of all states, from which the automaton can get to a state from Q_1 under action of one input symbol.
- $Q_3 = Q \backslash (Q_1 \cup Q_2)$ is the set of all other states.

Let us consider an example of an automaton which satisfies the conditions of Lemma 5. Let $X = \{0, 1\}$ and let the Moore diagram of the automaton be shown in Figure 2. We have $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$, $Q_1 = \{q_3, q_4, q_5\}$, $\Pi^{\infty}(Q_1) = \{q_4, q_5\}$, $Q_2 = \{q_1, q_2\}$, $Q_3 = \{q_0\}$.

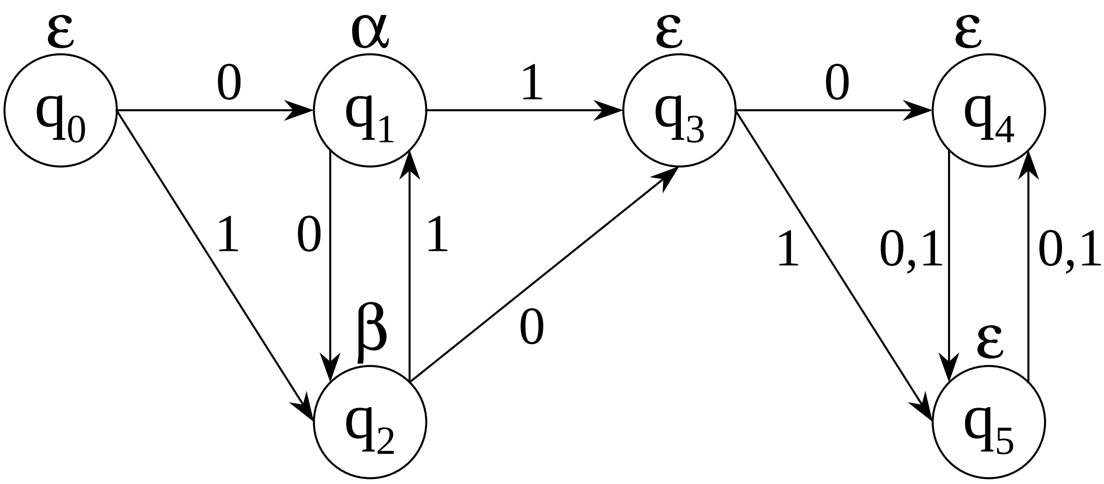

Fig. 2: Example of an automaton for Lemma 5

Let us prove that $\Pi(Q_1) \subseteq Q_1$, and all states $q \in \Pi^\infty(Q_1)$ are states without branches. Indeed, let $q \in \Pi(Q_1)$, i.e. $q = \pi(x, q_1), x \in X, q_1 \in Q_1$ and also for q_1 there exists $k_1 = k(q_1)$ such that for each word $w_1 \in X^*$ of length greater then or equal to k_1 ($|w_1| \geqslant k_1$) all states of the form $\pi(w_1, q_1)$ will be states without branches. Set $k = k(q) = k_1 - 1$ (if $k_1 = 0$, than set $k = 0$). Then for each word $w \in X^*$ of length greater than or equal to k ($|w| \geqslant k$) we have $\pi(w, q) = \pi(w, \pi(x, q_1)) = \pi(wx, q_1)$.

Since the length of the word is greater than or equal to $(k+1)$, i.e. this length is greater than or equal to k_1, the state $\pi(w, q)$ is a state without branches. Therefore, it follows from the definition of Q_1 that $q \in Q_1$. Since q is arbitrary state from the set $\Pi(Q_1)$, we conclude that $\Pi(Q_1) \subseteq Q_1$. Since the set Q_1 is finite, there exists $k_{\max} = \max\limits_{q \in Q_1}\{k(q)\}$ and $\Pi^{k_{\max}}(Q_1)$ contains only states without branches, hence all states in the set $\Pi^\infty(Q_1)$ are states without branches. By lemma condition, $Q \neq Q_1$, hence the sets Q_1 and Q_2 are not empty, but the set Q_3 can be empty.

We will assign the transformation ε to each state from Q_1 and Q_3. For each state $q \in Q_2$ the set $X_q = \{x \in X | \pi(x, q) \in Q_2 \cup Q_3\}$ is not empty (otherwise $q \in Q_1$), but it does not coincide with the alphabet X. For each $q \in Q_2$ we choose an arbitrary element $x_q \in X_q$, and we assign to the state q the transformation

$$\lambda_q(x) = \begin{cases} x, & x \in X_q; \\ x_q, & x \notin X_q. \end{cases}$$

Fix an arbitrary state $\overline{q} \in Q_3 \cup Q_2$, the word $w = 0^*$ and the transformation $\overline{f} = f_{\overline{q}}$.

The set $\Pi^\infty(Q_1)$ is the union of all disjoint cycles which consist of states without branches, Q_1 is the set of all states, from which the automaton always moves to a state from $\Pi^\infty(Q_1)$ after a while. Since $\Pi(Q_1) \subseteq Q_1$, the property (2) holds. Thus for all $q \in Q_1$ and $v \in X^*$ the equality $\pi(v, q) \in Q_1$ holds, i.e. the automaton moves from state which belongs Q_1 only to a state from Q_1.

Consider an action of an automaton transformation $f_{\overline{q}}$ on arbitrary infinite word of the form

$$v = v'0^* \in X^\omega,\ v' \in X^* \tag{3}$$

which contains only finite number of nonzero symbols. Let us prove that the automaton starting with the state $\overline{q}$ under action of this word goes to a state from the set Q_1. Indeed, either $\pi(v', \overline{q}) \in Q_1$, or if $\pi(v', \overline{q}) \in Q_2 \cup Q_3$, then the automaton starts to work at the state $\pi(v', \overline{q})$, and processes the input sequence that consists of zeroes. Since there is no 0-cycles in $Q_2 \cup Q_3$, the automaton goes to a state from the set Q_1. We will denote by $v^{(n)} = v_1 v_2 \dots v_n$ the initial segment of v and by $q^{(n)} = \pi\left(v^{(n-1)}, \overline{q}\right)$ the state to which the automaton goes under action of first $n-1$ symbols of word v. Let $c = c(v)$ be the minimal number such that $q^{(c+1)} \in Q_1$. It is obvious, that $c \geqslant 1$, $q^{(c)} \in Q_2$. For all $r < c-1$, we have $q^{(r)} \in Q_2 \cup Q_3$, and for all $r' \geqslant c$, we have $q^{(r')} \in Q_1$.

It follows from construction of the output function that the automaton changes symbol of an input word if and only if it goes from a state from the set Q_2 to a state from the set Q_1. Therefore $f_{\overline{q}}(v)$ differs from v only at the c-th position. The word $f_{\overline{q}}(v)$ also has form (3), so we can apply similar arguments to it, i.e. $f_{\overline{q}}(f_{\overline{q}}(v))$ differs from $f_{\overline{q}}(v)$ only in the position $c(f_{\overline{q}}(v))$. Moreover, $c(f_{\overline{q}}(v)) > c(v)$, since the automaton "fixes" symbol at position $c(v)$, after which the automaton goes to state from Q_1. Processing the input word $f_{\overline{q}}(v)$, the automaton works in the same way until it reaches the position $c(v) - 1$, i.e. $q^{(r)}(f_q(v)) = q^{(r)}(v), r \leqslant c(v)$. But since $\pi\left(v_{c(v)}, q^{(c(v))}(v)\right) = q^{(c(v)+1)}(v) \in Q_1$ and the symbol $x_{q^{(c(v))}(v)} \in X_{q^{(c(v))}(v)}$ is located in $c(v)$-th position of the word $f_{\overline{q}}(v)$, we see that

$$q^{(c(v)+1)}(f_{\overline{q}}(v)) = \pi\left(x_{q^{(c(v))}(v)}, q^{(c(v))}(f_{\overline{q}}(v))\right) = \pi\left(x_{q^{(c(v))}(v)}, q^{(c(v))}(v)\right) \notin Q_1.$$

Therefore the automaton moves to a state that belongs to Q_1 and changes the symbol later, i.e. $c(f_{\overline{q}}(v)) > c(v)$.

Applying sequentially the automaton transformation $\overline{f} = f_{\overline{q}}$ to the word $w = 0^*$ (this word has the form (3)), we obtain the words of the form (3) $f_{\overline{q}}(w), f_{\overline{q}}(f_{\overline{q}}(w)), \dots, f_{\overline{q}}^{(i)}(w), \dots$. Each of these words differs from previous word only by one symbol (zero has been changed to nonzero symbol). Moreover, the position of this symbol is located farther and farther from the beginning of word. This proves infinite order of $f_{\overline{q}}$ as well as infiniteness of semigroup generated by the automaton. □

The output function constructed by the rules described in Lemma 5 for the example given above is shown in Figure 2. The automaton transformations f_{q_0}, f_{q_1}, f_{q_2} have infinite order, so the semigroup defined by the automaton also has infinite order. Indeed, $f_{q_2}(0^*) = 10^*$, $f_{q_2}^2(0^*) = f_{q_2}(10^*) = 1010^*$ and so on.

Lemma 6. *Let π be a transition function such that there exist a cycle $q_0, q_1, \dots, q_{k-1}$, marked by 0, a state $q \in \{q_0, q_1, \dots, q_{k-1}\}$ and a symbol $x \in X \setminus \{0\}$ such that $\pi(x, q) = q' \notin \{q_0, q_1, \dots, q_{k-1}\}$. Then there exists the output function λ such that $A = (X, Q, \pi, \lambda)$ generates an infinite automaton transformation semigroup.*

Proof. Without loss of generality we can assume $q = q_0$, $x = 1$, i.e. $\pi(q_0, 1) = q' \notin \{q_0, q_1, \dots, q_{k-1}\}$. We will assign the transformation α to the states $q_0, q_1, \dots, q_{k-1}$, and the transformation β to the state q'. See Figure 3 for an example in the case $k = 4$ (only important transitions are shown). Set $w = 1^*$.

Consider the action of automaton transformation $f_{q_j}(0^i 1 v)$, where $v \in X^w, i \geqslant 0, 0 \leqslant j < k-1$, $j + i \equiv 0 \,(\mathrm{mod} k)$. From the last congruence it follows that $\pi(0^i, q_j) = q_0$. Therefore under action of i zeroes the automaton goes from state q_j to state q_0 and outputs i zeroes.

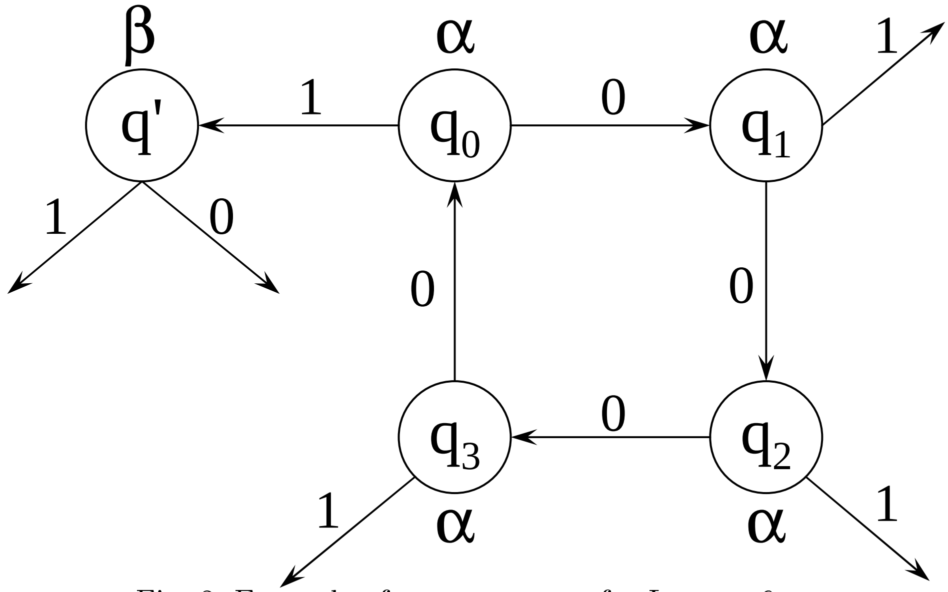

Fig. 3: Example of an automaton for Lemma 6

Then under the symbol 1, the automaton goes from state q_0 to state q' and outputs the symbol 0. After that the automaton emits the symbol 1. Thus

$$f_{q_j}\left(0^i 1v\right) = f_{q_j}\left(0^i\right) f_{q_0}(1) f_{q'}(v) = 0^i 0 f_{q'}(v) = 0^{i+1} 1 v'$$

for some $v' \in X^w$.

Let $\overline{f} = f_{q_1} \circ f_{q_2} \circ \cdots \circ f_{q_{k-1}} \circ f_{q_0}$ and consider $\overline{f}\left(0^{ik}1v\right)$, where i is a nonnegative integer. We obtain

$$\begin{aligned} f_{q_0}\left(0^{ik}1v\right) &= 0^{ik+1}1v_1 \text{ since } 0 + ik \equiv 0\,(\mathrm{mod} k) \\ f_{q_{k-1}}\left(0^{ik+1}1v_1\right) &= 0^{ik+2}1v_2 \text{ since } k-1+ik+1 \equiv 0\,(\mathrm{mod} k)) \\ &\cdots \\ f_{q_1}\left(0^{ik+k-1}1v_1\right) &= 0^{ik+k}1v_k \end{aligned}$$

where $v_1, \ldots, v_k \in X^\omega$. Therefore $\overline{f}\left(0^{ik}1v\right) = f_{q_1}\left(\cdots\left(f_{q_{k-1}}\left(f_{q_0}\left(0^{ik}1v\right)\right)\right)\cdots\right) = 0^{(i+1)k}1v'$ for some word $v' \in X^\omega$.

We have $\overline{f}(w) = \overline{f}(1^*) = 0^k 1 v_1, \ldots$. By induction, $\overline{f}^i(w) = 0^{ik}1v_i$, $v_i \in X^\omega$, which proves infinite order of $\overline{f} \in S_A$. Therefore semigroup S_A defined by the automaton A is infinite. □

Lemma 7. *Let π be a transition function such that there is no a cycle C, marked by 0, a state $q \in C$ and a symbol $x \in X \setminus \{0\}$ such that $\pi(x, q) \notin C$. Assume that there exists a 0-cycle $q_0, q_1, \ldots, q_{k-1}$ which contains a branch state. Then there exists an output function λ such that $A = (X, Q, \pi, \lambda)$ generates an infinite automaton transformation semigroup.*

Proof. Without loss of generality we can assume that q_0 is a branch state. Thus there exists $x \in X \backslash \{0\}$ such that $\pi(x, q_0) \neq \pi(0, q_0) = q_1$. We can assume $x = 1$. Consider cyclic part of sequence $q'_0 = q_0, q'_1 = \pi(1, q'_0), \ldots, q'_{i+1} = \pi(1, q'_i), \ldots$ of transitions under symbol 1 from the state q_0. This part has the form $q'_t, q'_{t+1}, \ldots, q'_{t+l} = q'_t$, where $t \geqslant 0, l \geqslant 1$. The case where this cyclic part does not contain all states $q_0, q_1, \ldots, q_{k-1}$ is similar to Lemma 6

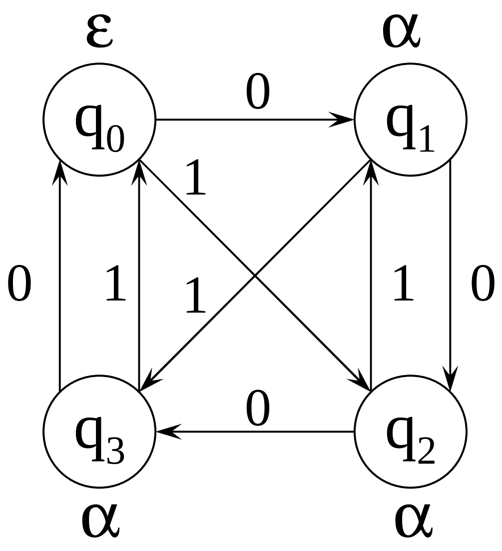

Fig. 4: Example of an automaton for Lemma 7

with interchanging of symbols 0 and 1 (i.e. there exists cycle marked by 1 with exit by 0 at least in one state). In the other case, we consider the set of states $\widetilde{Q} = \{q_0, q_1, \ldots, q_{k-1}\}$, the alphabet $\widetilde{X} = \{0, 1\}$ and the corresponding restriction of the transition function. They define group automaton with two different permutations ς_0 и ς_1 of states under actions of symbols of alphabet. Here $\varsigma_x(q) = \pi(x, q), x \in \widetilde{X}, q \in \widetilde{Q}$ and ς_1 is an permutation because it is surjective transformation of a finite set on itself. These two permutations are cycles of length k, so their orders are k. Moreover, ς_0 is a cyclic permutation, that is $\varsigma_0(q_i) = q_{(i+1) \bmod k}$. We have $\varsigma_0(q_0) \neq \varsigma_1(q_0)$ because q_0 is a branch state.

We will assign the transformation ε to the state q_0, transformation α to the states $q_1, \ldots, q_{k-1}$ and the transformation ε to other states. Thus we have defined the two automata $A = (X, Q, \pi, \lambda)$ and $\widetilde{A} = \left(\widetilde{X}, \widetilde{Q}, \widetilde{\pi}, \widetilde{\lambda}\right)$, where $\widetilde{\pi}\colon \widetilde{X} \times \widetilde{Q} \to \widetilde{Q}$ and $\widetilde{\lambda}\colon \widetilde{X} \times \widetilde{Q} \to \widetilde{X}$ are corresponding restrictions of functions π and λ (these function are well defined because $\pi\left(\widetilde{X}, \widetilde{Q}\right) \subset \widetilde{Q}$ and $\lambda\left(\widetilde{X}, \widetilde{Q}\right) \subset \widetilde{X}$). See Figure 4 for an example of $\widetilde{A}$. The semigroup generated by $\widetilde{A}$ is a subsemigroup of the factor semigroup of the semigroup generated by A. Let us prove the infinity of semigroup generated by $\widetilde{A}$, whence the infinity of semigroup generated by A concludes. To this end we set $w = 1^*$, $\overline{f} = f_{q_0}$.

Then we have $\overline{f}(w) = f_{q_0}(1^*) = \left(10^{k-1}\right)^*$ since the order of ς_1 equals k, i.e. the automaton will return to the state q_0 after reading k symbols. Let $b_0 = 0^k, b_1 = 10^{k-1}, b_0, b_1 \in \widetilde{X}^*$, $B_u = b_{u_1} b_{u_2} \ldots \in \left(\widetilde{X}^\omega \cup \widetilde{X}^*\right)$, where $u = u_1 u_2 \ldots \in \left(\widetilde{X}^\omega \cup \widetilde{X}^*\right)$ is a finite or infinite word. We have

$$\pi(b_0, q_i) = \varsigma_0^k(q_i) = q_{(i+k) \bmod k} = q_i, \lambda(b_0, q_i) = b_0, \text{ and } \pi(b_1, q_i) = \varsigma_0^{k-1}(\varsigma_1(q_i))$$

for each integer i such that $0 \le i \le k-1$. Also $\lambda(b_1, q_0) = b_1$, $\lambda(b_1, q_i) = b_0$ for each integer i such that $1 \le i \le k-1$. Note that $g = \varsigma_0^{k-1} \circ \varsigma_1 = \varsigma_0^{-1} \circ \varsigma_1$ is not the identity permutation. Moreover, $g(q_0) = \varsigma_0^{-1}(\varsigma_1(q_0)) \neq q_0$, since otherwise $\varsigma_1(q_0) = \varsigma_0(g(q_0)) = \varsigma_0(q_0)$, that contradicts the assumption that $\varsigma_0(q_0) \neq \varsigma_1(q_0)$. Let the order of the permutation g be equal to r $(r > 1)$. In this case, $\pi(B_u, q_0) = q_0$ for a finite word $u \in \widetilde{X}^*$ if and only if the quantity of ones in the word u is divisible by r, since each block of the form b_1 acts on state as permutation g, and each block of the form b_0 acts on state as identity permutation. Therefore $f_{q_0}(B_u) = B_{u'}$, where $u = u_1u_2\ldots \in \widetilde{X}^\omega$, $u' = u'_1u'_2\ldots \in \widetilde{X}^\omega$,

$$u'_i = \begin{cases} 1, & \text{if } u_i = 1 \text{ and } \pi\left(B_{u_1u_2\ldots u_{i-1}}, q_0\right) = q_0 \\ 0, & \text{otherwise,} \end{cases}$$

i.e.

$$u'_i = \begin{cases} 1, & \text{if } u_i = 1 \text{ and the count of ones in the word } u_1u_2\ldots u_{i-1} \text{ is divisible by } r, \\ 0, & \text{otherwise.} \end{cases}$$

Thus in the word u only the first, $r+1$-th, $2r+1$-th and so on letters remain, and all other letters are changed to 0.

Let us prove by induction that for all $j \geqslant 1$, $f_{q_0}^j(1^*) = B_{u^{(j)}}$, where $u^{(j)} = \left(10^{\left(r^{j-1}-1\right)}\right)^*$. For $j=1$ we have $u^{(1)} = \left(10^{\left(1^{1-1}-1\right)}\right)^* = (10^0)^* = 1^*$, $f_{q_0}(1^*) = B_{1^*} = B_{u^{(1)}}$. Assume that for $j = t$ the equality $f_{q_0}^t(1^*) = B_{u^{(t)}}, u^{(t)} = \left(10^{\left(r^{t-1}-1\right)}\right)^*$ holds. Then for $j = t+1$ we have

$$f_{q_0}^{t+1}(1^*) = f_{q_0}\left(f_{q_0}^t(1^*)\right) = f_{q_0}\left(B_{u^{(t)}}\right) = B_{u'},$$

$$u^{(t)} = \left(10^{\left(r^{t-1}-1\right)}\right)^* = \Bigg(\underbrace{10^{\left(r^{t-1}-1\right)} \cdot 10^{\left(r^{t-1}-1\right)} \cdot \ldots \cdot 10^{\left(r^{t-1}-1\right)}}_{r}\Bigg)^*.$$

Acting by transformation f_{q_0} we obtain

$$u' = \Bigg(\underbrace{10^{\left(r^{t-1}-1\right)} \cdot 00^{\left(r^{t-1}-1\right)} \cdot \ldots \cdot 00^{\left(r^{t-1}-1\right)}}_{r}\Bigg)^* = \left(10^{\left(r^t-1\right)}\right)^* = u^{(t+1)}.$$

Hence the words $f_{q_0}^j(1^*), j = 1, 2, \ldots$ are pairwise different. It follows that the order of f_{q_0} is infinite. Therefore the semigroup defined by $\widetilde{A}$ is infinite. Thus the semigroup defined by A is also infinite. □

Proof of Theorem 3. Let condition 2 hold, i.e. there exists a cyclic branch state for π. By Lemma 4, this state belongs to $\Pi^\infty(Q)$, so condition 3 holds.

Let condition 3 hold, i.e. $\Pi^\infty(Q)$ contains a branch state. Let us prove that condition 2 holds. On the contrary, suppose that all cyclic states are states without branches. Let $q \in Q$ be an arbitrary cyclic state, i.e. for some nonempty word $w = w_1\ldots w_k \in X^*$ the equality $\pi(w, q) = q$ holds. Then $q' = \pi(w_1, q)$ is a cyclic state, since $\pi(w_2\ldots w_kw_1, q') = \pi(w_2\ldots w_kw_1, \pi(w_1, q)) = \pi(w_1w_2\ldots w_kw_1, q) = \pi(w_1, q) = q'$. Since q by assumption is a state without branches, for all $x \in X$ we have $\pi(x, q) = \pi(w_1, q) = q'$, i.e. the automaton being in a cyclic state goes only to cyclic states. Thus the set $\Pi^\infty(Q)$, being equal to the set of states accessible from the cyclic ones, is equal to the set of all cyclic states. Since there is no branch cyclic state, it follows that there is no branch state which belongs to $\Pi^\infty(Q)$. This contradicts to condition 3.

Let us prove by contradiction that condition 3 follows from condition 1. Suppose that a transition function $\pi\colon X \times Q \to Q$ defines an infinite semigroup but $\Pi^\infty(Q)$ consists only of states without branches. By Theorem 2 an arbitrary automaton $A = (X, Q, \pi, \lambda)$ with the transition function π defines finite semigroup. This contradicts to our assumption.

It suffices to prove that condition 1 follows from condition 3. Let condition 3 hold, i.e. $\Pi^\infty(Q)$ contains a branch state. By the above, the transition function π satisfies conditions of one of Lemmas 5, 6, 7. Therefore, there exists an output function λ such that $A = (X, Q, \pi, \lambda)$

defines an infinite automaton transformation semigroup. Thus the transition π generates an infinite automaton transformation semigroup, which proves the theorem. □

6. Conclusion. A large number of automata of finite growth are described in this paper, namely automata with limitary cycle. It is possible to make a conclusion that they have finite growth judging only by transition function. For all other transition functions automata that generate infinite semigroups are constructed.

There are three interesting ways to extend the results of the paper:

1. for a transition function without limitary cycle to build an invertible automaton which defines infinite group;
2. to characterize all automata of finite growth;
3. to construct a criterion similar to Theorem 3 for other classes of growth: polynomial, intermediate, exponential.

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